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COMPACT SETS WITH DENSE ORBIT IN 2^X

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Abstract. Let X be a compact metric space without isolated points, and let 2^X be the hyperspace of all nonempty closed subsets of X endowed with the Hausdorff metric. Let $f : X \to X$ be a continuous mapping such that the induced map $2^f : 2^X \to 2^X$ is transitive. We show in this note that there exists a Cantor set C contained in X such that its orbit under 2^f is dense in 2^X. Furthermore, for each infinite compact set B with $B \subset C$, the orbit of B under 2^f is dense in 2^X as well.

1. Introduction and Some Definitions

Let $X = (X, d)$ be a compact metric space without isolated points. Let $f : X \to X$ be a continuous mapping. As usual, $\mathbb{N}$ denotes the set of all positive integers. Let f^0 be identity map in X, $f^1 = f$, and for each $n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$.

Given a point x in X, the *orbit of x under f* is the set

$$o(x, f) = \{f^n(x) : n \geq 0\}.$$

We say that $f : X \to X$ is

- *transitive* if, for each pair of nonempty open subsets of X, U, and W, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap W \neq \emptyset$;
- *weakly mixing* if, for each four nonempty open subsets of X, A, B, C, and D, there exists $n \in \mathbb{N}$ such that $f^n(A) \cap C \neq \emptyset$ and $f^n(B) \cap D \neq \emptyset$;
- *mixing* if, for each pair of nonempty open subsets of X, U, and W, there exists $N \in \mathbb{N}$ such that for each $n \geq N$, we have that $f^n(U) \cap W \neq \emptyset$.

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Notice that if $f : X \to X$ is weakly mixing, then it is transitive; and if f is mixing, then it is weakly mixing.

Let 2^X denote the set of all nonempty compact subsets of X endowed with the Hausdorff metric H induced by d. The space 2^X is compact and contains no isolated points; see [7].

Let $2^f : 2^X \to 2^X$ be the map in the hyperspace 2^X induced by f. For each $n \in \mathbb{N}$ and for each $A \in 2^X$, $\left(2^f\right)^n(A) = f^n(A)$.

The next result is already known; see [2] and [10].

Theorem 1.1. *The following statements are equivalent.*

- *f is weakly mixing.*
- *2^f is weakly mixing.*
- *2^f is transitive.*

The statement $2^f : 2^X \to 2^X$ is transitive implies that there exists an element A in 2^X with $o(A, 2^f)$ dense in 2^X; see [3]. Such a compact set A has an interesting elastic quality: Under iterations of f, it manages to travel as close as we want to any nonempty compact set of X. In this note, we are interested in discovering some topological properties of this compact set A.

The following is known: If $o\left(A, 2^f\right)$ is dense in 2^X, then for each $n \geq 0$, $f^n(A)$ is not finite and its interior, $int\left(f^n(A)\right)$, is empty; see [1]. Also, it is not difficult to show that every element of A has dense orbit under f. As far as we know, nothing else has been said in the literature about these kinds of sets.

This is our main result: Let $f : X \to X$ be a continuous mapping such that the induced map $2^f : 2^X \to 2^X$ is transitive. Then there exists a Cantor set C contained in X such that its orbit under 2^f is dense in the hyperspace 2^X. Furthermore, for each B in 2^X, $B \subset C$, with infinite cardinality; the orbit of B under 2^f is dense in 2^X as well.

2. Some Lemmas

From now on we assume that $X = (X, d)$ is a compact metric space without isolated points, and $f : X \to X$ is a continuous map.

Let x be in X and $\varepsilon > 0$. Then

$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}.$$

Given a nonempty subset of X, say A,

$$diam(A) = \sup\{d(x, y) : x \in A,\ y \in A\}.$$

Let $\mathcal{A} = \{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of nonempty subsets of X. Then

$$mesh(\mathcal{A}) = \sup\{diam(A_\lambda) : \lambda \in \Lambda\}.$$

Given a finite collection $(A_1, A_2, \ldots, A_n)$ of nonempty subsets of X, we define

$$\langle A_1, A_2, \ldots, A_n \rangle = \left\{ B \in 2^X : B \subset \cup_{i=1}^n A_i, \text{ and } \forall i,\ B \cap A_i \neq \emptyset \right\}.$$

If for each i, $1 \leq i \leq n$, A_i is closed, then $\langle A_1, A_2, \ldots, A_n \rangle$ is a closed subset of 2^X. Furthermore, if

$$mesh\left(\{A_1, \ldots, A_n\}\right) < \varepsilon,$$

for some $\varepsilon > 0$, then $diam\left(\langle A_1, A_2, \ldots, A_n \rangle\right) < \varepsilon$.

The family of all sets $\mathcal{U} = \langle A_1, A_2, \ldots, A_n \rangle$ where all A_i are nonempty open subsets of X form a base of a topology in 2^X known as the *Vietoris topology*. This topology and the Hausdorff metric in 2^X are compatible.

In the proof of Theorem 3.1, we use the following notation: If

$$\mathcal{U} = \langle A_1, A_2, \ldots, A_n \rangle,$$

then the sets that define $\mathcal{U}$ are denoted by $\mathcal{U}^i = A_i$, $1 \leq i \leq n$.

In preparation of our main result, we need some lemmas. The first of them is due to Harry Furstenberg [5]. There is another proof of this result in [8].

Lemma 2.1. *Let $f : X \to X$ be a weakly mixing map and let $M \geq 2$. Let $(U_1, \ldots, U_M)$ and $(V_1, \ldots, V_M)$ be two collections of nonempty open subsets of X, each of them with M elements. Then there exists $n \in \mathbb{N}$ such that for each i, $1 \leq i \leq M$, $f^n(U_i) \cap V_i \neq \emptyset$.*

Lemma 2.2. *Let $f : X \to X$ be a weakly mixing map. Let M and N be in $\mathbb{N}$, with $M \geq 2$. Let*

$$(W_1, W_2, \ldots, W_M), (U_{1,1}, U_{1,2}, \ldots, U_{1,M}), \ldots, (U_{N,1}, U_{N,2}, \ldots, U_{N,M}),$$

be $N + 1$ collections of nonempty open subsets of X, each of them with M, not necessary distinct, elements.

Then there exist M closed subsets of X, $B_1, \ldots, B_M$, such that

(1) *for each i, $1 \leq i \leq M$,*

$$\emptyset \neq int(B_i) \subset B_i \subset W_i,$$

and

(2) *for each $A \in \langle B_1, \ldots, B_M \rangle$ and for each j, $1 \leq j \leq N$, there exists $n \in \mathbb{N}$ such that*

$$f^n(A) \in \langle U_{j,1}, \ldots, U_{j,M} \rangle.$$

Furthermore, for each i, $1 \leq i \leq M$, $f^n(A \cap B_i) \subset U_{j,i}$.

Proof. Fix $M \geq 2$. We use induction on $N \in \mathbb{N}$.

Step 1. Let $N = 1$. Consider two collections

$$(W_1, W_2, \ldots, W_M) \text{ and } (U_{1,1}, U_{1,2}, \ldots, U_{1,M}),$$

of nonempty open subsets of X. By Lemma 2.1, there exists $n \in \mathbb{N}$ such that for each i,

$$f^n(W_i) \cap U_{1,i} \neq \emptyset.$$

Consider $W_i^* = W_i \cap f^{-n}(U_{1,i})$. Since each W_i^* is open and nonempty, there exists a closed set B_i with $int(B_i) \neq \emptyset$ and $B_i \subset W_i^*$.

Let $A \in \langle B_1, \ldots, B_M \rangle$. It follows that $f^n(A) \in \langle U_{1,1}, \ldots, U_{1,M} \rangle$, and for each i, $1 \leq i \leq M$,

$$f^n(A \cap B_i) \subset U_{1,i}.$$

Step 2. Now let us assume that the claim is valid for N. Consider $(N+1)+1$ collections of nonempty open subsets of X,

$$(W_1, W_2, \ldots, W_M), (U_{1,1}, U_{1,2}, \ldots, U_{1,M}), \ldots$$

$$\ldots, (U_{N,1}, U_{N,2}, \ldots, U_{N,M}), (U_{N+1,1}, U_{N+1,2}, \ldots, U_{N+1,M}).$$

Consider collections $(U_{N,1}, \ldots, U_{N,M})$ and $(U_{N+1,1}, \ldots, U_{N+1,M})$. By lemma 2.1, there exist M nonempty open subsets of X, $U_1, \ldots, U_M$, with $U_i \subset U_{N,i}$, $1 \leq i \leq M$, and a natural number m such that

$$f^m(U_i) \subset U_{N+1,i}.$$

Now consider the $N+1$ collections

$$(W_1, W_2, \ldots, W_M), (U_{1,1}, U_{1,2}, \ldots, U_{1,M}), \ldots$$

$$\ldots, (U_{N-1,1}, U_{N-1,2}, \ldots, U_{N-1,M}), (U_1, U_2, \ldots, U_M).$$

By hypothesis, there exist M closed subsets of X, $B_1, \ldots, B_M$,

$$\emptyset \neq int(B_i) \subset B_i \subset W_i, \ 1 \leq i \leq M,$$

such that for each $A \in \langle B_1, \ldots, B_M \rangle$ and for each j, $1 \leq j \leq N-1$, there exists $n \in \mathbb{N}$ with $f^n(A) \in \langle U_{j,1}, \ldots, U_{j,M} \rangle$, and $f^n(A \cap B_i)$ is contained in the corresponding $U_{j,i}$.

Also, for each $A \in \langle B_1, \ldots, B_M \rangle$, there exists n' so that

$$f^{n'}(A) \in \langle U_1, \ldots, U_M \rangle$$

and $f^{n'}(A \cap B_i) \subset U_i$, for each i.

Note that condition $U_i \subset U_{N,i}$ implies that

$$f^{n'}(A) \in \langle U_{N,1}, \ldots, U_{N,M} \rangle$$

and $f^{n'}(A \cap B_i) \subset U_{N,i}$, for each i.

Now, since $f^m(U_i) \subset U_{N+1,i}$,

$$f^{n'+m}(A) \in \langle U_{N+1,1}, \ldots, U_{N+1,M} \rangle$$

and, since $f^{n'}(A \cap B_i) \subset U_i$,

$$f^{n'+m}(A \cap B_i) \subset f^m(U_i) \subset U_{N+1,i}\,,$$

for each i, $1 \le i \le M$. This completes the proof. □

Given $A \in 2^X$, $|A|$ stands for the cardinality of A.

For each n in $\mathbb{N}$, let

$$F_n(X) = \left\{A \in 2^X : |A| \le n\right\}.$$

Notice that for each n, $F_n(X) \subset F_{n+1}(X)$, and the union $F(X) = \cup_{n=1}^{\infty} F_n(X)$ is a dense set in 2^X.

Let $\varepsilon > 0$, $N \in \mathbb{N}$, and $A \in 2^X$. We say that $o(A, 2^f)$ is *ε-close to* $F_N(X)$ provided that for each $B \in F_N(X)$, there is $n \ge 0$ such that $H(B, f^n(A)) < \varepsilon$. Note that if $o(A, 2^f)$ is ε-close to $F_N(X)$, then it is ε-close to $F_j(X)$ for $1 \le j \le N$.

Lemma 2.3. *Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty compact subsets of 2^X such that for each $A \in \mathcal{D}_n$, $o(A, 2^f)$ is ε_n-close to $F_n(X)$.

Then for each $A \in \cap_{n=1}^{\infty} \mathcal{D}n$, $o(A, 2^f)$ is dense in 2^X.

Proof. Let $A \in \cap_{n=1}^{\infty} \mathcal{D}_n$, $B \in 2^X$, and $\varepsilon > 0$.

There exist m points $b_1, b_2, \ldots, b_m$ in B so that

$$B \subset \cup_{i=1}^{m} B_{\frac{\varepsilon}{2}}(b_i)\,.$$

It follows that $H(B, \{b_1, b_2, \ldots, b_m\}) < \frac{\varepsilon}{2}$.

Take j such that $m \le j$ and $\varepsilon_j < \frac{\varepsilon}{2}$. Since A is in $\mathcal{D}_j$, there exists $n \ge 0$ such that

$$H(f^n(A), \{b_1, b_2, \ldots, b_m\}) < \varepsilon_j < \frac{\varepsilon}{2}.$$

Thus, $H(B, f^n(A)) < \varepsilon$. □

Considering the induced map 2^f, Lemma 2.3 gives us a procedure which helps us to find a compact set of X with dense orbit in 2^X.

3. A Cantor Set with Dense Orbit

We are ready for our main result.

Theorem 3.1. *Let $f : X \to X$ be a map such that the induced map $2^f : 2^X \to 2^X$ is transitive. Then there exists a Cantor set C in X such that its orbit $o(C, 2^f)$ is dense in 2^X. Furthermore, for each infinite compact set B contained in C, the orbit $o(B, 2^f)$ is dense in 2^X as well.*

Proof. Let $\varepsilon_0 = diam(X)$, and for each $n \in \mathbb{N}$, let $\varepsilon_n = \frac{\varepsilon_0}{2^n}$.

Consider the following family of finite open covers of X: For each n in $\mathbb{N}$, let

$$\widehat{U}_n = \{U_{n,1}, U_{n,2}, \ldots, U_{n,r_n}\},$$

where each element $U_{n,i} \neq \emptyset$ and $mesh(\widehat{U}_n) < \varepsilon_n$.

Step 1. Let W_0 and W_1 be two nonempty open subsets of X with $W_0 \cap W_1 = \emptyset$ and $mesh(\{W_0, W_1\}) < \varepsilon_1$. Let

$$\Lambda_1 = \{1, 2, \ldots, r_1\}^2 = \{(s,t) : s, t \in \{1, 2, \ldots, r_1\}\}.$$

Consider the following $r_1^2 + 1$ collections of open sets:

$$(W_0, W_1) \text{ and } \{(U_{1,s}, U_{1,t}) : (s,t) \in \Lambda_1\}.$$

By Lemma 2.2, there exist two closed subsets of X, C_0, and C_1, with the following properties:

(1) $\emptyset \neq int(C_i) \subset C_i \subset W_i$, $i = 0, 1$. Hence, $C_0 \cap C_1 = \emptyset$.

(2) For each $A \in \langle C_0, C_1 \rangle$ and for each (s,t) in Λ_1, there exists $n \in \mathbb{N}$ such that

$$f^n(A) \in \langle U_{1,s}, U_{1,t} \rangle.$$

Furthermore,

$$f^n(A \cap C_0) \subset U_{1,s}$$

and

$$f^n(A \cap C_1) \subset U_{1,t}.$$

Let $\mathcal{C}_1 = \langle C_0, C_1 \rangle$. Then $diam(\mathcal{C}_1) < \varepsilon_1$.

Note the following: For each $A \in \mathcal{C}_1$, $o(A, 2^f)$ is ε_1-close to $F_2(X)$. For, given any $\{a, b\} \in F_2(X)$, there exists (s,t) in Λ_1 such that $a \in U_{1,s}$ and $b \in U_{1,t}$. Since there exists n so that

$$f^n(A) \in \langle U_{1,s}, U_{1,t} \rangle,$$

we conclude that $H(\{a,b\}, f^n(A)) < \varepsilon_1$.

Step 2. Let $W_{0,0}$, $W_{1,0}$, $W_{0,1}$, and $W_{1,1}$ be 2^2 nonempty open subsets of X with the following properties:

(1) $W_{0,0} \cap W_{1,0} = \emptyset$ and $W_{0,1} \cap W_{1,1} = \emptyset$.

(2) $W_{0,0} \cup W_{1,0} \subset C_0$ and $W_{0,1} \cup W_{1,1} \subset C_1$.

(3) $mesh(\{W_{0,0}, W_{1,0}, W_{0,1}, W_{1,1}\}) < \varepsilon_2$.

Let

$$\Lambda_2 = \{1, 2, \ldots, r_2\}^{2^2} = \{(s_1, s_2, s_3, s_4) : s_i \in \{1, 2, \ldots, r_2\}\}.$$

Consider the following $(r_2)^{2^2} + 1$ collections of open sets:

$$(W_{0,0}, W_{1,0}, W_{0,1}, W_{1,1}),$$

$$\{(U_{2,s_1}, U_{2,s_2}, U_{2,s_3}, U_{2,s_4}) : (s_1, s_2, s_3, s_4) \in \Lambda_2\}.$$

By Lemma 2.2, there exist 2^2 closed sets $C_{0,0}$, $C_{1,0}$, $C_{0,1}$, and $C_{1,1}$ with the following properties:

(1) $\emptyset \neq int(C_{i,j}) \subset C_{i,j} \subset W_{i,j}$, for $(i,j) \in \{0,1\}^2$.

(2) For each $A \in \langle C_{0,0}, C_{1,0}, C_{0,1}, C_{1,1} \rangle$ and for each (s_1, s_2, s_3, s_4) in Λ_2, there exists $n \in \mathbb{N}$ such that

$$f^n(A) \in \langle U_{2,s_1}, U_{2,s_2}, U_{2,s_3}, U_{2,s_4} \rangle ,$$

with

$$f^n(A \cap \mathcal{C}_2^i) \subset U_{2,s_i},\ 1 \leq i \leq 2^2,$$

where $\mathcal{C}_2 = \langle C_{0,0}, C_{1,0}, C_{0,1}, C_{1,1} \rangle$.

Note that $diam(\mathcal{C}_2) < \varepsilon_2$ and $\mathcal{C}_2 \subset \mathcal{C}_1$.

Let $A \in \mathcal{C}_2$ and $\{a_1, a_2, a_3, a_4\} \in F_{2^2}(X)$.

Then there exists an element (s_1, s_2, s_3, s_4) of Λ_2 such that $a_i \in U_{2,s_i}$. Since there exists n so that

$$f^n(A) \in \langle U_{2,s_1}, U_{2,s_2}, U_{2,s_3}, U_{2,s_4} \rangle ,$$

we conclude that $H(\{a_1, a_2, a_3, a_4\}, f^n(A))$ is less than ε_2.

Thus, for every A in $\mathcal{C}_2$, the orbit $o(A, 2^f)$ is ε_2-close to $F_{2^2}(X)$.

Step 3. Assume we have already defined $\mathcal{C}_k$ with the following properties:

(1) $\mathcal{C}_k$ is of the form $\langle C_{0,0,\ldots,0}, \ldots, C_{1,1,\ldots,1} \rangle$. The 2^k closed sets of X that define $\mathcal{C}_k$ are indexed in this way:

$$\left\{ C_{a_1,a_2,\ldots,a_k} : (a_1, a_2, \ldots, a_k) \in \{0,1\}^k \right\}.$$

(2) For each $(a_1, a_2, \ldots, a_k) \in \{0,1\}^k$,

$$\emptyset \neq int\,(C_{a_1,a_2,\ldots,a_k}) \subset C_{a_1,a_2,\ldots,a_k} \subset C_{a_2,\ldots,a_k},$$

and $diam(C_{a_1,a_2,\ldots,a_k}) < \varepsilon_k$.

(3) $diam(\mathcal{C}_k) < \varepsilon_k$ and $\mathcal{C}_k \subset \mathcal{C}_{k-1}$.

(4) For each pair $(a_1, a_2, \ldots, a_k) \neq (b_1, b_2, \ldots, b_k)$, in $\{0,1\}^k$,

$$C_{a_1,a_2,\ldots,a_k} \cap C_{b_1,b_2,\ldots,b_k} = \emptyset.$$

(5) For each $A \in \mathcal{C}_k$ and each $(s_1, s_2, \ldots, s_{2^k})$ in

$$\Lambda_k = \{1, 2, \ldots, r_k\}^{2^k},$$

there exists $n \in \mathbb{N}$ such that

$$f^n(A) \in \left\langle U_{k,s_1}, U_{k,s_2}, \ldots, U_{k,s_{2^k}} \right\rangle ,$$

with

$$f^n\left(A \cap \mathcal{C}_k^j\right) \subset U_{k,s_j},\ \text{for } 1 \leq j \leq 2^k.$$

The last statement implies the following: If $A \in \mathcal{C}_k$, then $o(A, 2^f)$ is ε_k-close to $F_{2^k}(X)$.

In order to define $\mathcal{C}_{k+1}$ consider, for each $(a_1, a_2, \ldots, a_k) \in \{0,1\}^k$, two nonempty open subsets of X, $W_{0,a_1,a_2,\ldots,a_k}$, and $W_{1,a_1,a_2,\ldots,a_k}$, with the following properties:

(1) $W_{0,a_1,a_2,\ldots,a_k} \cap W_{1,a_1,a_2,\ldots,a_k} = \emptyset$.
(2) $W_{0,a_1,a_2,\ldots,a_k} \cup W_{1,a_1,a_2,\ldots,a_k} \subset C_{a_1,a_2,\ldots,a_k}$.
(3) $mesh(\{W_{0,a_1,a_2,\ldots,a_k}, W_{1,a_1,a_2,\ldots,a_k}\}) < \varepsilon_{k+1}$.

Let

$$\Lambda_{k+1} = \{1, 2, \ldots, r_{k+1}\}^{2^{k+1}},$$

and consider the $(r_{k+1})^{2^{k+1}} + 1$ collections of open sets:

$$(W_{0,0,\ldots,0}, W_{1,0,\ldots,0}, \ldots, W_{0,1,\ldots,1}, W_{1,1,\ldots,1}),$$

$$\left\{\left(U_{k+1,s_1}, U_{k+1,s_2}, \ldots, U_{k+1,s_{2^{k+1}}}\right) : (s_1, s_2, \ldots, s_{2^{k+1}}) \in \Lambda_{k+1}\right\}.$$

By Lemma 2.2, there exist 2^{k+1} closed sets of X,

$$\left\{C_{b_1,b_2,\ldots,b_{k+1}} : (b_1, b_2, \ldots, b_{k+1}) \in \{0,1\}^{k+1}\right\},$$

with the following properties:

(1) For each $(b_1, b_2, \ldots, b_{k+1}) \in \{0,1\}^{k+1}$,

$$\emptyset \neq int\left(C_{b_1,b_2,\ldots,b_{k+1}}\right) \subset C_{b_1,b_2,\ldots,b_{k+1}} \subset W_{b_1,b_2,\ldots,b_{k+1}}.$$

(2) For each

$$A \in \langle C_{0,0,\ldots,0}, \ldots, C_{1,1,\ldots,1}\rangle = \mathcal{C}_{k+1}$$

and for each

$$(s_1, s_2, \ldots, s_{2^{k+1}}) \in \Lambda_{k+1},$$

there exists $n \in \mathbb{N}$ such that

$$f^n(A) \in \left\langle U_{k+1,s_1}, U_{k+1,s_2}, \ldots, U_{k+1,s_{2^{k+1}}}\right\rangle$$

and

$$f^n\left(A \cap \mathcal{C}^i_{k+1}\right) \subset U_{k+1,s_i},$$

for $1 \leq i \leq 2^{k+1}$. Hence, for every $A \in \mathcal{C}_{k+1}$, the orbit $o(A, 2^f)$ is ε_{k+1}-close to $F_{2^{k+1}}(X)$.

Also notice that $\mathcal{C}_{k+1} \subset \mathcal{C}_k$ and $diam\left(\mathcal{C}_{k+1}\right) < \varepsilon_{k+1}$.

Step 4. Following the previous procedure, we define a decreasing sequence of compact subset of 2^X, $\{\mathcal{C}_k\}_{k=1}^{\infty}$.

Let

$$\left\{C_{a_1,a_2,\ldots,a_k} : (a_1, a_2, \ldots, a_k) \in \{0,1\}^k\right\}$$

be the 2^k compact subsets of X that define $\mathcal{C}_k$.

Let

$$C = \cap_{k=1}^{\infty} \left(\cup \left\{ C_{a_1,a_2,\ldots,a_k} : (a_1, a_2, \ldots, a_k) \in \{0,1\}^k \right\} \right).$$

Then C is a Cantor set contained in X. Also note that for every k, $C \in \mathcal{C}_k$. Thus, by Lemma 2.3, its orbit $o(C, 2^f)$ is dense in 2^X.

Since $diam(\mathcal{C}_k) < \varepsilon_k$ and $\lim_{k\to\infty} \varepsilon_k = 0$, then $\cap_{k=1}^{\infty} \mathcal{C}_k$ is a set with just one point. Therefore, $\cap_{k=1}^{\infty} \mathcal{C}_k = \{C\}$.

Step 5. Let B be an infinite compact set contained in C. Our goal is to show that $o(B, 2^f)$ is dense in 2^X.

It is enough to prove the following: Given $E \in F(X)$ and $\varepsilon > 0$, there exists n such that

$$H(E, f^n(B)) < \varepsilon.$$

Let $j \in \mathbb{N}$, $E = \{x_1, x_2, \ldots, x_j\} \in F_j(X)$, and $\varepsilon > 0$.

Since B is contained in C, for every k,

$$B \subset \cup \left\{ C_{a_1,a_2,\ldots,a_k} : (a_1, a_2, \ldots, a_k) \in \{0,1\}^k \right\}.$$

Since B is infinite, there exists N so that

$$l = \left| \left\{ i : B \cap \mathcal{C}_N^i \neq \emptyset \right\} \right| \geq j,$$

and $\varepsilon_N < \varepsilon$.

Let $n_1 < n_2 < \cdots < n_l$ be the natural numbers such that

$$\left\{ i : B \cap \mathcal{C}_N^i \neq \emptyset \right\} = \{n_1, n_2, \ldots, n_l\}.$$

For each i, $1 \leq i \leq 2^N$ and $i \notin \{n_1, n_2, \ldots, n_l\}$, take one point y_i in $\mathcal{C}_N^i$. Let D be the set formed with those points y_i, and let $A = B \cup D$. Note that $A \in \mathcal{C}_N$.

Recall that $\widehat{U}_N$ is an open cover of X. Thus, for each x_i in E there exists m_i, $1 \leq m_i \leq r_N$, so that $x_i \in U_{N,m_i}$. Therefore,

$$E \in \left\langle U_{N,m_1}, U_{N,m_2}, \ldots, U_{N,m_j} \right\rangle.$$

Now take an element $(s_1, s_2, \ldots, s_{2^N})$ in $\{1, 2, \ldots, r_N\}^{2^N}$ with the following two properties:

(1) $s_{n_1} = m_1$, $s_{n_2} = m_2$, ... , $s_{n_j} = m_j$.
(2) For each $i \notin \{n_1, n_2, \ldots, n_j\}$, let $s_i = m_1$.

Since A is in $\mathcal{C}_N$, there exists n such that

$$f^n(A) \in \left\langle U_{N,s_1}, U_{N,s_2}, \ldots, U_{N,s_{2^N}} \right\rangle,$$

and for each i, $1 \leq i \leq 2^N$,

$$f^n \left(A \cap \mathcal{C}_N^i \right) \subset U_{N,s_i}.$$

It follows that

$$f^n(B) \subset f^n(A) \subset \cup_{i=1}^{2^N} U_{N,s_i} = U_{N,m_1} \cup U_{N,m_2} \cup \cdots \cup U_{N,m_j},$$

and for each t, $1 \leq t \leq j$,

$$\emptyset \neq f^n\left(B \cap \mathcal{C}_N^{n_t}\right) \subset f^n\left(A \cap \mathcal{C}_N^{n_t}\right) \subset U_{N,s_{n_t}} = U_{N,m_t}.$$

Then for each t, $1 \leq t \leq j$, $f^n(B) \cap U_{N,m_t} \neq \emptyset$.

Thus,

$$f^n(B) \in \left\langle U_{N,m_1}, U_{N,m_2}, \ldots, U_{N,m_j} \right\rangle.$$

Since $diam\left(\left\langle U_{N,m_1}, U_{N,m_2}, \ldots, U_{N,m_j} \right\rangle\right) < \varepsilon_N < \varepsilon$, we obtain that $H(E, f^n(B)) < \varepsilon$. □

4. An Example

Theorem 3.1 produces a Cantor set C and a wide range of elements of 2^X with the property of having a dense orbit in 2^X under the induced map 2^f. Any of these elements is compact and contained in C. Therefore, all of them are totally disconnected. This raises the following question: Does there exist a map $f : X \to X$ and an element E of 2^X so that the orbit $o(E, 2^f)$ is dense in 2^X and E is not totally disconnected? The answer is yes. We present in this section such an example.

We use the following result. Its proof is not difficult so we leave it to the reader.

Lemma 4.1. *Let A and B be two elements of 2^X such that $o(A, 2^f)$ is dense in 2^X and*

$$\lim_{n\to\infty} H\left(\left(2^f\right)^n(A), \left(2^f\right)^n(B)\right) = 0.$$

Then $o(B, 2^f)$ is dense in 2^X as well.

Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Matrix A has the eigenvalues $\lambda = \frac{3+\sqrt{5}}{2} > 1$ and $\mu = \frac{1}{\lambda}$ and the eigenvectors

$$v_\lambda = \left(\frac{1+\sqrt{5}}{2}, 1\right) \text{ and } v_\mu = \left(\frac{1-\sqrt{5}}{2}, 1\right).$$

Thus, T expands by a factor of λ in the direction of v_λ and contracts by a factor of μ in the direction of v_μ.

Since T preserves the integer lattice $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$, it induces a map of the torus $X = \mathbb{R}^2/\mathbb{Z}^2$, $f : X \to X$. Note that f is a hyperbolic toral automorphism. We refer the reader to [4] for definition and basic properties of this kind of maps.

For each $x \in X$, there is a line through x, $W^s(x)$, called the stable manifold of x, with the following property: If L is an arc such that $x \in L$ and L is contained in $W^s(x)$, then f contracts the length of L, $l(L)$, by a factor of μ. Therefore,

$$\lim_{n\to\infty} l\left(f^n(L)\right) = 0.$$

For a proof of the next proposition, see chapter two of [4].

Proposition 4.2. *Any hyperbolic toral automorphism is mixing.*

Since $f : X \to X$ is mixing, it is weakly mixing, and the induced map $2^f : 2^X \to 2^X$ is transitive. Let $C \in 2^X$ be a Cantor set with $o(C, 2^f)$ dense in 2^X. Take a point x in C and consider L an arc so that $x \in L$ and $L \subset W^s(x)$. Let $E = C \cup L$. Note the following: E is an element of 2^X and E is not totally disconnected. Since $\lim_{n\to\infty} l\left(\left(2^f\right)^n (L)\right) = 0$, then

$$\lim_{n\to\infty} H\left(\left(2^f\right)^n (C), \left(2^f\right)^n (E)\right) = 0.$$

Therefore, the orbit $o(E, 2^f)$ is dense in 2^X.

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