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A NOTE ON PARATOPOLOGICAL GROUPS WITH COUNTABLE NETWORKS OF SEQUENTIAL NEIGHBORHOODS

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Abstract. In this paper, we discuss generalized metric properties of paratopological groups. We prove that a paratopological group is sn-metrizable if and only if it is so-metrizable. Moreover, we pose some questions concerning generalized metric properties on paratopological groups.

1. Introduction

A *semitopological group* G is a group G with a topology such that the product map of $G \times G$ into G is separately continuous. If G is a semitopological group and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous, then G is called a *quasitopological group*. A *paratopological group* G is a group G with a topology such that the product map of $G \times G$ into G is jointly continuous. If G is a paratopological group and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous, then G is called a *topological group*. However, there exists a paratopological group which is not a topological group; the Sorgenfrey line ([7, Example 1.2.2]) is such an example. Paratopological groups were discussed and many results have been obtained [2, 3, 4, 5, 12, 15, 16, 17, 18]. Obviously, each semitopological group is homogeneous, so it is enough to define the topology at one point and then translate it.

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In this paper, we mainly discuss the generalized metric properties on paratopological groups and pose some questions concerning generalized metric properties on paratopological groups. In section 3, we prove that a paratopological group is sn-metrizable if and only if it is so-metrizable. In section 4, we mainly discuss some questions concerning generalized metric properties on paratopological groups.

2. Preliminaries

Let X be a space. For $P \subset X$, the set P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P. P is a *sequentially open* subset of X if P is a sequential neighborhood of x in X for each $x \in P$. X is said to be a *sequential space* [8] if each sequentially open subset is open in X.

Definition 2.1. Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$; (b) the family $\mathscr{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathscr{P}_x$, and if $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathscr{P}_x$.

(1) The family $\mathscr{P}$ is called a *sn-network* (*sequential-neighborhood network*) [14] for X if each element of $\mathscr{P}_x$ is a sequential neighborhood of x in X for each $x \in X$. X is called *snf-countable* [14], if X has a *sn*-network $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable. A regular space X is called *sn-metrizable* [14] if X has an σ-locally finite *sn*-network.

(2) The family $\mathscr{P}$ is called a *so-network* (*sequentially-open network*) [14] for X if each element of $\mathscr{P}_x$ is a sequentially open neighborhood of x in X for each $x \in X$. X is called *sof-countable* [14], if X has an *so*-network $\mathscr{P}$ such that each $\mathscr{P}_x$ is countable. A regular space X is called *so-metrizable* [14] if X has an σ-locally finite *so*-network.

(3) The family $\mathscr{P}$ is called a *weak base* for X [1] if, for every $A \subset X$, the set A is open in X whenever for each $x \in A$ there exists $P \in \mathscr{P}_x$ such that $P \subset A$. The space X is *weakly first-countable* if $\mathscr{P}_x$ is countable for each $x \in X$.

It is easy to see that [14]
(1) weakly first-countable spaces $\Leftrightarrow$ snf-countable and sequential spaces;
(2) weak bases $\Rightarrow$ sn-networks for a space X;
(3) sn-networks $\Rightarrow$ weak bases for a sequential space X;
(4) every sequential and sof-countable space is first-countable.

All spaces are Hausdorff unless stated otherwise. The symbol $\mathbb{N}$ denotes the natural numbers. The letter e denotes the neutral element of a group. Readers may refer to [3, 7, 9] for notations and terminology not explicitly given here.

3. sn-METRIZABILITY IN PARATOPOLOGICAL GROUPS

Let G be a snf-countable paratopological group. Then it is easy to see that G has a sn-network $\{V_n(x) : x \in X, n \in \mathbb{N}\}$ such that the following conditions are satisfied:

(1) each $V_n(x)$ is a sequential neighborhood of x;
(2) $\{V_n(x) : n \in \mathbb{N}\}$ is a network at x;
(3) $V_{n+1}(x) \subset V_n(x)$ for each $n \in \mathbb{N}$ and $x \in X$.

Therefore, we will always assume that a sn-network of an snf-countable paratopological group satisfies the above conditions.

The following two lemmas are an easy exercise.

Lemma 3.1. *Suppose that $\{V_n(x) : n \in \mathbb{N}, x \in G\}$ and $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ are two sn-networks in the snf-countable paratopological group G. Then, for each $x \in G$ and $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $W_m(x) \subset V_n(x)$.*

Lemma 3.2. *Suppose that $\{V_n(x) : n \in \mathbb{N}, x \in G\}$ is a sn-network in the snf-countable paratopological group G. For each $x \in G$ and each $n \in \mathbb{N}$, put $W_n(x) = x \cdot V_n(e)$. Then $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is a sn-network in G.*

Lemma 3.3. *Suppose that $\{V_n(x) : n \in \mathbb{N}, x \in G\}$ is a sn-network in the snf-countable paratopological group G. For each $x \in G$ and each $n \in \mathbb{N}$, put $W_n(x) = x \cdot V_n(e) \cdot V_n(e)$. Then $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is a sn-network in G.*

Proof. By Lemmas 3.1 and 3.2, we can assume that $V_n(x) = x \cdot V_n(e)$, for each $x \in G$ and each $n \in \mathbb{N}$. Since G is joint continuity, it is easy to see that $\{W_n(x) : n \in \mathbb{N}, x \in G\}$ is a sn-network in G. □

Theorem 3.4. *Every snf-countable paratopological group G is sof-countable.*

Proof. Let $\{V_n(x) : n \in \mathbb{N}, x \in G\}$ be a sn-network in G. For each $x \in G$ and $n \in \mathbb{N}$, we can assume tha $V_n(x) = x \cdot V_n(e)$ by Lemma 3.2. Let $U_n = \{x \in V_n(e) : x \cdot V_k(e) \subset V_n(e) \text{ for some } k \in \mathbb{N}\}$. Obviously, we have $e \in U_n \subset V_n(e)$. Next we show that U_n is sequentially open in G. Indeed, take any $y \in U_n$ and a sequence $\{y_n : n \in \mathbb{N}\}$ converging to y. Then $y \cdot V_k(e) \subset V_n(e)$ for some $k \in \mathbb{N}$. By Lemmas 3.1, 3.2 and 3.3, it is easy to see that there exists an $m \in \mathbb{N}$ such that $(y \cdot V_m(e)) \cdot V_m(e) \subset y \cdot V_k(e)$. Hence $(y \cdot V_m(e)) \cdot V_m(e) \subset V_n(e)$, which implies that $V_m(y) = y \cdot V_m(e) \subset U_n$. Since $V_m(y)$ is a sequential neighborhood at y, the sequence $\{y_n : n \in \mathbb{N}\}$ is eventually in $V_m(y)$, hence eventually in U_n Therefore, the set U_n is sequentially open in G. Thus $\{U_m : m \in \mathbb{N}\}$ is a sequentially open neighborhood network at e. Then G is sof-countable. □

Corollary 3.5. *Every regular sn-metrizable paratopological group is so-metrizable.*

Corollary 3.6. *Every sequentially snf-countable paratopological group is first-countable.*

Corollary 3.7. *If G is a weakly first-countable paratopological group, then G is first-countable.*

Proof. Since a weakly first-countable space is snf-countable and sequential, it follows from Theorem 3.4 that G is sof-countable. Then G is first-countable since G is sequential space. □

Corollary 3.8. *If G is a weakly first-countable topological group, then G is metrizable.*

A related concept for sn-networks is *cs*-networks.

Definition 3.9. Let $\mathscr{P}$ be a family of subsets of a space X. The family $\mathscr{P}$ is called a *cs-network* [10] for $x \in X$ if whenever a sequence $\{x_n\}_n$ converges to x and U is open in X and contains x, there exist $m \in \mathbb{N}$ and $P \in \mathscr{P}$ such that $\{x\} \cup \{x_n : n \geqslant m\} \subset P \subset U$. If every point of X has a countable *cs*-network, then we call X *csf-countable.*

It is easy to see that [14]
(1) snf-countable spaces $\Rightarrow$ csf-countable spaces;
(2) weak bases $\Rightarrow$ sn-networks $\Rightarrow$ cs-networks for a space X.

Example 3.10. There exists a csf-countable topological group G such that G is not snf-countable.

Proof. Let X be a convergent sequence, and let G be the free Abelian topological group $A(X)$. Then $A(X)$ is a countable k_ω-space[1] [3, Corollary 7.4.2], and hence $A(X)$ is csf-countable and sequential since a countable k_ω-space is a sequential space with a countable cs-network [19]. However, $A(X)$ is not metrizable [3, Theorem 7.1.20]. Then G is not snf-countable since a sequential snf-countable space is first-countable by Corollary 3.6, and hence G would be metrizable, which is a contradiction. □

[1]A quotient image of a topological sum of countably many compact spaces is called a k_ω-space.

4. Open questions

In view of Theorem 3.4, we have the following question.

Question 4.1. *Let G be a snf-countable semitopological group or a quasitopological group. Is G sof-countable?*

Definition 4.2. Let (X, τ) be a topological space. We define a *sequential closure-topology* σ_τ [8] on X as follows: $O \in \sigma_\tau$ if and only if O is a sequentially open subset in (X, τ). The topological space (X, σ_τ) is denoted by σX.

It is easy to see that σG is a quasitopological group for a topological group G.

The following question was posed by Y. Ge during the 1st Topology Forum held in Zhangzhou, PRC.

Question 4.3. *Let G be a topological group. Is σG a topological group?*

The following theorem is a partial answer to Question 4.3.

Theorem 4.4. *Let G be a snf-countable topological group. Then σG is a topological group.*

Proof. It follows from Corollary 3.5 that G is sof-countable. Let $\{V_n : n \in \mathbb{N}\}$ be a decreasing so-network at point e. Therefore, $\{V_n : n \in \mathbb{N}\}$ is a neighborhood base at point e in σG. Indeed, let U be a sequentially open set in G containing e; then there exists $n \in \mathbb{N}$ such that $V_n \subset U$. For, suppose that $V_n \not\subseteq U$ for each $n \in \mathbb{N}$. Then we can take a $x_n \in V_n \setminus U$ for each $n \in \mathbb{N}$. Obviously, the sequence $\{x_n\}_n$ converges to e. Since U is a sequentially open neighborhood at e, the sequence $\{x_n\}_n$ eventually in U. However, $\{x_n : n \in \mathbb{N}\} \cap U = \emptyset$, which is a contradiction. By the joint continuity of the operation in G, it is easy to see that $\{V_n \cdot V_n : n \in \mathbb{N}\}$ is also a decreasing so-network at e. For each $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $V_m^2 \subset V_n$. For, suppose that $V_m \cdot V_m \not\subseteq V_n$ for each $m \in \mathbb{N}$. Then we can take a point $y_m \in V_m \cdot V_m \setminus V_n$ for each $m \in \mathbb{N}$. Obviously, the sequence $\{y_m\}_m$ converges to e. Since V_n is a sequentially open neighborhood at e, the sequence $\{y_m\}_m$ eventually in V_n. However, $\{y_m : m \in \mathbb{N}\} \cap V_n = \emptyset$, which is a contradiction. Hence, the operation in σG is jointly continuous. So σG is a topological group. □

A regular space is called $\aleph$ if it has a σ-locally finite *cs*-network. A space X is said to have a G_δ*-diagonal* if the diagonal $\Delta = \{(x, x) : x \in X\}$ can be represented as the intersection of a countable family of open neighborhoods of Δ in $X \times X$.

Question 4.5. *Is every snf-countable topological group an $\aleph$-space?*

Assuming Martin's Axiom, E.V. Douwen constructed in [6] an infinite countably compact topological group G without non-trivial convergent sequences. Obviously, G is snf-countable. Suppose that G is an $\aleph$-space. Then G has a G_δ-diagonal [9, Theorem 4.6], and G is metrizable since it is well known that a countably compact space with a G_δ-diagonal is compact metrizable [9, Theorem 2.14]. Since G has no non-trivial convergent sequences, G is discrete, and therefore is finite since G is compact, which is contradiction with G being infinite. Therefore, G is not an $\aleph$-space.

A subset B of a paratopological group G is called *ω-narrow* in G if, for each neighborhood U of the neutral element of G, there is a countable subset F of G such that $B \subset FU \cap UF$.

Question 4.6. *Does every snf-countable ω-narrow topological group have a countable sn-network?*

Definition 4.7. Let X be a space and $\{\mathscr{P}_n\}_n$ a sequence of collections of open subsets of X.

(1) $\{\mathscr{P}_n\}_n$ is called a *quasi-development* for X if for every $x \in U$ with U open in X, there exists an $n \in \mathbb{N}$ such that $x \in \mathrm{st}(x, \mathscr{P}_n) \subset U$.
(2) $\{\mathscr{P}_n\}_n$ is called a *development* for X if $\{\mathrm{st}(x, \mathscr{P}_n)\}_n$ is a neighborhood base at x in X for each point $x \in X$.
(3) X is called *quasi-developable* (resp. *developable*), if X has a quasi-development (resp. *development*).
(4) X is called *Moore*, if X is regular and developable.
(5) A space X has a *uniform base* if and only if it is a metacompact developable space.

Recently, P.Y. Li, L. Mou and S.Z. Wang [18] have proved that a Moore paratopological group need not be metrizable. Therefore, C. Liu posed the following question in a private communication with the author in this paper.

Question 4.8. *Is every regular paratopological group with a uniform base metrizable?*

Let (X, τ) be a topological space. A function $g : \mathbb{N} \times X \to \tau$ satisfies that $x \in g(n, x)$ for each $x \in X, n \in \mathbb{N}$. A space X is a β-space [9] if there is a function $g : \mathbb{N} \times X \to \tau$ such that if $x \in g(n, x_n)$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}$ has a cluster point in X.

In [16], the authors proved that each first-countable β-space is developable. Therefore, we have the following question.

Question 4.9. *Is every quasi-developable paratopological (semitopological) group a β-space?*

In [13], F. Lin and C. Liu proved that a Baire quasi-developable paratopological group is a topological group. However, the following is still open.

Question 4.10. *Is every Baire quasi-developable semitopological (or quasitopological) group a topological group?*

Definition 4.11. Let X be a space. If there exists a sequence of open covers $\{\mathscr{U}_n\}_n$ satisfying the following condition:

($\sharp$) For each $x \in X$ and a sequence $\{x_n\}$, if $x_n \in \mathrm{st}^2(x, \mathscr{U}_n)$ then $\{x_n\}$ has a cluster point in X.

Then X is called a wM-*space*.

It is still open whether a wM-space with a G_δ-diagonal is metrizable [11]. Therefore, we have the following question.

Question 4.12. *Let G be a paratopological group with a G_δ-diagonal. If G is a wM-space, is it metrizable?*

A regular space X is said to be a σ-*space* if X has a σ-locally finite network.

Question 4.13. *Let G be a normal paratopological group. If G is a k-space and a σ-space, is it paracompact?*

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