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COMPACTLY GENERATED QUASITOPOLOGICAL HOMOTOPY GROUPS WITH DISCONTINUOUS MULTIPLICATION

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Abstract. For each integer $Q \geq 1$, there exists a path connected metric compactum X such that the homotopy group $\pi_Q(X,p)$ is compactly generated but not a topological group (with the quotient topology).

1. Introduction

We answer in the negative the question posed in [7] (and in [1, Problem 5.1]) of whether higher homotopy groups (with a certain natural quotient topology) are guaranteed to be topological groups. For each $Q \geq 1$, we exhibit a compactly generated counterexample.

The familiar homotopy group $\pi_Q(X,p)$ becomes a topological space endowed with the quotient topology induced by the natural surjective map $\Pi_Q : M_Q(X,p) \to \pi_Q(X,p)$. ($M_Q(X,p)$ denotes the space of based maps, with the compact open topology, from the Q-sphere S^Q into X.) It is an open problem to understand when $\pi_Q(X,p)$ is or is not a topological group with the standard operations.

Straightforward proofs [6] show that the topology of $\pi_Q(X,p)$ is an invariant of the homotopy type of the underlying space X, that $\pi_Q(X,p)$ is a quasitopological group (i.e., multiplication is continuous separately in each coordinate and group inversion is continuous), and that each map $f : X \to Y$ induces a continuous homomorphism $f_* : \pi_Q(X,p) \to \pi_Q(Y,f(p))$.

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If X has strong local properties (for example, if X is locally n-connected for all $0 \leq n \leq Q$), then $\pi_Q(X,p)$ is discrete [3],[2],[6], and hence a topological group.

If $Q \geq 2$, the group $\pi_Q(X,p)$ is abelian, and this fact has the capacity to nullify structural pathology present when $Q = 1$. For example, H. Ghane, Z. Hamed, B. Mashayekhy, and H. Mirebrahimi [7] show if $Q \geq 2$, then $\pi_Q(X,p)$ is a topological group if X is the Q-dimensional version of the 1-dimensional Hawaiian earring.

If $Q = 1$, the standard group multiplication in $\pi_1(X,p)$ can fail to be continuous if $\Pi_1 \times \Pi_1 : M_1(X,p) \times M_1(X,p) \to \pi_1(X,p) \times \pi_1(X,p)$ fails to be a quotient map. Recent counterexamples of Jeremy Brazas [1, Example 4.22] and Paul Fabel [4] show $\pi_1(X,p)$ fails to be a topological group if X is the union of large circles joined at a common point and parameterized by the rationals, or if X is the 1-dimensional Hawaiian earring, respectively.

For the main result, given $Q \geq 1$, we obtain a space X as the union of convergent line segments $L_n \to L$, joined at the common point p, with a small Q-sphere S_n attached to the end of each segment L_n. This yields the following.

Theorem 1.1. *For each $Q \in \{1,2,3,...\}$, there exists a compact path connected metric space X such that, with the quotient topology, $\pi_Q(X,p)$ is compactly generated and multiplication is discontinuous in $\pi_Q(X,p)$.*

2. Definitions

If Y is a space and if $A \subset Y$, then the set A is *closed under convergent sequences* if A enjoys the following property: If the sequence $\{a_1, a_2, ...\} \subset A$ and if $a_n \to a$, then $a \in A$.

The space Y is a *sequential space* if Y enjoys the following property: If $A \subset Y$ and if A is closed under convergent sequences, then A is a closed subspace of Y.

If Y and Z are spaces, the surjective map $q : Y \to Z$ is a *quotient map* if, for every subset $A \subset Z$, the set A is closed in Z if and only if the preimage $q^{-1}(A)$ is closed in Y.

Given a space X and $p \in X$ and an integer $Q \geq 1$, let $\pi_Q(X,p)$ denote the familiar Qth homotopy group of X based at p.

To topologize $\pi_Q(X,p)$, let $M_Q(X,p)$ denote the space of based maps $f : (S^Q, 1_Q) \to (X,p)$ from the Q-sphere S^Q into X, and impart $M_Q(X,p)$ with the compact open topology.

Let $\Pi_Q : M_Q(X,p) \to \pi_Q(X,p)$ be the canonical quotient map such that $\Pi_Q(f) = \Pi_Q(g)$ if and only if f and g belong to the same path component of $M_Q(X,p)$, and declare $U \subset G$ to be open if and only if $\Pi_Q^{-1}(U)$ is open in $M_Q(X,p)$.

A *Peano continuum* is a compact locally path connected metric space.

If Q is a positive integer, a *closed Q-cell* is any space homeomorphic to $[0,1]^Q$, and the *Q-sphere* S^Q is the quotient of $[0,1]^Q$ by identifying to a point the Q-1 dimensional boundary $[0,1]^Q \backslash (0,1)^Q$.

3. Basic Properties of $\pi_Q(X,p)$

Lemma 3.1. *If X is a metrizable space, then $M_Q(X,p)$ is a metrizable space, and $\pi_Q(X,p)$ is a sequential space.*

Proof. Since X is metrizable and since S^Q is a compact, the uniform metric shows $M_Q(X,p)$ is metrizable. Moreover, since S^Q is compact, the compact open topology coincides with the metric topology of uniform convergence in $M_Q(X,p)$. Thus, $\pi_Q(X,p)$ is the quotient of a metric space, and hence $\pi_Q(X,p)$ is a sequential space [5]. □

Lemma 3.2. *Suppose X is metrizable and suppose $z_n \to z$ in $\pi_Q(X,p)$. Then there exists $n_1 < n_2 ...$ and a convergent sequence $\alpha_{n_k} \in M_Q(X,p)$ such that $\Pi_Q(\alpha_{n_k}) = z_{n_k}$.*

Proof. If there exists N such that $z_n = z$ for all $n \geq N$, then there exists $\alpha \in M_Q(X,p)$ such that $\Pi_Q(\alpha) = z$ (since Π_Q is surjective). Let $\alpha_n = \alpha$ for all $n \geq N$, and thus the constant sequence $\{\alpha_n\}$ converges.

If no such N exists, then obtain a subsequence $\{x_n\} \subset \{z_n\}$ such that $x_n \neq z$ for all n. To see that $\{x_1, x_2, ...\}$ is not closed in $\pi_Q(X,p)$, note $z \notin \{x_1, x_2, ...\}$ and, since $x_n \to z$, it follows that z is a limit point of the set $\{x_1, x_2, ...\}$. Thus, since Π_Q is a quotient map, $\Pi_Q^{-1}\{x_1, x_2, ...\}$ is not closed in $M_Q(X,p)$. Obtain a limit point $\beta \in \overline{\Pi_Q^{-1}\{x_1, x_2, ...\}} \backslash \Pi_Q^{-1}\{x_1, x_2, ...\}$. Since $M_Q(X,p)$ is metrizable (Lemma 3.1), there exists a sequence $\beta_k \to \beta$ such that $\{\beta_k\} \subset \Pi_Q^{-1}\{x_1, x_2, ...\}$. Moreover, (refining $\{\beta_k\}$ if necessary), there exists a subsequence $\{m_1, m_2, ...\} \subset \{1, 2, ...\}$ such that $\Pi_Q(\beta_k) = x_{m_k}$, and if $k < j$, then $m_k < m_j$. Let $x_{m_k} = z_{n_k}$ and let $\alpha_{n_k} = \beta_k$. □

4. Main Result

Fixing a positive integer Q, the goal is to construct a path connected compact metric space X such that if $\pi_Q(X,p)$ has the quotient topology, then $\pi_Q(X,p)$ is compactly generated, but the standard group multiplication is not continuous in $\pi_Q(X,p)$.

The idea to construct X is to begin with the cone over a convergent sequence $\{w, w_1, w_2, ...\}$ (i.e., we have a sequence of convergent line segments $L_n \to L$, joined at a common endpoint $p \in L_n$), and then we attach a Q-sphere S_n of radius $\frac{1}{10^n}$ to the opposite end of each segment L_n at w_n.

Specifically, for $Q \geq 1$, let R^{Q+1} denote Euclidean space of dimensions $Q+1$ with Euclidean metric d. Let $p = (0,0,...,0)$ and let $w_n = (\frac{1}{n}, 1, 0, ...0)$ and let $w = (0,1,0,...,0)$. Consider the line segment $L_n = [p, w_n]$, and observe for each n, if $i \neq n$, then $d(w_n, w_i) \geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2+n}$.

Let $L = [p, w]$ and let $\gamma_n : L \to L_n$ be the linear bijection fixing p and observe $\gamma_n \to id|L$ uniformly.

Let $c_n = (\frac{1}{n}, 1 + \frac{1}{10^n}, 0, ..., 0)$. Let S_n denote the Euclidean Q-sphere such that $x \in S_n$ if and only if $d(x, c_n) = \frac{1}{10^n}$. Let $q_n = (\frac{1}{n}, 1 + \frac{2}{10^n}, 0, ..., 0)$ and notice $S_n \cap L_n = \{w_n\}$.

Let $X_n = \cup_{k=1}^{n} (L_k \cup S_k)$.

Let $X = L \cup X_1 \cup X_2 ....$

Define retracts $R_n : X \to X_n$ such that

$$R_n(x_1, x_2, ..., x_{n+1}) = (x_1^*, x_2, ..., x_{n+1})$$

with x_1^* minimal such that $im(R_n) \subset X_n$.

Notice $R_n \to id_X$ uniformly, and for all n, we have $R_n R_{n+1} = R_n$. Hence, we obtain the natural homomorphism

$$\phi : \pi_Q(X, p) \to \lim_{\leftarrow} \pi_Q(X_n, p)$$

defined via $\phi([\alpha]) = \{R_{n*}([\alpha])\}$.

For $Q \geq 1$, let $G = \pi_Q(X, p)$. Thus, if $Q = 1$, then G is the free group on generators $\{x_1, x_2, ...\}$ and if $Q \geq 2$, then G is the free abelian group on generators $\{x_1, x_2, ...\}$. Let $* : G \times G \to G$ denote the familiar multiplication.

Elements of G admit a canonical form as maximally reduced words in the letters $\{x_1, x_2, ...\}$ in the format $x_{n_1}^{k_1} * x_{n_2}^{k_2} * ... * x_{n_m}^{k_m}$ with $n_i \neq n_{i+1}$. If $Q \geq 2$, then G is abelian and we can further require $n_i < n_j$ whenever $i < j$.

Define $l : G \to \{0, 1, 2, 3, ...\}$ such that $l(x_{n_1}^{k_1} * x_{n_2}^{k_2} ... * x_{n_m}^{k_m}) = m$.

Let G_N denote the subgroup of G such that if $x_{n_1}^{k_1} * x_{n_2}^{k_2} ... * x_{n_m}^{k_m} \in G_N$, then $n_i \leq N$ for all i.

Let $\phi_N : G \to G_N$ denote the natural epimorphism such that $\phi_N(g)$ is the reduced word obtained from g after deleting all letters x_i of index $i > N$.

Lemma 4.1. *The homomorphism* $\phi : \pi_Q(X, p) \to \lim_{\leftarrow} \pi_Q(X_n, p)$ *is continuous and one to one, and the space* $\pi_Q(X, p)$ *is* T_2.

Proof. Recall $G = \pi_Q(X,p)$ and note $\pi_Q(X_n,p)$ is canonically isomorphic to G_n. To see that ϕ is continuous, first recall, in general, a map $\alpha : Y \to Z$ [6] induces a continuous homomorphism $\alpha_* : \pi_Q(Y,y) \to \pi_Q(Z,\alpha(z)))$ and, in particular, the retractions $R_n : X \to X_n$ induce continuous epimorphisms $R_{n*} : G \to G_n$. By definition, $\phi = \{R_{n*}\}$, and hence ϕ is continuous since $\lim_{\leftarrow} G_n$ enjoys the product topology.

To see that ϕ is one to one, suppose $[f] \in \ker\phi$. Since $f : S^Q \to X$ is a map and since S^Q is a Peano continuum, then $im(f)$ is a Peano continuum, and hence $im(f)$ is locally path connected, and, in particular, $im(f) \cap \{q_1, q_2, ...\}$ is finite.

Obtain N such that $im(f) \cap \{q_1, q_2, ...\} \subset \{q_1, ..., q_N\}$. Notice X_N is a strong deformation retract of $X \backslash \{q_{N+1}, q_{N+2}, ...\}$ (since for $i \geq N+1$, $S_i \backslash \{q_i\}$ is contractible to p, and we can contract to p simultaneously for $k \in \{1,2,3,...\}$ the subspaces $L \cup L_{N+k} \cup (S_{N+k} \backslash \{q_{N+k}\})$. In particular, under the strong deformation retraction collapsing $X \backslash \{q_{N+1}, q_{N+2}, ...\}$ to X_N, f deforms in X to $R_N \circ f$, and, by assumption, $R_N \circ f$ deforms in X_N to the constant map (determined by p). Hence, f is inessential in X and this proves ϕ is one to one.

Since X_n is locally contractible, $\pi_Q(X_n,p)$ is discrete [6]. Hence, $\lim_{\leftarrow} \pi_Q(X_n,p)$ is metrizable and, in particular, T_2. Thus, G is T_2 since G injects continuously into the T_2 space $\lim_{\leftarrow} G_n$. □

Lemma 4.2. *Suppose $\{g, g_1, g_2, ...\} \subset G$. Suppose $g_n \to g$. Then $\{l(g_n)\}$ is bounded and $\phi_N(g_n) \to \phi_N(g)$.*

Proof. Suppose $g_n \to g$. Then $\phi(g_n) \to \phi(g)$ in $\lim_{\leftarrow} G_n$, since ϕ is continuous. This means precisely that for each $N \geq 1$, the sequence $\phi_N(g_n) \to \phi_N(g)$.

To prove $\{l(g_n)\}$ is bounded, suppose, to obtain a contradiction, $\{l(g_n)\}$ is not bounded. Select a subsequence $\{y_n\} \subset \{g_n\}$ such that $l(y_n) \to \infty$. By Lemma 3.2, there exists a subsequence $\{z_n\} \subset \{y_n\}$ and a convergent sequence of maps $\{\alpha_n\} \subset M_Q(X,p)$ such that $\Pi(\alpha_n) = z_n$. Let $z_n = x_{s_1}^{k_1} * x_{s_2}^{k_2} ... * x_{s_{m_n}}^{k_{m_n}}$ and let $Z_n = \{q_{s_1}, q_{s_2}, ... q_{s_{m_n}}\}$. Thus, Z_n consists of the tops of the corresponding spheres in X, and hence $Z_n \subset im(\alpha_n)$. Obtain a surjective map $\gamma : [0,1] \to S^Q$ (such that $\gamma(\{0,1\}) = 1_Q$) and notice there exists $s_0 = 0 < t_1 < s_1 < t_2 < s_2 ... < t_{m_n} < 1 = s_{m_n}$ such that $\alpha_n \gamma(s_i) = p$ and $\alpha_n(t_i) \in Z_n$. Note $d(q_i, p) > \frac{1}{2}$. Thus, since $m_n \to \infty$, the maps $\{\alpha_n\}$ are not equicontinuous, contradicting the fact that the convergent sequence $\{\alpha_n\}$ is equicontinuous. □

Remark 4.3. $\pi_Q(X,p)$ is compactly generated. (Select a convergent sequence of generators $v_1, v_2, ... \subset M_Q(X,p)$, such that $[v_n]$ generates the cyclic group $\pi_Q(L_n \cup S_n, p))$ and $[v_n] \to e$ and e denotes the identity in $\pi_Q(X,p))$

Theorem 4.4. *Multiplication* $* : \pi_Q(X,p) \times \pi_Q(X,p) \to \pi_Q(X,p)$ *is not continuous.*

Proof. Recall $G = \pi_Q(X,p)$ and consider the following doubly indexed subset $A \subset G$. Let A consist of the union of all reduced words of the form $x_n^k * x_{k+1} * x_{k+2} * ... * x_{k+n}$, taken over all pairs of positive integers n and k.

To prove that $* : G \times G \to G$ is not continuous, it suffices to prove that A is closed in G and that $*^{-1}(A)$ is not closed in $G \times G$.

To prove A is closed in G, since G is a T_2 sequential space (see lemmas 3.1 and 4.1), it suffices to prove every convergent sequence in A has its limit in A. Suppose $g_m \to g$ and $g_m \in A$ for all m. Let $g_m = x_{n_m}^{k_m} * x_{k_m+1} * x_{k_m+2} * ... * x_{k_m+n_m}$. Notice $l(g_m) \geq n_m$. Thus, by Lemma 4.2, the sequence$\{n_m\}$ is bounded. For each n_i, by Lemma 4.2, the sequence $\phi_{n_i}(g_m)$ is eventually constant, and thus the sequence $\{k_m\}$ is bounded (since every subsequence of $x_{n_i}^1, x_{n_i}^2, x_{n_i}^3, ...$ diverges in G_{n_i}). Thus, $\{g_1, g_2, ...\}$ is a finite set, and hence (since G is T_1) $\{g_1, g_2, ...\}$ is closed in G. Thus, $g \in \{g_1, g_2, ...\}$, and hence A is closed in G.

Let $e \in G$ denote the identity of G. To prove $*^{-1}(A)$ is not closed in $G \times G$, we will show $(e,e) \notin *^{-1}(A)$ and (e,e) is a limit point of $*^{-1}(A)$. Note $e * e = e$ and $e \notin A$ since $l(e) = 0$ and $l(x) \geq 1$ for all $x \in A$. Thus, $(e,e) \notin *^{-1}(A)$.

To see that (e,e) is a limit point of $*^{-1}(A)$, suppose $U \subset G$ is open and suppose $e \in U$. Let $V = \Pi^{-1}(U) \subset M_Q(X,p)$. First, we show there exists N such that $x_n^k \in U$ for all $n \geq N$ and for all k, argued as follows.

Obtain a closed Q-cell $B \subset S^Q$ and recall $w = (0,1,0,...,0)$. Obtain an inessential map $\alpha \in M_Q(X,p)$ such that $im(\alpha) \subset L$ and such that $\alpha^{-1}(w) = B$. Let $k_1, k_2, ...$ be any sequence of integers and consider the sequence $x_1^{k_1}, x_2^{k_2}, ... \subset G$. For each $n \geq 1$, obtain $\alpha_n \in M_Q(X,p)$ such that $\Pi(\alpha_n) = x_n^{k_n}$, such that $\alpha_n|(S^Q \backslash B) = \gamma_n(\alpha|(S^Q \backslash B))$, and such that $\alpha_n(B) \subset S_n$. Hence, $\alpha_n \to \alpha$, and thus $\Pi(\alpha_n) \to \Pi(\alpha)$. Hence, $x_n^{k_n} \to e$. Since $\{k_i\}$ was arbitrary, it follows there exists N such that $x_n^k \in U$ for all $n \geq N$ and all k.

Obtain N as above and, for each $k \geq 1$, define $v_k = x_{k+1} * ... * x_{k+N}$. To see that the sequence $v_k \to e$, select N disjoint closed Q-cells $B_1, B_2, ..., B_N \subset S^Q$ (and, if $Q = 1$, we also require the closed intervals satisfy $B_j < B_{j+1}$ for $1 \leq j \leq N-1$). Construct $\beta : S^Q \to L$ such that $\beta_i^{-1}(w) = B_1 \cup B_2 ... \cup B_N$. Note β is inessential since L is contractible. For each $i \in \{1, ..., N\}$ and for each k, let $f_{i,k} : S^Q \to X$ satisfy $\Pi(f_{i,k}) = x_{k+i}$, and $f_{i,k}|(S^Q \backslash B_i) = \gamma_k(\beta|(S^Q \backslash B_i))$. Now let $\beta_k =$

$\beta|(S^1\backslash(B_1 \cup ... \cup B_N)) \cup f_{1,k}|B_1 \cup f_{2,k}|B_2 ... \cup f_{N,k}|B_N$. Notice $\beta_k \to \beta$ uniformly and $\Pi(\beta_k) = v_k$ and $\Pi(\beta) = e$, and thus $v_k \to e$. In particular, there exists $K > N$ such that $v_K \in U$. Thus, $(x_N^K, v_K) \in (U, U)$ and $x_N^K * v_K \in A$. This proves (e, e) is a limit point of $*^{-1}(A)$, and thus $*^{-1}(A)$ is not closed in $G \times G$. □

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