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by

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MODELS OF HYPERSPACES

ALEJANDRO ILLANES

Abstract. In this expository paper we discuss most of what is known about models of hyperspaces of metric continua.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. Given a continuum X, with metric d, we consider the following *hyperspaces* of X.

$$2^X = \{A \subset X : A \text{ is nonempty and closed in } X\},$$
$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\},$$
$$C(X) = C_1(X).$$

All the hyperspaces are considered with the *Hausdorff metric* H [17, Definition 2.1 and Theorem 2.2] defined as

$$H(A, B) = \max\{\max\{d(a, B) : a \in A\} \max\{d(b, A) : b \in B\}\},$$

where $d(a, B) = \min\{d(a, b) : b \in B\}$.

The hyperspace $F_n(X)$ is known as the *n-symmetric product* of X. The hyperspace $F_1(X)$ is an isometric copy of X inserted in each one of the hyperspaces.

In the theory of hyperspaces it is very useful to have geometric ideas of how they look. Since they are defined as certain classes of subsets of a given space, this task is not easy. For this reason, we try to construct models for them. A *model* for a given hyperspace $\mathcal{K}(X)$ is a topologically equivalent space, where the elements are points instead of subsets.

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From the geometric point of view, models of hyperspaces is a very attractive subject. Moreover, models are a very powerful tool to suggest properties and results on hyperspaces. Unfortunately, as we will see, there are only a few hyperspaces that can be modeled.

In this paper we present a survey of what has been made on models of hyperspaces of metric continua. Previous versions, in Spanish, of this topic can be found in [13, Chapter 3] and [14]. In this paper we update the information with new developed models. Here, we privilege the geometric ideas; for example, we do not prove the continuity of any function. Some of the models included here are explained with more detail in Chapter II of [17].

2. The unit interval, $C(X)$

The simplest continuum is the unit interval $[0,1]$. Notice that

$$C([0,1]) = \{[a,b] : 0 \leq a \leq b \leq 1\}.$$

It is easy to check that the function $\varphi : C([0,1]) \to \mathbb{R}^2$ ($\mathbb{R}^2$ is the Euclidean plane) given by $\varphi([a,b]) = (a,b)$ is a homeomorphism between $C([0,1])$ and the triangle $T = \{(a,b) \in \mathbb{R}^2 : 0 \leq a \leq b \leq 1\}$, represented in Figure 1.

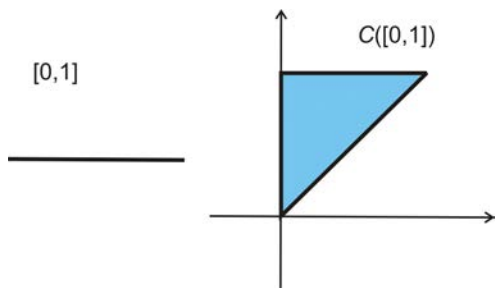


Figure 1

Thus, we can say that this triangle is a model for $C([0,1])$. Observe that the set of elements in $C([0,1])$ that contain 0 (intervals of the form $[0,b]$) is represented by an edge of T. Similarly, the set of elements of $C([0,1])$ containing 1 is represented by another edge of T. The set of singletons $F_1([0,1])$ is represented on the third edge of T (the diagonal).

Sometimes it is more useful to represent $C([0,1])$ by using the map $\psi : C([0,1]) \to \mathbb{R}^2$ given by $\psi([a,b]) = (\frac{a+b}{2}, b-a)$. Notice that the image of ψ is the triangle illustrated in Figure 2.

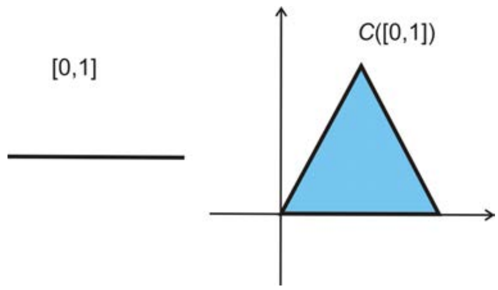


FIGURE 2

3. The circle, $C(X)$

Let S^1 be the unit circle in $\mathbb{R}^2$, centered at the origin. For each subarc A of S^1 let $m(A)$ be the middle point of A in S^1 and let $L(A)$ be the length of A. Then define $\varphi : C(S^1) \to \mathbb{R}^2$ by

$$\varphi(A) = \begin{cases} [1 - (L(A)/2\pi)]m(A), & \text{if } A \neq S^1, \\ (0,0), & \text{if } A = S^1. \end{cases}$$

It is easy to check that φ is a homeomorphism between $C(S^1)$ and the unit disc. Thus, a model for $C(S^1)$ is this disc.

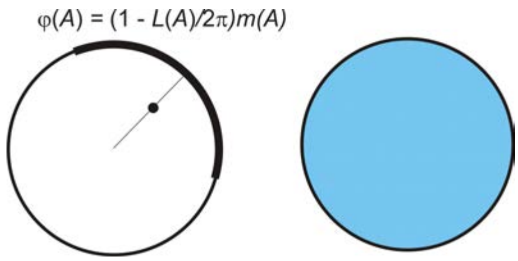


FIGURE 3

Take a point $p \in S^1$. For later use, we need to identify the image under φ of the set $\mathcal{C} = \{A \in C(S^1) : p \in A\}$. By the homogeneity of S^1 we suppose that $p = (0,1)$. The best way to visualize $\mathcal{C}$ is recognizing its boundary, which is given by the set $\{A \in C(S^1) : A$ is a subarc of S^1

and p is an end point of $A\} \cup \{S^1, \{p\}\}$. We start at $\{p\}$, we consider arcs having p as their end point and draw the images of them under φ, then we obtain the curve in Figure 4. Now we see that $\mathcal{C}$ has the shape of a heart.

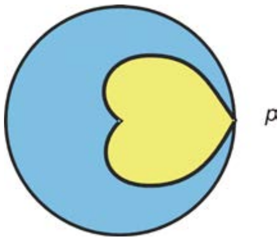


FIGURE 4

4. THE SIMPLE TRIOD, $C(X)$

Another simple continuum is the *simple triod* T defined as the union of three arcs L_1, L_2 and L_3, called the *legs* of T, joined by a point v called the *vertex* of T (Figure 5). The hyperspace $C(T)$ is the union of $C(L_1)$, $C(L_2)$, $C(L_3)$ and $C_v(T) = \{A \in C(T) : v \in A\}$. By the model in section 2, each set $C(L_i)$ can be represented as a convex triangle. Given an element A of $C_v(T)$, A is uniquely determined by the lengths of the intersections of A with the legs of T. So, they can be represented by a vector with three coordinates (a, b, c).

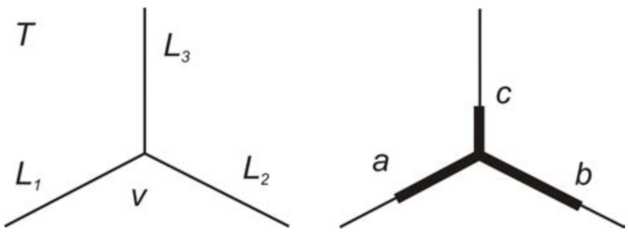


FIGURE 5

Varying the three lengths a, b and c we obtain a convex cube in $\mathbb{R}^3$. Thus $C(T)$ is the union of a convex cube in $\mathbb{R}^3$ with three convex triangles attached, as it is pictured in Figure 6.

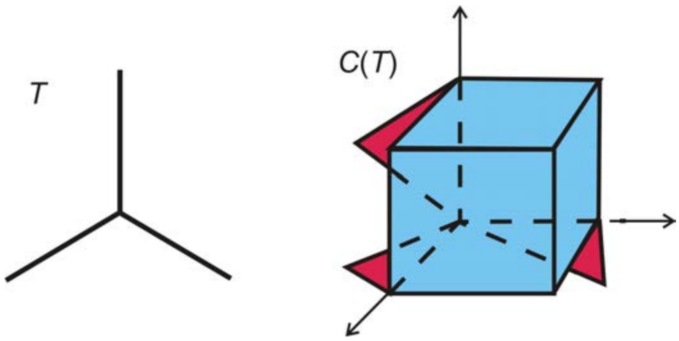


FIGURE 6

5. The noose, $C(X)$

The following continuum we consider is the *noose* N which is the union of a simple closed curve S and an arc J intersecting at a point v that is an end point of J. The hyperspace $C(N)$ is the union of $C(S)$, $C(J)$ and $C_v(N) = \{A \in C(N) : v \in A\}$. By the previous examples, the set $C(J)$ can be represented as a convex triangle and $C(S)$ can be represented as a disc. Moreover, the elements A of $C_v(N)$ are uniquely determined by $A \cap S$ and by the length of $A \cap J$.

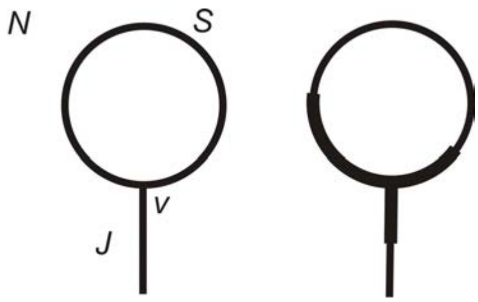


FIGURE 7

For each element $B \in C(S)$ such that $v \in B$, we can enlarge B by using a subarc of J containing v. Thus, for each such B, in the model of $C(N)$ we have to put an arc. As we saw in section 3, the set $C_v(S)$ of all such elements B is a two cell with the shape of a heart. Hence, a model for $C_v(S)$ is the cylinder $C_v(S) \times [0, 1]$. To this cylinder we add the disc

$C(S)$ and the convex triangle $C(J)$. Now, it is not difficult to see that a model for $C(N)$ is the space represented in Figure 8.

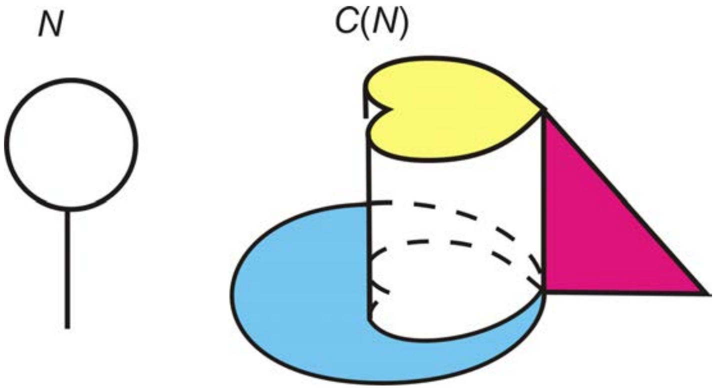


FIGURE 8

6. No more locally connected models for $C(X)$ in $\mathbb{R}^3$

A *finite graph* is a continuum that can be put as a finite union of arcs whose pairwise intersections are finite. A *simple n-od* is a finite graph G that is the union of n arcs emanating from a single point, v, and otherwise disjoint from one another. The point v is called the *vertex* of G.

Now consider the continuum $\mathcal{H}$ with the shape of the letter H. Let J be the transversal arc, as in Figure 9.

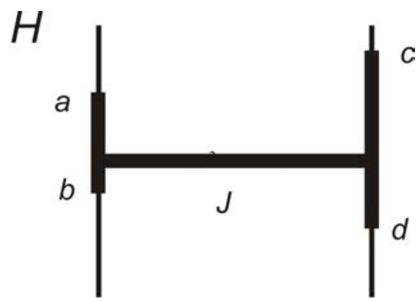


FIGURE 9

Since the subcontinua of $\mathcal{H}$ containing J can be enlarged in four independent directions, thus obtaining four lengths a, b, c and d, $C(\mathcal{H})$ contains a 4-cell and $C(\mathcal{H})$ cannot be embedded in $\mathbb{R}^3$.

If X contains a simple 4-od with vertex v, similarly as we did with the simple triod, it can be seen that $C_v(X) = \{A \in C(X) : v \in A\}$ is a 4-cell. Thus $C(X)$ is not embeddable in $\mathbb{R}^3$.

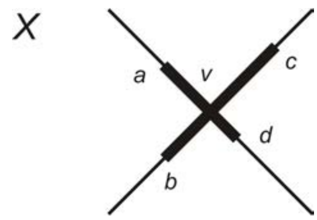
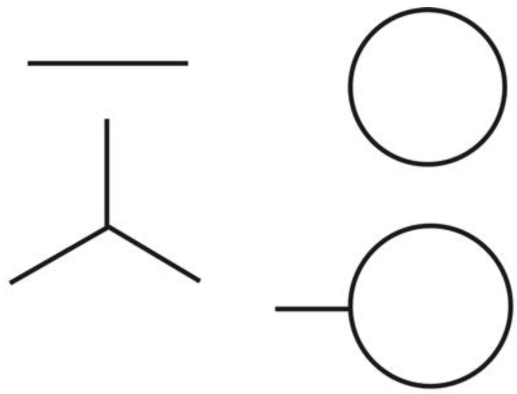


FIGURE 10

Let Z be a locally connected continuum such that $C(Z)$ is embeddable in $\mathbb{R}^3$. Then $C(Z)$ is finite dimensional. Thus, (see the historical remarks in [17, p. 44]) Z is a finite graph. By the paragraphs above, Z contains neither two ramification points nor a simple 4-od. This implies that Z has at most one ramification point and it is of order at most 3. Therefore, Z is either an arc, a simple closed curve, a simple triod or a noose. Therefore, if Z is a locally connected continuum, then $C(Z)$ is embeddable in $\mathbb{R}^3$ if and only if Z is one of the continua described in sections 2, 3, 4 or 5.



Locally connected continua X such that $C(X)$ is embeddable in R^3

FIGURE 11

7. More continua X for which $C(X)$ is embeddable in $\mathbb{R}^3$

There are more continua X for which $C(X)$ is embeddable in $\mathbb{R}^3$. For example, S. B. Nadler, Jr. showed that there are exactly eight hereditarily decomposable continua X such that cone(X) is homeomorphic to $C(X)$. These continua are pictured in Figure 12.

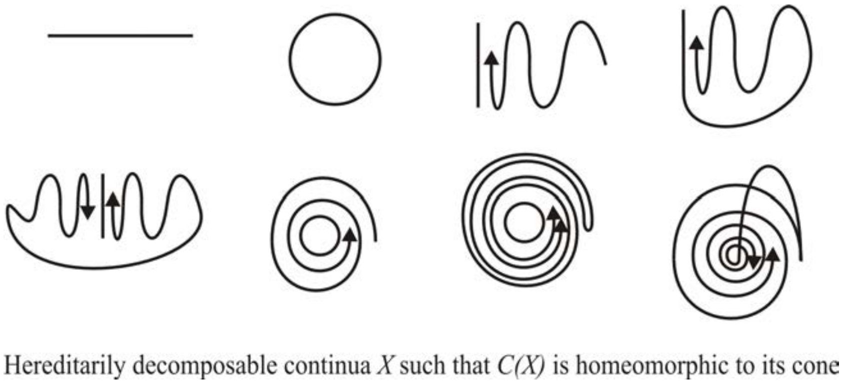


Figure 12

Since almost all of them can be embedded in $\mathbb{R}^2$, their hyperspace $C(X)$ can be embedded in $\mathbb{R}^3$. Another example X such that $C(X)$ is embeddable in $\mathbb{R}^3$ is the Buckethandle continuum (see Figure 29, p. 193 of [17]) for which it is also known that $C(X)$ is homeomorphic to its cone. One more example X is the continuum consisting of a simple triod with a ray surrounding it, X is illustrated in Figure 13.

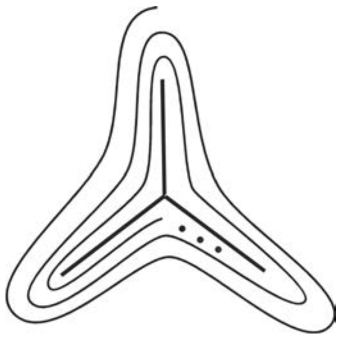


Figure 13

The model for $C(X)$ is in Figure 14.

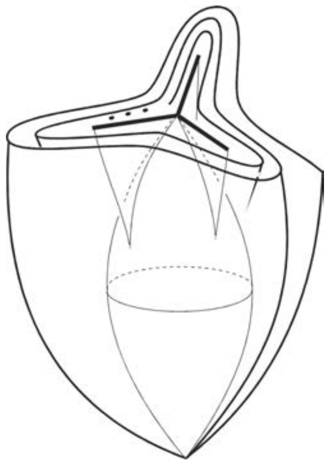


FIGURE 14

This model consists of a solid rocket (homeomorphic to the cube with three triangles of Figure 6) surrounded by an infinite sheet converging to it. This model was useful for showing a tree-like continuum X such that its hyperspace $C(X)$ does not have the fixed point property (see [15]). The general problem of characterizing those continua X for which $C(X)$ is embeddable in $\mathbb{R}^3$ seems to be very difficult. In fact, an answer to the following old problem by J. Krazinkiewicz is not known.

Problem 7.1. (see Question 3.5 of [24]) Is it true that if $C(X)$ is embeddable in $\mathbb{R}^3$, then X is embeddable in $\mathbb{R}^2$?

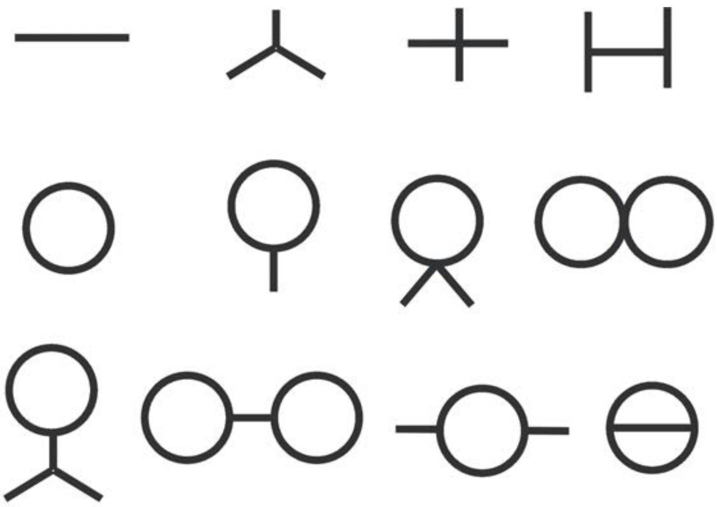
We can also ask a similar question as Problem 7.1 for $n \geq 3$, that is, we can ask if the fact that $C(X)$ is embeddable in $\mathbb{R}^{n+1}$ implies that X is embeddable in $\mathbb{R}^n$. This question can be easily solved since if $n \geq 3$ and $C(X)$ is embeddable in $\mathbb{R}^{n+1}$, then $C(X)$ is finite dimensional. This implies that (see Corollary 73.11 of [17]) X is 1-dimensional, so X is embeddable in $\mathbb{R}^3$.

An *n-od* in a continuum X is a subcontinuum B of X for which there exists a subcontinuum A of B such that $B - A$ has at least n components. It is known [17, Theorem 70.1] that X contains an n-od if and only if $C(X)$ contains an n-cell. Very recently, V. Martínez de la Vega and N. Ordoñez have found a characterization of locally connected continua X for which $C(X)$ is embeddable in $\mathbb{R}^4$ (and $\mathbb{R}^5$).

8. Locally connected continua X for which $C(X)$ is embeddable in $\mathbb{R}^4$ and $\mathbb{R}^5$

Theorem 8.1. [21] *Let X be a locally connected continuum. Then the following are equivalent.*

(a) $C(X)$ is embeddable in $\mathbb{R}^4$,
(b) $\dim(C(X)) \leq 4$,
(c) X contains no 5-ods,
(d) $C(X)$ contains no 5-cells,
(e) X is one of the continua in Figure 15.



Locally connected continua X such that $C(X)$ is embeddable in R^4

Figure 15

Theorem 8.2. [21] *Let X be a locally connected continuum. Then the following are equivalent.*

(a) $C(X)$ is embeddable in $\mathbb{R}^5$,
(b) $\dim(C(X)) \leq 5$,
(c) X contains no 6-ods,
(d) $C(X)$ contains no 6-cells,
(e) X is one of the continua in Figure 15 or $X = Z \cup J$, where Z is of one of the continua in Figure 15 and J is an arc such that $Z \cap J = \{p\}$ (for some $p \in Z$) and p is an end point of J.

The proofs of Theorems 8.1 and 8.2 depend on the construction of the models of $C(X)$ (in $\mathbb{R}^4$) of the continua in Figure 16.

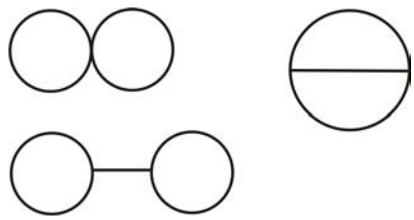


FIGURE 16

The construction of the models of the continua in Figure 16 is difficult. In [21], V. Martínez de la Vega and N. Ordoñez gave explicit formulas for embedding their hyperspaces $C(X)$ in $\mathbb{R}^4$. In particular, the formulas for the θ-curve are really complex. So this procedure seems not to be useful for proving a similar result for $\mathbb{R}^n$, for $n \geq 6$. The following question remains open.

Question 8.3. [21, Problem 2]. Given $n \geq 6$ and a continuum X, are the following equivalent?
(a) $C(X)$ is embeddable in $\mathbb{R}^n$,
(b) $\dim(C(X)) \leq n$.

More results and questions on the topic of embedding the hyperspace $C(X)$ in some space $\mathbb{R}^n$ can be found in Chapter III of [24].

9. Infinite dimensional models for $C_n(X)$

In the literature, we can find some models for the hyperspace $C(X)$, in the case that $C(X)$ is infinite dimensional. For example, C. Eberhart and S. B. Nadler, Jr. constructed models for smooth fans in [8]. In [17, Example 6.1] the hyperspace is constructed of the continuum called F_ω, which is the continuum that is the union of countably infinitely many arcs J_1, J_2,... satisfying the following conditions: All the arcs J_i emanate from a single point, v, and are otherwise disjoint from one another and $\lim J_n = \{v\}$. The most important result about infinite dimensional models is the one given by the following fundamental theorem.

Theorem 9.1. *([7] and [6] for the case $n \geq 2$). Let X be a continuum. Then the following are equivalent.*
(a) $C(X)$ is homeomorphic to the Hilbert cube,
(b) X is locally connected and each arc in X has empty interior,
(c) $C_n(X)$ is homeomorphic to the Hilbert cube.

10. $C_n([0,1])$ FOR $n \geq 2$

R. M. Schori has shown that $C_2([0,1])$ is a 4-cell by using the following argument [10, Lemma 2.2]. Let $C_0^1 = \{A \in C_2([0,1]) : 0, 1 \in A\}$ and $C_1 = \{A \in C_2([0,1]) : 1 \in A\}$. The typical elements of C_0^1 are of the form $A = [0,a] \cup [b,1]$, where $0 \leq a \leq b \leq 1$. We can define $\varphi(A) = (a,b)$. Then φ is not a function since $\varphi([0,1]) = \varphi([0,a] \cup [a,1]) = (a,a)$ for each $a \in [0,1]$. The image of φ is the triangle T in Figure 1. If we identify the diagonal Δ of T to a point we obtain the space T/Δ and now φ is a well defined homeomorphism between C_0^1 and T/Δ. This proves that C_0^1 is a 2-cell. It is easy to show that the function $\psi : C_0^1 \times [0,1] \to C_1$ given by $\psi(A,t) = t + (1-t)A$ is continuous, surjective and its only nondegenerate fiber is the set $C_0^1 \times \{1\}$. Thus, C_1 is homeomorphic to the cone of C_0^1. Hence, C_1 is a 3-cell. Finally, the function $\sigma : C_1 \times [0,1] \to C_2([0,1])$ given by $\sigma(A,t) = tA$ is continuous, surjective and its only nondegenerate fiber is $C_1 \times \{0\}$. Hence, $C_2([0,1])$ is homeomorphic to the cone over C_1. Therefore, $C_2([0,1])$ is a 4-cell.

In [11], it has been shown that, if $n \geq 3$, then $\{A \in C_n([0,1]) : A$ has a neighborhood in $C_n(X)$ that is a $2n$-cell$\} = C_n([0,1]) - C_{n-1}([0,1])$. In particular, this implies that $C_n([0,1])$ is not a $2n$-cell. The author has constructed a model for $C_3([0,1])$ and he is able to show that $C_3([0,1])$ can be embedded in $\mathbb{R}^6$. The following problem remains unsolved.

Problem 10.1. Is $C_n([0,1])$ embeddable in $\mathbb{R}^{2n}$ for each (for some) $n \geq 4$?

11. $C_n(S^1)$ FOR $n \geq 2$

In [12] it is shown that $C_2(S^1)$ is the cone over a solid torus. The proof is difficult and it seems that it cannot be generalized for $n \geq 3$. Nothing else is known for $C_n(S^1)$ ($n \geq 3$). The following questions are interesting.

Problem 11.1. (a) Find a model for $C_3(S^1)$; (b) Is $C_n(S^1)$ the cone over a continuum for some (for all) $n \geq 3$?; (c) Is $C_n(S^1)$ embeddable in $\mathbb{R}^{2n}$ for each (for some) $n \geq 3$?

In [19, Theorem 3.1] it was shown that if X is a simple m-od, then $C(X)$ is the cone over the set $\{A \in C_n(X) : A$ contains an end point of $X\}$. In [20] V. Martínez de la Vega proved that if G is a finite graph such that $C_n(G)$ is a cone for some $n \geq 2$, then G is either an m-od or a simple closed curve. So, the answer to Problem2.8 (b) could give a characterization of those finite graphs G for which $C_n(G)$ is a cone.

12. Models for 2^X

In the early 1920's, in Poland it was conjectured that $2^{[0,1]}$ is a Hilbert cube. It was not until the 1970's when this problem was solved by D. W. Curtis and R. M. Schori who proved the following fundamental theorem.

Theorem 12.1. [7]. *Let X be a continuum. Then the following are equivalent.*

(a) X is locally connected,

(b) 2^X is homeomorphic to the Hilbert cube.

Although models of some very specific examples have been constructed for 2^X, the only significant result about models for 2^X is Theorem 12.1.

13. The unit interval, $F_n(X)$

As before, in the topic of models, the simplest continuum is the unit interval $[0,1]$.

Let $\varphi : F_2([0,1]) \to \mathbb{R}^2$ be given by $\varphi(\{a,b\}) = (\min\{a,b\}, \max\{a,b\})$. Clearly, φ is an embedding whose image is the convex triangle $\{(a,b) \in \mathbb{R}^2 : 0 \leq a \leq b \leq 1\}$. Thus, $F_2([0,1])$ is a 2-cell.

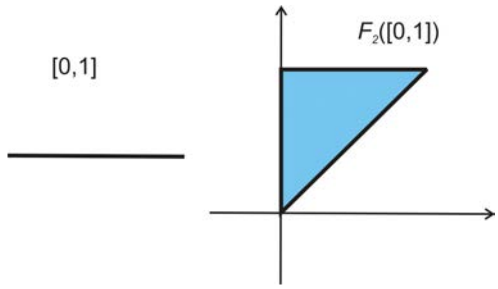


Figure 17

In order to construct a model for $F_3([0,1])$ let us consider again the map $\varphi : F_3([0,1]) \to \mathbb{R}^2$ given by $\varphi(A) = (\min A, \max A)$. Then φ is a continuous function whose image is the triangle T in Figure 17. Given $(a,b) \in T$, the fiber $\varphi^{-1}((a,b))$ is the set $\{\{a,b,c\} : a \leq c \leq b\}$. In the case that $a < b$, the set $\{\{a,b,c\} : a \leq c \leq b\}$ is a simple closed curve since c runs over the interval $[a,b]$ and $\{a,a,b\} = \{a,b,b\}$. In the case that $a = b$, $\varphi^{-1}((a,b)) = \{\{a,b,c\} : a \leq c \leq b\} = \{\{a\}\}$. Thus to obtain a model for $F_3([0,1])$ we need to put a circle at each point $(a,b) \in T$ such that $a < b$. This can be realized by taking the revolution body that can be obtained by rotating T around its diagonal. Therefore, $F_3([0,1])$ is a 3-cell.

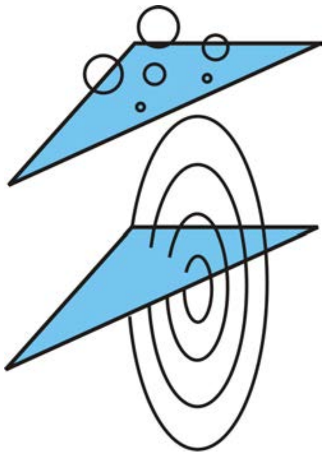


FIGURE 18

In the case that $n \geq 4$, K. Borsuk and S. Ulam, in the first paper about symmetric products, proved that $F_n([0,1])$ is not an n-cell [3, Theorem 7]. A detailed study of the hyperspaces $F_n([0,1])$ was made by R. N. Andersen, M. M. Marjanović and R. M. Schori in [1]. In particular, in Theorem 2.1 of [1], it was shown that $F_4([0,1])$ is homeomorphic to cone$(D^2) \times [0,1]$, where D_2 is the *Dunce hat*. Recall that D_2 is the space that can be obtained by identifying the edges of a convex triangle according to the arrows shown in Figure 19.

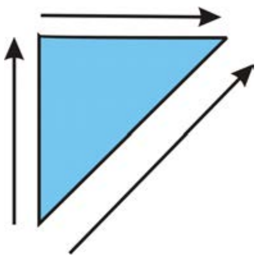


FIGURE 19

First, identify two of the arrows to obtain a cone; second, identify the vertex of the cone to a point in its base to obtain the space in Figure 20; finally, identify the two simple bold closed curves in Figure 20.

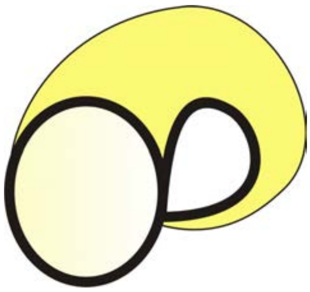


FIGURE 20

It is easy to see that D_2 can be constructed in $\mathbb{R}^3$. So, $F_4([0,1])$ is embeddable in $\mathbb{R}^5$. This answers the following question for the case $n = 4$. This question is open for $n \geq 5$.

Question 13.1. [3, last paragraph]. Is $F_n([0,1])$ embeddable in $\mathbb{R}^{n+1}$ for every $n \geq 5$?

14. The circle, $F_n(X)$

The symmetric product $F_2(S^1)$ is the Moebius Strip. We can see this by using the following argument. Let $\mathcal{NA} = \{\{p,q\} \in F_2(S^1) : p \neq -q\}$. Let $A : \mathcal{NA} \to C(S^1)$, $m : \mathcal{NA} \to S^1$, $L : \mathcal{NA} \to \mathbb{R}$ and $\varphi : \mathcal{NA} \to \mathbb{R}^2$ be given by

$$\begin{aligned} A(\{p,q\}) &= \text{the shorter arc joining } p \text{ and } q \text{ in } S^1, \\ m(\{p,q\}) &= \text{the middle point of } A(\{p,q\}), \\ L(\{p,q\}) &= \text{the length of } A(\{p,q\}), \text{ and} \\ \varphi(\{p,q\}) &= (1 - (\tfrac{1}{2\pi} L(\{p,q\})))m(\{p,q\}). \end{aligned}$$

Notice that φ is a homeomorphism between $\mathcal{NA}$ and the annulus $R = \{z \in \mathbb{R}^2 : \frac{1}{2} < |z| \leq 1\}$. If we want to extend φ to the set $\mathcal{A} = \{\{z,-z\} \in F_2(S^1) : z \in S^1\}$, by continuity and depending on the way we approximate $\{z,-z\}$ by elements $\{p,q\} \in \mathcal{NA}$, we should define $\varphi(\{z,-z\})$ with two values, namely, $\varphi(\{z,-z\}) = w$ or $-w$, where $2w$ is the point obtained by rotating z by $\frac{\pi}{2}$. To solve this ambiguity, we need to identify points w and $-w$. Notice that the points w are the points of the circle $B = \{u \in \mathbb{R}^2 : |u| = \frac{1}{2}\}$. The quotient space obtained from $R_0 = R \cup B$ by the identification is the Moebius strip M. In Figure 21 we show how this strip can be obtained. We start with the annulus and we cut it by two arrows a and b. Then we make the transformations marked in Figure 21 to get the strip M.

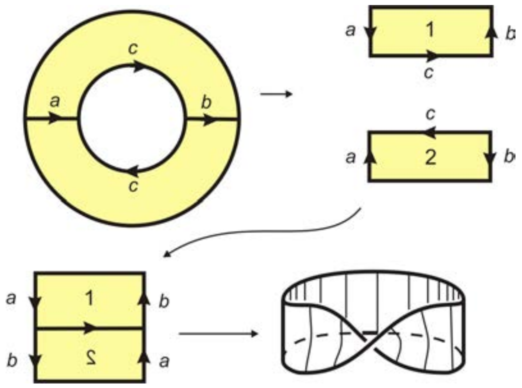


Figure 21

For further use, we need to represent in M the set $D_p = \{\{p, z\} : z \in S^1\}$. According to the definition of φ, the image of this set consists of two arcs B_1 and B_2 in the annulus R_0. If we follow the transformations that we have made to obtain the Moebius strip M, we can see that D_p is homeomorphic to a simple closed curve B that touches the boundary of M in exactly one point. This simple closed curve is represented at the end of Figure 22. It is important to remark that in this representation of the Moebius strip the curve B can be pictured entirely without dotted lines.

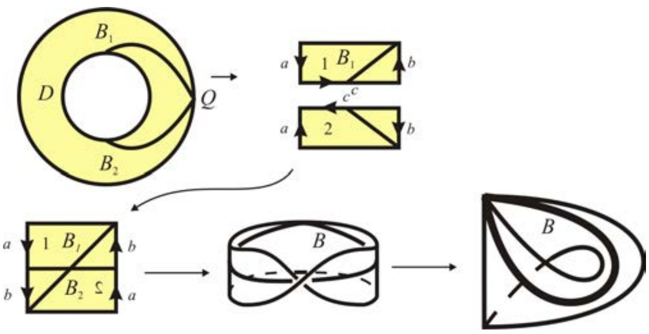


Figure 22

As we have seen, some models of hyperspaces are easy to construct. There are other more complicated examples for which a specific approach is necessary. To illustrate how difficult may it be to construct a model, let us mention that K. Borsuk made a mistake. He published a paper [2] claiming that $F_3(S^1)$ is homeomorphic to $S^1 \times S^2$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$. Three years later, R. Bott [4] corrected this fact by proving that $F_3(S^1)$ is homeomorphic to S^3. J. Mostovoy pointed out that an interesting illustration of the non-triviality of Bott's theorem is the result attributed to E. Schepin which says that the embedding knot $(F_1(S^1))$ is a trefoil knot (see Theorem 2 of [23]).

Even when no models for $F_n(S^1)$ $(n \geq 4)$ have been constructed, in [25] and [27] some topological properties of these spaces have been studied.

15. The simple triod, $F_2(X)$

Consider a simple 3-od Y as illustrated in Figure 23. Let $J_1 = L_1 \cup L_2$, $J_2 = L_2 \cup L_3$ and $J_3 = L_3 \cup L_1$. Note that $F_2(T) = F_2(L_1) \cup F_2(L_2) \cup F_2(L_3)$. Since each J_i is an arc, we know that $F_2(J_i)$ can be viewed as a convex triangle. Thus, to obtain a model for $F_2(Y)$ we need to take the three triangles $F_2(J_1)$, $F_2(J_2)$ and $F_2(J_3)$ and identify the points that represent elements of $F_2(Y)$ appearing in more than one triangle. For example, $F_2(J_1) \cap F_2(J_2) = F_2(L_2)$ which is a subtriangle in both triangles $F_2(J_1)$ and $F_2(J_2)$. In Figure 23, we picture the triangles $F_2(J_1)$, $F_2(J_2)$ and $F_2(J_3)$ with the parts that need to be identified. The resulting space is a convex triangle with three wings attached to it.

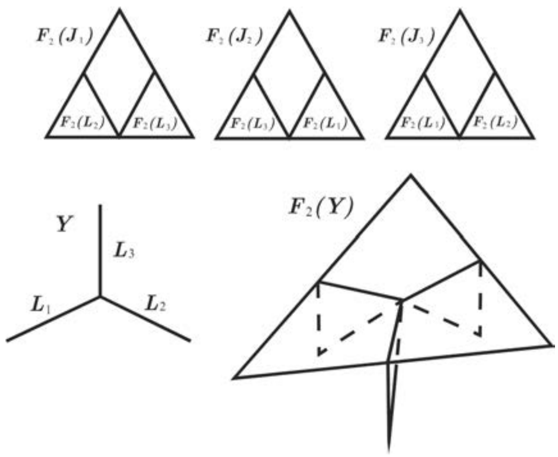


FIGURE 23

E. Castañeda has recently found a model for $F_3(Y)$ [26]. He showed that $F_3(Y)$ is the cone over a torus with four disks attached to it, one as an "equator" and the three other ones as "meridians" (Figure 24).

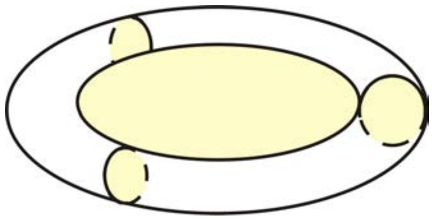


FIGURE 24

16. The simple 4-od, $F_2(X)$

Let X be a simple 4-od with vertex v, as in the Figure 25, where L is one of the legs. Let T be the simple triod obtained by removing the leg L from X. By the previous example, $F_2(T)$ is a convex triangle with three wings. Note that $F_2(X) = F_2(T) \cup F_2(L) \cup \langle T, L\rangle$, where $\langle T, L\rangle = \{\{p, q\} \in F_2(X) : p \in T \text{ and } q \in L\}$. Observe that $\langle T, L\rangle$ is homeomorphic to $T \times L$, $\langle T, L\rangle \cap F_2(T) = \{\{v, p\} \in F_2(X) : p \in T\}$ is a simple triod located on the convex triangle and $\langle T, L\rangle \cap F_2(L) = \{\{v, q\} \in F_2(X) : q \in L\}$ corresponds to the middle arc in $T \times L$. Thus, to obtain a model for $F_2(X)$ we have to put together three pieces, namely the triangle with wings, $T \times L$ and a convex triangle. The final model is illustrated in Figure 25, where the space $(T \times L) \cup F_2(J)$ is attached to the triangle with wings by the simple triod on the triangle.

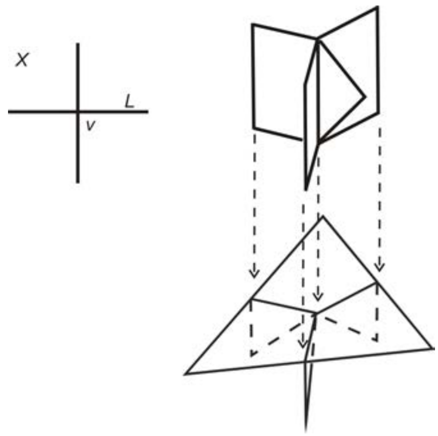


FIGURE 25

17. The noose, $F_2(X)$

Recall that the noose N is the union of a simple closed curve S and an arc J joined by a point v that is an end point of J. As in the previous example, $F_2(N) = F_2(S) \cup F_2(J) \cup \langle S, J\rangle$, where $\langle S, J\rangle = \{\{p, q\} \in F_2(N) : p \in S \text{ and } q \in J\}$. Note that $\langle S, J\rangle$ is homeomorphic to $S \times J$, $\langle S, J\rangle \cap F_2(S) = \{\{v, q\} \in F_2(N) : q \in S\}$ is a simple closed curve in $F_2(N)$ as the one we have constructed in the example of the Moebius strip (Figure 22) and $\langle S, J\rangle \cap F_2(J) = \{\{v, p\} \in F_2(N) : p \in J\}$ is an arc in the cylinder $S \times J$. Therefore, the model for $F_2(N)$ can be obtained as it is illustrated in Figure 26, where the arrows indicate how the simple closed curve in the strip is attached to the base of the cylinder. Notice that, since we can see the curve in the strip, the rest of the strip can be pushed down in such a way that this attachment can be done in the space $\mathbb{R}^3$.

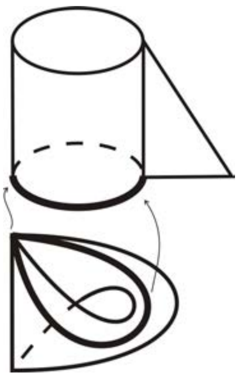


Figure 26

18. The figure eight continuum, $F_2(X)$

Let Y be the "figure eight" continuum. That is, Y is the union of two simple closed curves S_1 and S_2 whose intersection is one-point set $\{v\}$. Note that $F_2(Y) = F_2(S_1) \cup F_2(S_2) \cup \langle S_1, S_2\rangle$, where $\langle S_1, S_2\rangle = \{\{p, q\} \in F_2(Y) : p \in S_1 \text{ and } q \in S_2\}$. Given a point $p \in S_1$, the set of points $R_p = \{\{p, q\} : q \in S_2\}$ is a simple closed curve. Thus $\bigcup\{R_p : p \in S_1\}$ is a union of pairwise disjoint simple closed curves. Since p runs over the simple closed curve S_1, we can see that $\langle S_1, S_2\rangle$ is homeomorphic to the product $S^1 \times S^1$. Therefore, $\langle S_1, S_2\rangle$ is a torus. Notice that $R_v = \{\{v, q\} : q \in S_2\}$ is an "equator" of the torus $\langle S_1, S_2\rangle$ and the set $R = \{\{p, v\} : p \in S_1\}$ is a "meridian" of the torus. Since $F_2(S_1) \cap \langle S_1, S_2\rangle = R$, $F_2(S_2) \cap \langle S_1, S_2\rangle = R_v$ and $F_2(S_1) \cap F_2(S_2) = \{\{v\}\}$, we have that $F_2(Y)$ is the union of a torus with two Moebius strips attached by R and R_v. The representation of the model for $F_2(Y)$ is illustrated in Figure 27.

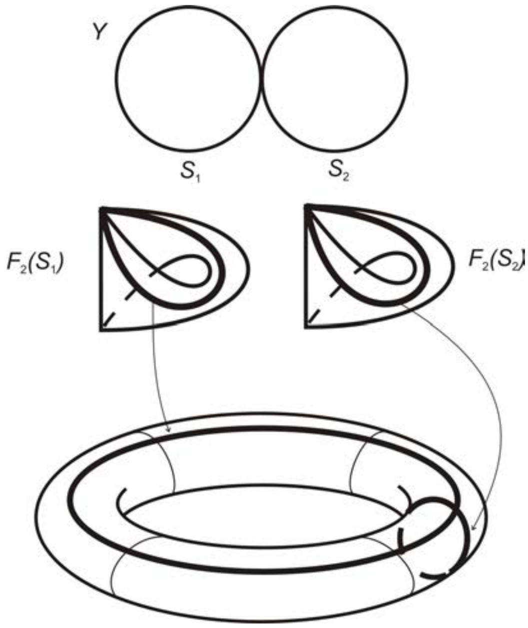
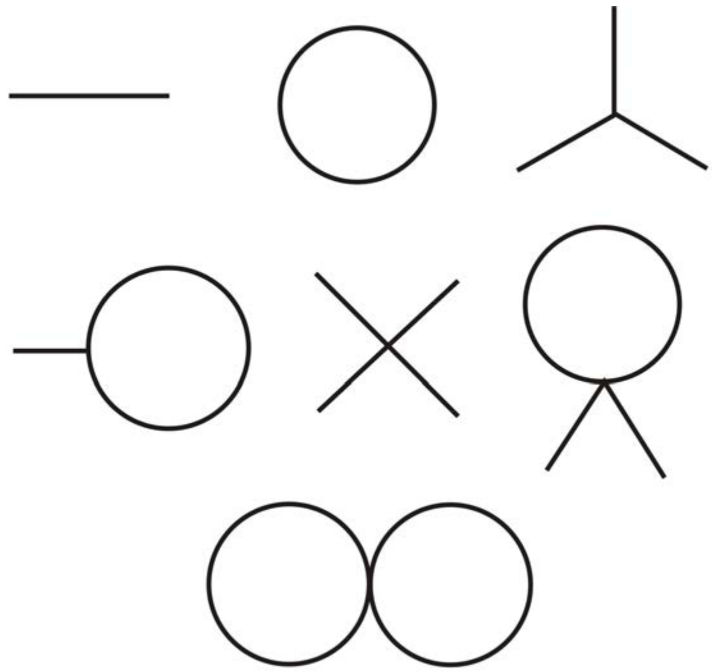


FIGURE 27

Note that, since each one of the sets R and R_v in the Moebius strips can be seen, the final model for $F_2(Y)$ can be constructed in $\mathbb{R}^3$ (in order to see this we can image the following: in order to attach $F_2(S_1)$ to the torus by the set R we can imagine the curve R in a vertical plane P in front of us in the space $\mathbb{R}^3$, then we can push the rest of $F_2(S_1)$ behind P in such a way that $F_2(S_1)$ is in one of the halves in which $\mathbb{R}^3$ is divided by P and that $F_2(S_1) \cap P = R$. Then we can make a rigid movement to put $F_2(S_1)$ on the upper part of the torus. The Moebius strip $F_2(S_2)$ must be attached to the torus from inside the tube.

We have shown that $F_2(Y)$ can be embedded in $\mathbb{R}^3$. E. Castañeda [5] proved that F_2(simple 5-od) contains a topological copy of the cone over the complete graph K_5 and then, using tools from low-dimensional topology, he showed that this cone is not embeddable in $\mathbb{R}^3$. Thus, F_2(simple 5-od) is not embeddable in $\mathbb{R}^3$. He also found that F_2(figure H-continuum) contains a topological copy of the topological cone over the complete bipartite graph $K_{3,3}$ and then he showed that F_2(figure H-continuum) is not embeddable in $\mathbb{R}^3$. Therefore, if a finite graph G either contains two vertices or it contains a vertex of order at least 5, then $F_2(G)$ is not embeddable in $\mathbb{R}^3$. The two last paragraphs can be resumed in the following theorem.

Theorem 18.1. [5, Theorem 3]. *Given a locally connected continuum X we have that $F_2(X)$ can be embedded in $\mathbb{R}^3$ if and only if X is embeddable in the figure eight continuum.*



Locally connected continua X such that $F_2(X)$ is embeddable in R^3

FIGURE 28

19. The $\sin(\frac{1}{x})$-continuum, $F_2(X)$

In [5, Problem 2] E. Castañeda asked if Theorem 18.1 can be extended for all continua X. In [16] the author constructed a model for $F_2(\sin(\frac{1}{x})$-continuum) and proved that it can be embedded in $\mathbb{R}^3$. In fact, he proved that if X is any compactification of the ray $[0, \infty)$ with an arc as the remainder, then $F_2(X)$ can be embedded in $\mathbb{R}^3$. To see how complex this model is, we describe some steps of the construction of $F_2(Z)$, where Z is the $\sin(\frac{1}{x})$-continuum.

(a) Put a copy of Z in $\mathbb{R}^2 \times \{0\}$ (Figure 29).

FIGURE 29

(b) Put a sequence $\{Z_n\}_{n=1}^{\infty}$ of copies of Z converging to Z in such a way that the remainder of each Z_n is R and the sets $Z - \{R\}$, $Z_1 - \{R\}$, $Z_2 - \{R\}$,... are pairwise disjoint (Figure 30).

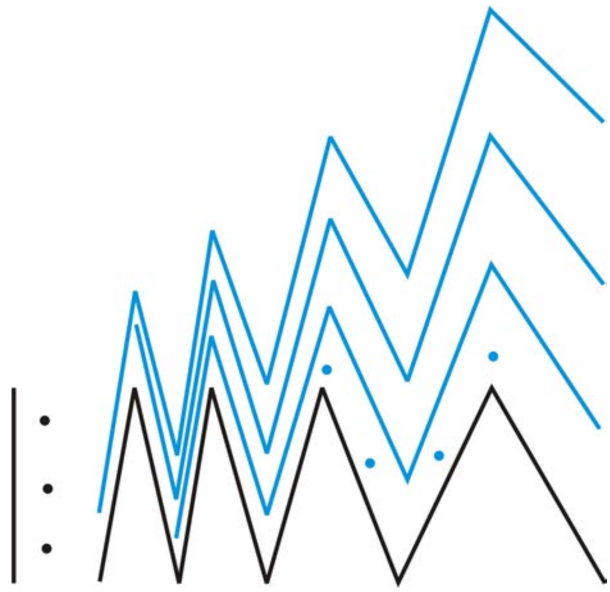


FIGURE 30

(c) Join some cuspides and some valleys in Figure 30 as it is shown in Figure 31. In this way infinitely many quadrilaterals are formed. Some of them point upward and some of them point downward.

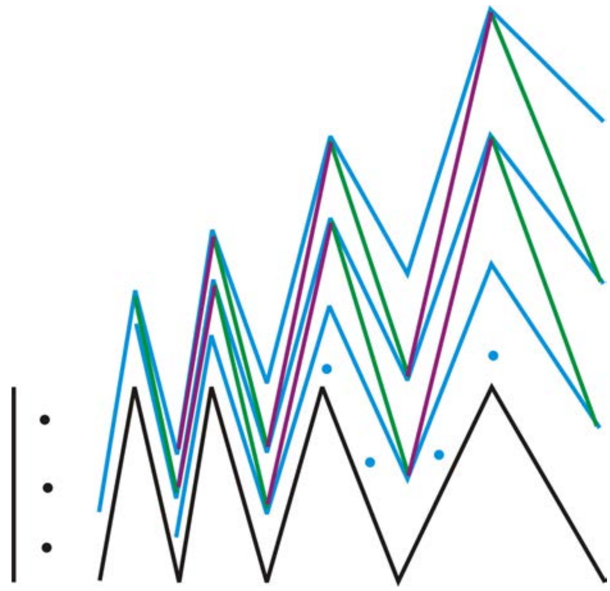


FIGURE 31

(d) For each one of these quadrilaterals and on each of its edges put a triangle as it is shown in Figure 32. For those quadrilaterals pointing upward, the triangles are constructed in $\mathbb{R}^2 \times [0, \infty)$ with one of their vertices in $\mathbb{R}^2 \times \{1\}$; for the quadrilaterals pointing downward, the triangles are constructed in $\mathbb{R}^2 \times (-\infty, 0]$, with vertices in $\mathbb{R}^2 \times \{-1\}$. On the segments of the first constructed copy of Z (Figure 29), we put triangles that make the whole set closed. This requires to put, on the segment R, two triangles T_1 and T_2, one with the third vertex in $\mathbb{R}^2 \times \{1\}$ and the other one in $\mathbb{R}^2 \times \{-1\}$. Thus we obtain a subcontinuum W of $\mathbb{R}^3$. Finally, the triangles T_1 and T_2 are folded (and identified) until they become a single triangle T contained in $\mathbb{R}^2 \times \{0\}$, which has R as an edge. By this, we obtain a map $\varphi : T_1 \cup T_2 \to T$. The rest of W $(W - (T_1 \cup T_2))$ follows continuously the folding movement in such a way that we extend φ so that $\varphi|_{W-(T_1 \cup T_2)} : W - (T_1 \cup T_2) \to \varphi(W - (T_1 \cup T_2))$ is a homeomorphism. The continuum resulting from W after identifying T_1 and T_2 is a model for $F_2(Z)$.

20. More questions

Problem 20.1. [16]. Characterize finite graphs G such that $F_2(G)$ is embeddable in $\mathbb{R}^4$.

FIGURE 32

Problem 20.2. [16]. Is it true that, for a finite graph G, $F_2(G)$ is embeddable in $\mathbb{R}^4$ if and only if G is embeddable in $\mathbb{R}^2$? Is $F_2(K_5)$ embeddable in $\mathbb{R}^4$?

The sufficiency in the first part of Problem 12 is true by the result in [22, Theorem 1] which says that $F_2([0,1]^2) = [0,1]^4$. In [22] one can find more results about $F_2([0,1]^m)$.

21. THE HILBERT CUBE, $F_n(X)$

We denote by Q the Hilbert cube. V. V. Fedorchuk proved that for each $n \geq 2$, $F_n(Q)$ is homeomorphic to Q [9, p. 223]. However, Q is not the only continuum X for which $F_n(X)$ is homeomorphic to Q. In [18, p. 139] it was shown that if X is the union of two Hilbert cubes joined by a point, then $F_2(X)$ is homeomorphic to Q. This is the only case we know in which two different spaces can have the same symmetric product.

Question 21.1. Do there exist two non-homeomorphic finite-dimensional continua X and Y such that $F_2(X)$ is homeomorphic to $F_2(Y)$?

Combining Fedorchuk's Theorem and Theorems 9.1 and 12.1, we can conclude that Q has the property that all its hyperspaces (2^Q, $C_n(Q)$ and $F_n(Q)$) are homeomorphic to Q. We do not know if Q is the only space with this property.

Question 21.2. Does there exist a continuum X, non-homeomorphic to the Hilbert cube such that X is homeomorphic to each one of its hyperspaces 2^X, $C_n(X)$ and $F_n(X)$ (for all n)?

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