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by

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# DEFINING TOPOLOGIES ON TREES

ALEKSANDER BŁASZCZYK

Abstract. In this paper we collect results concerning topologies defined on trees. We also introduce the new concept of a topology on a tree generated by a collection of filters. Some applications are presented as well.

## 1. Definitions and basic construction

For a partial order $(X, \leq)$ and $x \in X$ we shall use the following notation:

$$[x, \to) = \{z \in X \colon x \leq z\},$$
$$(\leftarrow, x) = \{z \in X \colon z < x\},$$
$$(x, \to) = \{z \in X \colon x < z\},$$
$$[x, y] = \{z \in X \colon x \leq z \leq y\}.$$

A *tree* is a partial order $(T, \leq)$ such that:

(1) there exists the least element (the root) in $T$,
(2) for every $t \in T$ the set $(\leftarrow, t)$ is well ordered by the relation $\leq$.

Clearly, for every $s \in T$ the set $[s, \to)$ is a tree in which the element $s$ is the root. As usual, the order type of the set $(\leftarrow, t)$ is called the height of $t$ in $T$ and denoted by $\mathrm{ht}(t, T)$.

If $(T, \leq)$ is a tree and $t \in T$ then

$$\mathrm{succ}_T(t) = \{s \in T \colon s \text{ is minimal in } (t, \to)\}$$

denotes the set of all *immediate successors* of the element $t$. If the tree $T$ is fixed we shall write $\mathrm{succ}(t)$ instead of $\mathrm{succ}_T(t)$.

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A tree $(T, \leq)$ is called *infinitely branching* whenever the set $\mathrm{succ}(t)$ is infinite for every $t \in T$. All trees considered here are assumed to be infinitely branching.

If a set $X$ is infinite, then $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a (free) filter whenever the following conditions hold true:

(1) $\{X \setminus F \colon |F| < \omega\} \subseteq \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,
(2) $(\forall F \in \mathcal{F})(\forall G \subseteq X)(F \subseteq G \Rightarrow G \in \mathcal{F})$,
(3) $(\forall F_1, F_2 \in \mathcal{F})(F_1 \cap F_2 \in \mathcal{F})$.

So, we assume that every filter contains the Fréchet filter, i.e. complements of all finite subsets of $X$ are elements of a filter.

Let $(T, \leq)$ be a tree and let $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$, where

$$\mathcal{F}_t \subseteq \mathcal{P}(\mathrm{succ}(t))$$

for every $t \in T$, be an indexed family of filters. For every $s \in T$ and every function

$$\varphi_s \in \prod\{\mathcal{F}_t \colon t \in [s, \rightarrow)\},$$

we consider the set

$$U_{s,\varphi_s} = \bigcup\{U^{\alpha}_{\varphi_s} \colon \alpha < \mathrm{ht}[s, \rightarrow)\},$$

where sets $U^{\alpha}_{\varphi_s} \subseteq T$ are defined as follows:

$$U^{0}_{\varphi_s} = \{s\},$$
$$U^{\alpha+1}_{\varphi_s} = U^{\alpha}_{\varphi_s} \cup \bigcup\{\varphi_s(t) \colon t \in U^{\alpha}_{\varphi_s} \text{ and } \mathrm{ht}(t, [s, \rightarrow)) = \alpha\},$$
$$U^{\alpha}_{\varphi_s} = \{t \in T \colon [s, t) \subseteq \bigcup\{U^{\beta}_{\varphi_s} \colon \beta < \alpha\}\}$$

if $\alpha$ is a limit ordinal.

For every tree $T$ and every indexed family $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ of filters we consider the collection

$$\mathcal{B}(T, \mathfrak{F}) = \{U_{s,\varphi_s} \colon s \in T \text{ and } \varphi_s \in \prod\{\mathcal{F}_t \colon t \in [s, \rightarrow)\}\}.$$

Some properties of the collection $\mathcal{B}(T, \mathfrak{F})$ are listed in the following lemmas:

**Lemma 1.1.** *For every $s \in T$ we have*

$$[s, \rightarrow) = U_{s,\chi_s},$$

*where for every $u \in [s, \rightarrow)$ we set $\chi_s(u) = \mathrm{succ}(u)$.*

*Proof.* In fact, it is easy to observe that for $u \in [s, \rightarrow)$ we have $u \in U^{\alpha}_{\chi_s}$ whenever $\mathrm{ht}(u, [s, \rightarrow)) = \alpha$. □

**Lemma 1.2.** *If $t \in U_{s,\varphi_s}$, then $[s, t] \subseteq U_{s,\varphi_s}$.*

*Proof.* Suppose $[s,t] \nsubseteq U_{s,\varphi_s}$. Let $p = \min\{u \in [s,t] \colon u \notin U_{s,\varphi_s}\}$ and let $\mathrm{ht}(p,[s,\rightarrow)) = \alpha$. Then $[s,p) \subseteq \bigcup\{U^{\beta}_{\varphi_s} \colon \beta < \alpha\}$. If $\alpha$ is a limit ordinal, then $p \in U^{\alpha}_{\varphi_s} \subseteq U_{s,\varphi_s}$; a contradiction. So, we can assume that $\alpha = \beta + 1$ and $p \in \mathrm{succ}(u)$ for some $u \in [s,t]$. In this case $u \in U^{\beta}_{\varphi_s}$ and $p \notin U^{\beta+1}_{\varphi_s}$. Therefore $p \notin \varphi_s(t)$, consequently, $[p,\rightarrow) \cap U_{s,\varphi_s} = \emptyset$. Again we get a contradiction since $t \geq p$ and $t \in U_{s,\varphi_s}$. □

**Lemma 1.3.** *If $t \notin U_{s,\varphi_s}$ and $s \notin U_{t,\psi_t}$ then $U_{s,\varphi_s} \cap U_{t,\psi_t} = \emptyset$.*

*Proof.* Suppose that $u \in U_{s,\varphi_s} \cap U_{t,\psi_t}$. Then, by the Lemma 1.2, we have $[s,u] \subseteq U_{s,\varphi_s}$ and $[t,u] \subseteq U_{t,\psi_t}$. Since the set $(\leftarrow,u)$ is well ordered we can assume that $s < t$. Hence we get $[s,t] \subseteq [s,u] \subseteq U_{s,\varphi_s}$; a contradiction since $t \notin U_{s,\varphi_s}$. □

**Lemma 1.4.** *Assume that $s,t \in T$ and*

$$\varphi_s = (F_u \colon u \in [s,\rightarrow)) \text{ and } \psi_t = (G_u \colon u \in [t,\rightarrow))$$

*are given families of filters. If $t \in U_{s,\varphi_s}$, then*

$$U_{s,\varphi_s} \cap U_{t,\psi_t} = U_{t,\xi_t},$$

*where $\xi_t(u) = F_u \cap G_u$ for every $u \in [t,\rightarrow)$.*

*Proof.* Note that if $u \in [t,\rightarrow)$, then $\xi_t(u) \subseteq \psi_t(u)$. Hence, by induction, we have $U^{\alpha}_{\xi_t} \subseteq U^{\alpha}_{\psi_t}$ for every $\alpha < \mathrm{ht}[t,\rightarrow)$. Consequently $U_{t,\xi_t} \subseteq U_{t,\psi_t}$. In a similar way we prove that if $\beta = \min\{\alpha \colon t \in U^{\alpha}_{\varphi_s}\}$, then $U^{\alpha}_{\xi_t} \subseteq U^{\beta+\alpha}_{\varphi_t}$ for $\alpha < \mathrm{ht}[t,\rightarrow)$. As a consequence we get $U_{t,\xi_t} \subseteq U_{s,\varphi_s}$.

For the proof of the remaining inclusion suppose that

$$u \in (U_{s,\varphi_s} \cap U_{t,\psi_t}) \setminus U_{t,\xi_t}.$$

Without loss of generality we can assume that $u$ is of the minimal height in $T$. By lemma 1.2 we have $[t,u) \subseteq U_{s,\varphi_s} \cap U_{t,\psi_t}$. Hence, by the minimality of $u$, we get $[t,u) \subseteq U_{t,\xi_t}$ and thus $u \in U_{t,\xi_t}$; a contradiction. Therefore $u$ is not a limit point in $T$. There exists $p \in U_{s,\varphi_s} \cap U_{t,\psi_t}$ such that $u$ is a successor of $p$. Then we have $u \in \varphi_s(p) \cap \psi_t(p) = F_p \cap G_p = \xi_t(p) \subseteq U_{t,\psi_t}$; a contradiction. □

From Lemma 1.3 and Lemma 1.4 we immediately obtain the following:

**Lemma 1.5.** *The family $\mathcal{B}(T,\mathfrak{F}) \cup \{\emptyset\}$ is closed under finite intersections.*

Let $(T,\leq)$ be a tree and let $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ be a family of filters such that

$$\mathcal{F}_t \subseteq \mathcal{P}(\mathrm{succ}(t))$$

for every $t \in T$.

**Definition 1.** The *tree topology* $\mathcal{T}_{\mathfrak{F}}$ on $T$ is the topology generated by the family $\mathcal{B}(T,\mathfrak{F})$. A tree endowed with the tree topology $\mathcal{T}_{\mathfrak{F}}$ is called an $\mathfrak{F}$*–tree.*

By Lemma 1.5 the family $\mathcal{B}(T,\mathfrak{F})$ is a base of the topology $\mathcal{T}_{\mathfrak{F}}$. Below we shall list basic properties of the tree topology. For the reader's convenience we recall some notions related to trees. For an ordinal $\alpha$ the $\alpha$-th level of the tree $(T,\leq)$ is defined as

$$\mathrm{Lev}_\alpha(T) = \{t \in T\colon\ \mathrm{ht}(t,T) = \alpha\}.$$

We shall also use

$$\mathrm{Lev}_{<\alpha}(T) = \{t \in T\colon\ \mathrm{ht}(t,T) < \alpha\}$$

and

$$\mathrm{Lev}_{\leq\alpha}(T) = \{t \in T\colon\ \mathrm{ht}(t,T) \leq \alpha\}.$$

The height of the tree $(T,\leq)$ is defined as

$$\mathrm{ht}(T) = \min\{\alpha\colon\ \mathrm{Lev}_\alpha(T) = \emptyset\}.$$

Since all trees considered here are infinitely branching, there are no maximal elements in $T$. Hence $\mathrm{ht}(T)$ is always a limit ordinal.

**Theorem 1.6.** *Let $(T,\leq)$ be an $\mathfrak{F}$–tree of height $\kappa \geq \omega$. Then the following conditions hold true:*

(1) *$T$ is a zero–dimensional dense in itself Hausdorff space,*
(2) *$T$ is nowhere compact, i.e. if $A \subseteq T$ is a compact subspace then* $\mathrm{Int}\,A = \emptyset$,
(3) $\mathrm{cl}\,\mathrm{Lev}_\alpha(T) = \mathrm{Lev}_{\leq\alpha}(T)$ *for every $\alpha < \kappa$,*
(4) $\mathrm{Int}\,\mathrm{Lev}_{\leq\alpha}(T) = \emptyset$ *for every $\alpha < \kappa$,*
(5) *if $A \subseteq T$ is a chain, then $A$ is closed and discrete,*
(6) *if $A \subseteq T$ is an antichain, then $A$ is discrete subspace of $T$.*

*Proof.* Let $\mathfrak{F} = (\mathcal{F}_t\colon t \in T)$ be the family of filters determining the topology $\mathcal{T}_{\mathfrak{F}}$. It is clear that every set $U_{t,\varphi_t} \in \mathcal{B}(T,\mathfrak{F})$ is infinite. Since $\mathcal{B}(T,\mathfrak{F})$ is a base, the $\mathfrak{F}$–tree $T$ has to be dense in itself.

For the proof of the condition (1) we shall show that every $U_{t,\varphi_t}$ is closed. In fact, assume that $s \notin U_{t,\varphi_t}$. If $t \notin [s,\rightarrow)$, then by Lemma 1.1 and Lemma 1.3, we have $[s,\rightarrow) \cap U_{t,\varphi_t} = \emptyset$.

If $t \in [s,\rightarrow)$ we take (the exactly one) element $u \in \mathrm{succ}(s)$ such that $u \leq t$. We take an arbitrary function $\psi_s \in \prod\{\mathcal{F}_p\colon p \in [s,\rightarrow)\}$ such that $\psi_s(s) = \mathrm{succ}(s) \setminus \{u\}$. Since $U_{t,\varphi_t} \subseteq [u,\rightarrow)$ and $U_{s,\psi_s} \cap [u,\rightarrow) = \emptyset$, the set $U_{s,\psi_s}$ is an open neighborhood of $s$ disjoint with $U_{t,\varphi_t}$

To complete the proof of the condition (1) it remains to show that elements of $\mathcal{B}(T,\mathfrak{F})$ separates points of $T$. Assume $s,t \in T$ are different.

If they are not comparable, then, by the Lemma 1.1 and Lemma 1.3, the intervals $[s,\to)$ and $[t,\to)$ are disjoint open neighborhoods of $s$ and $t$, respectively. If $s<t$ we take a function $\psi_s \in \prod\{\mathcal{F}_p \colon p \in [s,\to)\}$ such that $\psi_s(s) = \operatorname{succ}(s) \setminus \{u\}$, where $u \leq t$ is a successor of $s$. Then $U_{s,\psi_s}$ and $[u,\to)$ are disjoint open neighborhoods of $s$ and $t$, respectively.

The condition (2) will follow immediately from the condition (5) since every basic set $U_{s,\psi_s}$ contains an infinite chain. So, it contains an infinite closed discrete set.

For the proof of the condition (3) we observe that every set $\operatorname{Lev}_{\leq\alpha}(T)$ is closed. In fact, if $\operatorname{ht}(t,T) > \alpha$, then $\operatorname{Lev}_{\leq\alpha}(T) \cap [t,\to) = \emptyset$. Since $\operatorname{Lev}_\alpha(T) \subseteq \operatorname{Lev}_{\leq\alpha}(T)$, we get $\operatorname{cl}\operatorname{Lev}_\alpha(T) \subseteq \operatorname{Lev}_{\leq\alpha}(T)$. For the proof of the remaining inclusion we take $t \in \operatorname{Lev}_\beta(T)$ for some $\beta < \alpha$ and we fix arbitrary basic neighborhood $U_{t,\varphi_t}$. By the definition of $U_{t,\varphi_t}$ we have $U_{t,\varphi_t} \cap \operatorname{Lev}_\alpha(T) \neq \emptyset$ since $\beta<\alpha$. Hence $t \in \operatorname{cl}\operatorname{Lev}_\alpha(T)$.

For the proof of the condition (4) one can use the same arguments as above. If $t \in \operatorname{Lev}_\beta(T)$, then sets $U_{t,\varphi_t}$ intersect every level $\operatorname{Lev}_\alpha(T)$ for $\alpha \geq \beta$. In particular $\operatorname{Int}\operatorname{Lev}_{\leq\alpha}(T) = \emptyset$.

For the proof of the property (5) we take a chain $A \subseteq T$ and a point $t \in T$. We can assume that $A$ is a maximal chain. If $A \cap [t,\to) = \emptyset$ or $A \cap [t,\to) = \{t\}$, then we are done. If $t \in A$ and there exists some $s \in A \cap (t,\to)$ then, by maximality of $A$, we have $[t,s] \subseteq A$. We choose a point $u \in \operatorname{succ}(t) \setminus [t,s]$. Then we consider a function $\varphi_t \in \prod\{\mathcal{F}_p \colon p \in [t,\to)\}$ such that $\varphi_t(t) = \operatorname{succ}(t) \setminus \{u\}$. Since $[u,\to) \cap U_{t,\varphi_t} = \emptyset$, by Lemma 1.2 we have $U_{t,\varphi_t} \cap (A \setminus \{t\}) = \emptyset$.

To prove that every antichain $A \subseteq T$ is discrete it is enough to observe that for every $s \in A$ we have $A \cap [s,\to) = \emptyset$. Then the conclusion follows from the Lemma 1.1. □

**Example 1.7.** Every countable $\mathfrak{F}$–tree has a continuous bijection onto the space of rational numbers.

*Proof.* Let $T$ be a countable tree and let $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ be a family of filters. Let us consider topology $\mathcal{T}$ generated by the family

$$\mathcal{P} = \{U_{t,\varphi_t} \colon t \in T \text{ and } \varphi_t(s) = \operatorname{succ}(s) \text{ for all but finitely many } s \in [t,\to)\}.$$

Clearly, $\mathcal{P}$ is a countable subset of $\mathcal{B}(T,\mathfrak{F})$. Analyzing proof of the point (1) of the Theorem 1.6 one can easily see that the space $(T,\mathcal{T})$ is a zero–dimensional dense in itself Hausdorff space. Moreover, an analysis of the point (2) of the same theorem shows that this space is nowhere compact. Therefore, by the theorem of Sierpiński [25], the space $(T,\mathcal{T})$ is homeomorphic to the rationals. □

An Aronszajn tree is called *special* if it is a countable union of antichains. The tree topology gives a wide variety of dense in itself spaces which are countable unions of closed discrete subspaces.

**Example 1.8.** Let $(T, \leq)$ be a special Aronszajn tree. Then for every $\mathfrak{F} = (\mathcal{F}_t : t \in T)$ the $\mathfrak{F}$–tree $T$ is an uncountable dense in itself space which is a countable union of closed discrete subspaces.

*Proof.* Since all levels of the Aronszajn tree $T$ are countable, for every $\alpha < \mathrm{ht}(T) = \aleph_1$ we have $\mathrm{Lev}_\alpha(T) = \{t^n_\alpha : n < \omega\}$. For every $n < \omega$ we consider the set $T_n = \{t^n_\alpha : \alpha < \aleph_1\}$. Clearly, $T = \bigcup\{T_n : n < \omega\}$. Now, if $A \subseteq T$ is an antichain, then $A = \bigcup\{A \cap T_n : n < \omega\}$ and for every $n < \omega$ the set $A \cap T_n$ is an antichain. Moreover, for every $s \in T$ the set $A \cap T_n \cap \mathrm{succ}(s)$ has at most one element. Consequently, all the sets $A \cap T_n$ are closed since for every $s \in T \setminus (A \cap T_n)$ there exists a set $F \in \mathcal{F}_s$ such that $F \subseteq T \setminus (A \cap T_n)$. By the Theorem 1.6, point (6), the set $A \subseteq T$ is discrete in $T$. Hence $A$ is a countable union of closed discrete sets. The proof is complete since $T$ is a countable union of antichains. □

The next theorem shows that from the point of view of separation axioms $\mathfrak{F}$–tries are very regular.

**Theorem 1.9.** *Every $\mathfrak{F}$-tree is a collectionwise normal space.*

*Proof.* Let $(T, \leq)$ be an $\mathfrak{F}$-tree for some $\mathfrak{F} = (\mathcal{F}_t : t \in T)$. For any discrete family $\{E_\xi : \xi \in \kappa\}$ of closed subsets of $T$ we set

$$U_\xi = \bigcup\{U_{s,\varphi_s} : s \in E_\xi \text{ and } U_{s,\varphi_s} \cap \bigcup\{E_\eta : \eta \in \kappa \setminus \{\xi\}\} = \emptyset\}.$$

Using Lemma 1.3 one can easily check that $E_\xi \subseteq U_\xi$ and $U_\xi \cap U_\eta = \emptyset$ whenever $\xi, \eta < \kappa$ are different. □

Let us recall that a filter $\mathcal{F} \subseteq \mathcal{P}(X)$ is a (free) ultrafilter whenever for every $F \subseteq X$ we have either $F \in \mathcal{F}$ or else $X \setminus F \in \mathcal{F}$. Let us also recall that a space $X$ is *extremally disconnected* whenever for any two disjoint open sets $U, V \subseteq X$ there is $\mathrm{cl}\, U \cap \mathrm{cl}\, V = \emptyset$.

If the collection $\mathfrak{F} = (\mathcal{F}_t : t \in T)$ consists of ultrafilters, then extending a bit of a theorem of Szymański [29], we obtain the following:

**Theorem 1.10.** *If $T$ is an $\mathfrak{F}$–tree with $\mathfrak{F} = (\mathcal{F}_t : t \in T)$ and $\mathrm{ht}(T) = \omega$, then $T$ is extremally disconnected iff for every $t \in T$ the filter $\mathcal{F}_t$ is an ultrafilter.*

*Proof.* Let us assume that the $\mathfrak{F}$–topology on $T$ is extremally disconnected, $t \in T$ is an arbitrary point and $\mathrm{succ}(t) = A \cup B$ with $A \cap B = \emptyset$. We have to show that either $A \in \mathcal{F}_t$ or $B \in \mathcal{F}_t$. Clearly, the sets

$$U = \bigcup\{[a, \rightarrow) : a \in A\} \text{ and } V = \bigcup\{[b, \rightarrow) : b \in B\}$$

are open and disjoint and $\operatorname{cl} U \cup \operatorname{cl} V = [t, \to)$. Since the space is extremally disconnected, the sets $\operatorname{cl} U$ and $\operatorname{cl} V$ are both open and one of them contains the point $t$. If $t \in \operatorname{cl} U$, then $A \in \mathcal{F}_t$. Otherwise $B \in \mathcal{F}_t$.

Now assume that $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ consists of ultrafilters. To prove that the $\mathfrak{F}$–tree $T$ is extremally disconnected we first show the following:

**Claim:** For every $A \subseteq T$ and $t \in T$ we have

$$t \in \operatorname{cl} A \setminus A \Rightarrow \{s \in \operatorname{succ}(t) \colon s \in \operatorname{cl} A\} \in \mathcal{F}_t.$$

In fact, suppose that $\{s \in \operatorname{succ}(t) \colon s \in \operatorname{cl} A\} \notin \mathcal{F}_t$. Then $B = \{s \in \operatorname{succ}(t) \colon s \notin \operatorname{cl} A\} \in \mathcal{F}_t$ and for every $s \in B$ there exists an open basic sets $U_s \in \mathcal{B}(T, \mathfrak{F})$ such that $U_s \cap A = \emptyset$. It is not difficult to show that the set

$$U = \{t\} \cup \bigcup \{U_s \colon s \in B\}$$

is an open neighborhood of $t$ and $U \cap A = \emptyset$; a contradiction.

Now, if $U \subseteq T$ is an open set and $t \in \operatorname{cl} U \setminus U$, then by the Claim we can construct inductively a basic neighborhood of $t$ contained in $\operatorname{cl} U$. This is possible since $\operatorname{ht}(T) = \omega$. □

The last theorem is not true without the assumption that $\operatorname{ht}(T) = \omega$. We have the following example:

**Example 1.11.** Let $(T, \leq)$ be a tree with the underlying set

$$T = \bigcup \{{}^{\alpha}\omega : \alpha < \omega + \omega\}$$

and the partial order given by

$$x \leq y \iff y \restriction \operatorname{dom}(x) = x,$$

and let $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ be a collection of filters. Then there exist disjoint sets $U, V \subseteq T$ such that $U, V \in \mathcal{T}_{\mathfrak{F}}$ and $\emptyset \in \operatorname{cl} U \cap \operatorname{cl} V$. In particular, the $\mathfrak{F}$–tree $T$ is not extremally disconnected.

*Proof.* Clearly, for every $s \in T$ we have $|\operatorname{succ}(s)| = \aleph_0$ and hence $|\mathcal{F}_s| \leq 2^{\aleph_0}$. On the other hand we have also $|\operatorname{Lev}_{<\omega}(T)| = \aleph_0$. Therefore we get

$$|\prod \{\mathcal{F}_t \colon t \in \operatorname{Lev}_{<\omega}(T)\}| = 2^{\aleph_0}.$$

Let us enumerate $\prod \{\mathcal{F}_t \colon t \in \operatorname{Lev}_{<\omega}(T)\} = \{\varphi_\alpha \colon \alpha < 2^{\aleph_0}\}$. Now we extend every $\varphi_\alpha$ to the whole $T$ as follows:

$$\psi_\alpha(s) = \begin{cases} \varphi_\alpha(s), & \text{if } s \in \operatorname{Lev}_{<\omega}(T), \\ \operatorname{succ}(s), & \text{in the other case.} \end{cases}$$

Then for every $\alpha < 2^{\aleph_0}$ we set $U_\alpha = U_{\emptyset,\psi_\alpha}$. Since $|\psi_\alpha(s)| = \aleph_0$, we have $|U_\alpha \cap \mathrm{Lev}_\omega(T)| = 2^{\aleph_0}$ for every $\alpha < 2^{\aleph_0}$. Therefore, for every $\alpha < 2^{\aleph_0}$ we can pick different points

$$s_\alpha, t_\alpha \in (U_\alpha \cap \mathrm{Lev}_\omega(T) \setminus (\{s_\beta\colon \beta < \alpha\} \cup \{t_\beta\colon \beta < \alpha\}).$$

Then the sets

$$U = \bigcup\{[s_\alpha, \to)\colon \alpha < 2^{\aleph_0}\} \text{ and } V = \bigcup\{[t_\alpha, \to)\colon \alpha < 2^{\aleph_0}\}$$

are open and disjoint. Let us fix a basic neighborhood $U_\varphi = U_{\emptyset,\varphi} \in \mathcal{B}(T, \mathfrak{F})$, where $\varphi \in \prod\{\mathcal{F}_t\colon t \in T\}$. There exists $\alpha < 2^{\aleph_0}$ such that

$$\varphi \upharpoonright \mathrm{Lev}_{<\omega}(T) = \varphi_\alpha = \psi_\alpha \upharpoonright \mathrm{Lev}_{<\omega}(T).$$

By the definition of basic open sets at limit ordinals, we have

$$U_\varphi^\omega = \{t \in T\colon [\emptyset, t) \subseteq \bigcup\{U_\varphi^\beta\colon \beta < \omega\}\},$$

which follows that

$$U_\alpha \cap \mathrm{Lev}_\omega(T) = U_{\emptyset,\psi_\alpha} \cap \mathrm{Lev}_\omega(T) = U_\varphi^\omega \cap \mathrm{Lev}_\omega(T).$$

This follows that $U \cap U_\varphi \neq \emptyset \neq V \cap U_\alpha$. The proof is complete. □

## 2. Special cases of $\mathfrak{F}$–trees and their modifications

We shall list (in chronological order) topologies which are special cases or modifications of tree topologies defined above. We shall also describe main properties of these constructions as well as the purposes for which they were considered.

**1. The Arhangel'skii-Franklin space $S_\omega$; 1968.**
The topology is defined at the set

$$\mathrm{Seq} = \omega^{<\omega} = \bigcup\{{}^n\omega : n < \omega\}$$

considered as a tree with the order given as follows: for all $x, y \in \mathrm{Seq}$ we declare

$$x \leq y \Longleftrightarrow y \upharpoonright \mathrm{dom}(x) = x.$$

The space $S_\omega$ is just the tree $T = \mathrm{Seq}$ equipped with the topology $\mathcal{T}_{\mathfrak{F}}$, where the indexed family $\mathfrak{F} = (\mathcal{F}_t\colon t \in T)$ is such that $\mathcal{F}_t = \{\omega \setminus F\colon |F| < \omega\}$ for every $t \in T$, i.e. $\mathfrak{F}$ consists of Fréchet filters. Since $\mathrm{ht}(\mathrm{Seq}) = \omega$, a set $U \subseteq S_\omega$ is open in the Arhangel'skii-Franklin topology whenever for every $t \in U$ almost all successors of $t$ are in $U$; see Arhangel'skii and Franklin [1].

The Arhangel'skii-Franklin space $S_\omega$ can be also described as the direct limit of a countable direct system in which the first space is the converging sequence and every next space in the sequence is obtained from the previous one by adding a convergent sequence to every isolated point; see also V.Kannan and M.Rajagopalan [16].

The space $S_\omega$ was investigated in context of the sequential spaces and the main theorem in Arhangelski and Franklin [1] was the following:

**Theorem 2.1** (Arhangel'skii–Franklin)**.** *The space $S_\omega$ is a countable, dense in itself zero-dimensional Hausdorff sequential space in which*

$$\min\{\alpha\colon (\forall A \subseteq X)(A^\alpha = \mathrm{cl}_X A)\} = \omega_1,$$

*where $A^\alpha = \bigcup\{A^\beta\colon \beta < \alpha\}$ for limit ordinals $\alpha$ and $A^{\alpha+1}$ is the set of all limits of sequences contained in $A^\alpha$.*

**2. The Levy construction of the space $\mathcal{S}_\mathcal{U}$; 1977.**

The Levy topology is defined also at the set Seq. It is generated by sets of the form

$$K_s^F = \{s\} \cup \bigcup\{[t, \to)\colon t \in F\},$$

where $s \in$ Seq and $F \subseteq \mathrm{succ}(s)$ is an element of a fixed ultrafilter $\mathcal{U}$; see R. Levy [20]. Hence, the set $K_s^F$ consists of a sequence $s = \langle s_0, \dots, s_{m-1}\rangle$ and all extensions of the sequence $s$ which are of the form

$$\langle s_0, \dots, s_{m-1}, n, t_{m+1}, \dots, t_k\rangle,$$

where $n \in F$ and $F \in \mathcal{U}$.

Let us now recall the Rudin–Keisler preorder of ultrafilters. If $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}(\omega)$ are ultrafilters, then

$$\mathcal{U} \leqslant \mathcal{V} \Leftrightarrow (\exists \varphi\colon \omega \to \omega)(\{A \subseteq \omega\colon \varphi^{-1}[A] \in \mathcal{V}\} = \mathcal{U}).$$

If $\mathcal{U} \leqslant \mathcal{V}$ and $\mathcal{V} \leqslant \mathcal{U}$, then $\mathcal{U}$ and $\mathcal{V}$ are of the same type, shortly $\mathcal{U} \equiv \mathcal{V}$. There are $2^{2^\omega}$ different types of ultrafilters. As a result, there are $2^{2^\omega}$ different Levy spaces.

Main result of the paper [20] is the following:

**Theorem 2.2** (Levy)**.** *For every ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\omega)$, the space $\mathcal{S}_\mathcal{U}$ is a countable zero-dimensional homogeneous Hausdorff space with $\chi(s, \mathcal{S}_\mathcal{U}) \geq \omega_1$ for each $s \in \mathcal{S}_\mathcal{U}$. Moreover, $\mathcal{S}_\mathcal{U}$ is homeomorphic to $\mathcal{S}_\mathcal{V}$ whenever $\mathcal{U}$ and $\mathcal{V}$ are of the same type in sense of the Rudin–Keisler preorder. In particular, there are $2^{2^\omega}$ non–homeomorphic countable regular spaces, each of which has no point of first countability.*

One can easily note that the Levy spaces $\mathcal{S}_\mathcal{U}$ are not extremally disconnected.

**3. The El'kin construction; 1980**

The most general construction of a topology generated by a family of filters seams to belongs to A.G. El'kin; see [11].

Let $X$ be an infinite set. For each $x \in X$ there is fixed a free filter $\mathcal{U}_x \subseteq \mathcal{P}(X)$ assigning the topology

$$\mathcal{T}_x = \{W \subseteq X : x \notin W \vee (\exists A \in \mathcal{U}_x)(\{x\} \cup A \subseteq W)\}.$$

in which $x$ is the unique non-isolated point. Then El'kin considered the topology

$$\mathcal{T} = \bigcap\{\mathcal{T}_x : x \in X\}.$$

This is a topology on $X$ and $(X, \mathcal{T})$ is extremally disconnected iff each $\mathcal{U}_x$ is an ultrafilter. However, the topology $\mathcal{T}$ is not Hausdorff in general. Making some (rather complicated) modifications El'kin obtained a new topology $\mathcal{T}^0$ which is Hausdorff, extremally disconnected and has other nice properties.

If instead of a general set $X$ we consider a tree $T$ of height $\omega$ and we replace in the El'kin's construction "a free filter $\mathcal{U}_x \subseteq \mathcal{P}(X)$" by "a filter $\mathcal{F}_t \subseteq \mathcal{P}(\mathrm{succ}(t))$", then we obtain a tree–topology on $T$.

**4. The Szymański's topology on** Seq$(X)$**; 1985.** The most elegant construction of a topology generated by a family of filters belongs to A. Szymański [29]. The underlying set is the tree

$$\mathrm{Seq}(X) = X^{<\omega} = \bigcup\{{}^nX : n < \omega\}$$

with the same order as in Seq, i.e.

$$s \leq t \Longleftrightarrow t \restriction \mathrm{dom}(s) = s.$$

If $X = \omega$ we shall simply write Seq instead of Seq$(\omega)$. Let $\mathcal{U} \subseteq \mathcal{P}(X)$ be a fixed free ultrafilter. The Szymański's topology on Seq$(X)$ is defined by declaring

$$W \subseteq \mathrm{Seq}(X) \text{ is open} \Leftrightarrow (\forall s \in W)(\exists A \in \mathcal{U})(s^\frown A \subseteq W),$$

where $s^\frown A = \{s^\frown a : a \in A\}$; see A. Szymański [29]. The symbol $s^\frown a$ denotes the *concatenation* of the sequence $s$ by an element $a$, i.e.,

$$s = \langle s_0, \dots s_{k-1}\rangle \Rightarrow s^\frown a = \langle s_0, \dots, s_{k-1}, a\rangle$$

or, in other words, if $s \in {}^k\omega$, then $s^\frown a = s \cup \{(k, a)\}$. Clearly, for every $s \in$ Seq$(X)$, the set succ$(s)$ can be identified with the set $X$ alone. Then the Szymański's topology on Seq$(X)$ is just the tree topology $\mathcal{T}_{\mathfrak{F}}$ on Seq$(X)$ for $\mathfrak{F} = (\mathcal{F}_t : t \in T)$, where $\mathcal{F}_t = \mathcal{U}$ for every $t \in$ Seq$(X)$.

For $X = \omega$ the same topology described in a bit different way was considered independently by V.Trnková [30]. Earlier a similar topology, not in Seq, but on the set $[\omega]^{<\omega}$ of all finite subsets of $\omega$ was considered by A. Louveau [22]. However, there was proved by J. E. Vaughan that the space considered by A. Louveau is homeomorphic to Seq with a suitable topology; see [31], Theorem 4.2.

The main theorem of A. Szymański [29] and the purpose for which he considered the space $\mathrm{Seq}(X)$ is the following:

**Theorem 2.3** ( Szymański)**.** *If $\lambda$ is a measurable cardinal and $\mathcal{U} \subset \mathcal{P}(\lambda)$ is a $\lambda$-complete ultrafiler on $\lambda$, then the space* $\mathrm{Seq}(\lambda)$ *is an extremally disconnected dense in itself Tychonoff space with the property that for every extremally disconnected space $X$ of size less than $\lambda$ the space $X \times$* $\mathrm{Seq}(\lambda)$ *is extremally disconnected as well.*

The Szymański's Theorem is motivated by the fact that if X and Y are infinite extremally disconnected compact spaces, then the space $X \times Y$ is not extremally disconnected.

It is not difficult to note that instead of a single ultrafilter $\mathcal{U}$ in the Szymański's construction one can consider an indexed family $\mathfrak{F} = (\mathcal{F}_t : t \in T)$ of filters on $X$. Then we get the $\mathcal{T}_{\mathfrak{F}}$ topology on the tree $T = \mathrm{Seq}(X)$. There is a number of very nice results about these spaces. Below we list the most interesting of them. As usual, $\beta X$ denotes the Stone–Čech compactification of $X$ and "incomparable" means incomparable in the sense of the Rudin–Keisler preorder whereas "type" means equivalent in the same preorder.

In chronological order there appeared the following results:

(1) (Dow–Gubbi–Szymański, 1988) If $T = \mathrm{Seq}(\omega)$ and $\mathfrak{F} = (\mathcal{F}_t : t \in T)$ consists of pairwise incomparable weak $P$–points, then the Stone–Čech compactification of the $\mathfrak{F}$–tree $T$ is an extremally disconnected compact rigid space; see [10],
(2) (Błaszczyk–Rajagopalan–Szymanski, 1993) If $T = \mathrm{Seq}(\omega)$ and $\mathfrak{F} = (\mathcal{F}_t : t \in T)$ consists of ultrafilters, then the $\mathfrak{F}$–tree $T$ is a dense in itself hereditarily extremally disconnected space; see [4],
(3) (Kato,1994) If $T = \mathrm{Seq}(\omega)$ and all ultrafilters in $\mathfrak{F} = (\mathcal{F}_t : t \in T)$ are the same, then the $\mathfrak{F}$–tree $T$ is a homogeneous; see [17],
(4) (Lindgren–Szymański, 1997) If $\mathcal{U}_1$ and $\mathcal{U}_2$ are selective ultrafilters of different types and $H(\mathcal{U}_i)$, for $i \in \{1, 2\}$, is the orbit of the point $\{\emptyset\}$ with respect to the set of all homeomorphisms of $\beta(\mathrm{Seq}(X))$ with the tree topology generated by $\mathcal{U}_i$, then the spaces $H(\mathcal{U}_i)$ are separable extremally disconnected countably compact and homogeneous, but the space $H(\mathcal{U}_1) \times H(\mathcal{U}_2)$ is not pseudocompact; see [21],
(5) (Vaughan, 2001) If $T = \mathrm{Seq}(\omega)$ and $\mathfrak{F} = (\mathcal{F}_t : t \in T)$, then the $\mathfrak{F}$–tree $T$ is homogeneous iff all the ultrafilters $\mathcal{F}_t$ are of the same type in sense of the Rudin–Keisler; see [31],
(6) (Vaughan, 2001) If $T = \mathrm{Seq}(\omega)$ and $\mathfrak{F} = (\mathcal{F}_t : t \in T)$ and $f : T \to T$ is continuous as a function of the $\mathfrak{F}$–tree $T$ and for every $t \in T$ the ultrafilter $\mathcal{F}_{f(t)}$ is not below $\mathcal{F}_t$ for all $t \in T$, then the function $f$ is locally constant; see [31],

(7) (Błaszczyk–Szymanski, 2001) If $T = \mathrm{Seq}(\omega)$ and $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$, then the Stone–Čech compactification of the $\mathfrak{F}$–tree $T$ has a semi–open mapping onto the Cantor cube $\{0,1\}^\omega$ iff the ultrafilters $\mathcal{F}_t$ are not nowhere dense; see [6],

**5. The construction of Juhász, Soukup and Szentmiklóssy; 2008** Let $(T, \leq)$ be a tree and let $F = (\mathcal{F}_t \colon t \in T)$ be a family of ultrafilters, where $\mathcal{F}_t \subseteq \mathrm{succ}(t)$ for all $t \in T$. The topology of I. Juhász, L. Soukup and Z. Szentmiklóssy on $T$ is defined as

$$\{U \subseteq T \colon (\forall t \in U)(U \cap \mathrm{succ}(t) \in \mathcal{F}_t)\};$$

see [13]. The tree $T$ endowed with the topology described above is denoted by $X(F)$; see also I. Juhász and M. Megidor [12]. It is not difficult to observe that the space $X(F)$ coincide with the topology of the $\mathfrak{F}$–tree whenever $\mathrm{ht}(T) = \omega$. However, in general the topology on $X(T)$ is richer than the topology of the $\mathfrak{F}$–tree. There was proved in [13] that the topology $\mathcal{T}_{\mathfrak{F}}$ is monotonically normal but the main result is the following:

**Theorem 2.4** (Juhász, Soukup and Szentmiklóssy)**.** *If all the ultrafilters in $\mathfrak{F}$ are $\lambda$-descendingly complete then the space $X(T)$ is $\lambda^+$-irresolvable, which means that no subspace of $T$ is $\lambda^+$-resolvable.*

Let us recall that a space is $\kappa$-resolvable for $\kappa > 1$ if it contains $\kappa$ many disjoint dense subsets. For $\kappa \geq \omega$ and $T = \kappa^{<\kappa}$, where

$$\kappa^{<\kappa} = \bigcup \{ {}^{\alpha}\kappa : \alpha < \kappa\},$$

and $T$ is considered with the natural order, i.e., $x \leq y$ whenever $y \restriction \mathrm{dom}(x) = x$ for $x, y \in \kappa^{<\kappa}$, and for a family $F = (\mathcal{F}_t \colon t \in T)$ of free ultrafilters, the space $X(F)$ was studied earlier by A. Brzeska, see [7]. In particular, she proved that such a space is normal.

## 3. Some applications of $\mathfrak{F}$–topologies

**1. Mappings of $\mathfrak{F}$-tries into itself.**

In this section we shall use the general assumption that if $(T, \leq)$ is an $\mathfrak{F}$-tree for some $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$, then for every $t \in T$, $\mathcal{F}_t$ is an ultrafilter.

**Lemma 3.1.** *If $(T, \leq)$ is an $\mathfrak{F}$-tree, then for every $s \in T$ and $A \subseteq \mathrm{succ}(s)$ we have*

$$A \in \mathcal{F}_s \Leftrightarrow \mathrm{cl}\, A = \{s\} \cup A.$$

*Proof.* For a basic set $U_{s,\varphi_s}$ we have $U^1\varphi_s = \{s\} \cup \varphi_s(s) \subseteq U_{s,\varphi_s}$ where $\varphi_s(s) \in \mathcal{F}_s$. Therefore, if $A \in \mathcal{F}_s$, then every open neighborhood of $s$ intersect $A$ so, $s \in \mathrm{cl}\, A$. Hence $\{s\} \cup A \subseteq \mathrm{cl}\, A$. To show that $\{s\} \cup A \supseteq \mathrm{cl}\, A$ assume that $t \in \mathrm{cl}\, A \setminus A$ and $t \neq s$. If $t < s$ then we pick $u \in \mathrm{succ}(t) \setminus (\leftarrow, s]$.

Let $\varphi_t \in \prod\{\mathcal{F}_p : p \in [t, \rightarrow)\}$ be such that $\varphi_t(t) = \operatorname{succ}(t) \setminus \{u\}$. Then we have $U_{t,\varphi_t} \cap \operatorname{succ}(s) = \emptyset$ and thus $t \notin \operatorname{cl} \operatorname{succ}(s)$. Since $A \subseteq \operatorname{succ}(s)$, we have $t \notin \operatorname{cl} A$. In the remaining case, i.e., $t \not< s$, we have $[t, \rightarrow) \cap \operatorname{succ}(s) = \emptyset$.

Assume that $A \subseteq \operatorname{succ}(s)$ and $\operatorname{cl} A = \{s\} \cup A$. If $A \notin \mathcal{F}_s$ then $\operatorname{succ}(s) \setminus A \in \mathcal{F}_s$ and we can consider a basic open set $U_{s,\varphi_s}$ such that $\varphi_s(s) = \operatorname{succ}(s) \setminus A$. Then we obtain $U_{s,\varphi_s} \cap A = \emptyset$, a contradiction which completes the proof. □

Later we shall assume additionally that $\operatorname{ht}(T) = \omega$ and there exists a cardinal $\kappa \geq \omega$ such that $|\operatorname{succ}(s)| = \kappa$ for all $s \in T$. Hence, by the Theorem 1.10, the $\mathfrak{F}$-tree $T$ is extremally disconnected.

The Rudin–Keisler preorder can also be defined at the set of all ultrafilters on every $\kappa \geq \omega$. If $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}(\kappa)$ are ultrafilters, then

$$\mathcal{U} \leqslant \mathcal{V} \Leftrightarrow (\exists \varphi \colon \kappa \to \kappa)(\{A \subseteq \kappa \colon \varphi^{-1}[A] \in \mathcal{V}\} = \mathcal{U});$$

see also Section 2.2 for the case $\kappa = \omega$. From the topological point of view, ultrafilters are points in $\beta\kappa$. Hence, $\mathcal{U} \leqslant \mathcal{V}$ iff there exists a function $f \colon \kappa \to \kappa$ such that $\mathcal{U} = f^*(\mathcal{V})$, where $f^* \colon \beta\kappa \to \beta\kappa$ is the Stone-Čech extension of $f$ over $\beta\kappa$. In fact, we have

$$f^*(\mathcal{V}) = \{A \subseteq \kappa \colon f^{-1}[A] \in \mathcal{V}\}.$$

Let us observe that for the inequality $\mathcal{U} \leqslant \mathcal{V}$ it is enough to have a surjection $f \colon A \to B$, where $A \in \mathcal{V}$ and $B \in \mathcal{U}$. In fact, we can extend the function $f$ to a function $g \colon \kappa \to \kappa$. Since $\mathcal{V} \in \operatorname{cl} A$ and $\mathcal{U} \in \operatorname{cl} B$ and $g[A] = B$, we have $g^*(\mathcal{V}) = \mathcal{U}$.

If $\mathcal{U} \leqslant \mathcal{V}$ and $\mathcal{V} \leqslant \mathcal{U}$, then $\mathcal{U}$ and $\mathcal{V}$ are of the same type, shortly $\mathcal{U} \equiv \mathcal{V}$. Since there exists $2^{2^\kappa}$ different ultrafilters on $\kappa$ and only $2^\kappa$ functions from $\kappa$ to $\kappa$, there exists $2^{2^\kappa}$ incomparable ultrafilters.

Some version of the next lemma was proved by Vaughan [31].

**Lemma 3.2.** *Assume $(T, \leq)$ is an $\mathfrak{F}$-tree for some $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ and $f \colon T \to T$ is continuous and $f(s) = t$. If $f[A] \in \mathcal{F}_t$ for some $A \in \mathcal{F}_s$, then $\mathcal{F}_t \leqslant \mathcal{F}_s$.*

*Proof.* By Lemma 3.1, Theorem 1.9 and Theorem 1.10, $\{s\} \cup A$ is a closed subspace of the extremally disconnected normal space $T$. Then the closure of $\{s\} \cup A$ in $\beta T$ equals to the Stone-Čech extension of $\{s\} \cup A$. Since $A$ is discrete, $s$ is a point in the remainder of $\beta A$. Analogously, $t$ is a point in the remainder of $\beta f[A]$. Since the point $s$ is in fact equal to the ultrafilter $\mathcal{F}_s$ in $\beta A$ and $t$ is equal to the ultrafilter $\mathcal{F}_t$ in $\beta f[A]$, we get $\mathcal{F}_t \leqslant \mathcal{F}_s$. □

**Theorem 3.3.** *Assume $(T, \leq)$ is an $\mathfrak{F}$-tree and $f \colon T \to T$ is a continuous closed mapping. If $\mathfrak{F}$ consists of pairwise incomparable ultrafilters, then $f$ is the identity.*

*Proof.* Suppose $f(s) = t \neq s$. By Lemma 3.1 we have $\operatorname{cl}\operatorname{succ}(s) = \{s\} \cup \operatorname{succ}(s)$. Since $f$ is closed we have

$$(*) \qquad \operatorname{cl} f[\operatorname{succ}(s)] = \{t\} \cup f[\operatorname{succ}(s)].$$

We claim that $f[\operatorname{succ}(s)] \cap \operatorname{succ}(t) \in \mathcal{F}_t$. In fact, otherwise

$$B = \operatorname{succ}(t) \setminus f[\operatorname{succ}(s)] \in \mathcal{F}_t.$$

By the condition $(*)$, for every $p \in B$ we can pick a basic neighborhood $U_p$ such that $U_p \cap f[\operatorname{succ}(s)] = \emptyset$. Then the set

$$U = \{t\} \cup \bigcup \{U_p \colon p \in B\}$$

is an open neighborhood of $t$ and $U \cap f[\operatorname{succ}(s)] = \emptyset$. Hence $f^{-1}[U] \cap \operatorname{succ}(s) = \emptyset$; we get a contradiction because $s \in \operatorname{cl}\operatorname{succ}(s)$. Now we set

$$A = f^{-1}[f[\operatorname{succ}(s)] \cap \operatorname{succ}(t)] \cap \operatorname{succ}(s).$$

Clearly, $f[A] = f[\operatorname{succ}(s)] \cap \operatorname{succ}(t) \in \mathcal{F}_t$. We claim that $A \in \mathcal{F}_s$. In fact, otherwise $\operatorname{succ}(s) \cap f^{-1}[\operatorname{succ}(t) \setminus f[\operatorname{succ}(s)]] \in \mathcal{F}_s$ and so $s \in \operatorname{cl} f^{-1}[\operatorname{succ}(t) \setminus f[\operatorname{succ}(s)]]$. Then we have

$$t \in f[\operatorname{cl} f^{-1}[\operatorname{succ}(t) \setminus f[\operatorname{succ}(s)]]] \subseteq \operatorname{cl}(\operatorname{succ}(t) \setminus f[\operatorname{succ}(s)]).$$

But the last condition is impossible since $\operatorname{succ}(t) \setminus f[\operatorname{succ}(s)] \notin \mathcal{F}_t$. Therefore $A \in \mathcal{F}_s$ and, by the Lemma 3.2, we have $\mathcal{F}_t \leqslant \mathcal{F}_s$ which leads to a contradiction. □

Jerry Vaughan [31] proved that if $\mathfrak{F} = (\mathcal{U}_s \colon s \in \mathrm{Seq})$ is a family of ultrafilters over $\omega$ and $f\colon \mathrm{Seq}(\omega) \to \mathrm{Seq}(\omega)$ is continuous with respect to the $\mathfrak{F}$–topology and $\mathcal{U}_{f(s)} \nleqslant \mathcal{U}_s$ for all $s \in \mathrm{Seq}$, then for every $s \in \mathrm{Seq}(\omega)$ there exists an open neighborhood $W_s$ of $s$ such that $f \restriction W_s$ is constant. In particular $f$ cannot be open. A special case of ths result was obtained by V. Kannan and M. Rajagopolan [16].

Using some ideas of Vaughan we obtain the following theorem corresponding to the Theorem 3.3:

**Theorem 3.4.** *Assume $(T, \leq)$ is an $\mathfrak{F}$-tree and $\mathfrak{F}$ consists of pairwise incomparable ultrafilters. If $U, V \subseteq T$ are open sets and $f\colon U \to V$ is an open surjection, then $U = V$ and $f$ is the identity.*

*Proof.* Let $\mathfrak{F} = (\mathcal{U}_t \colon t \in T)$. Suppose there exists a point $s \in U$ such that $f(s) = t \in V \setminus \{s\}$. We claim the following:

**Claim:** There exists a set $A_s \in \mathcal{U}_s$ such that $f[A_s] = \{t\}$.

For the proof of the claim let us consider the set

$$A = \{p \in \operatorname{succ}(s) \colon f(p) = t\}.$$

We shall show that $A \in \mathcal{U}_s$. Suppose that $B = \operatorname{succ}(s) \setminus A \in \mathcal{U}_s$. Since $f$ is continuous, there exits a set $B' \in \mathcal{U}_s$ such that $f[B'] \subseteq [t \to)$.

Hence $C = B \cap B' \in \mathcal{U}_s$ and $f[C] \subseteq (t, \to)$ since $f(p) \neq t$ for every $p \in B$. Let as consider the set

$$D = \{p \in C\colon f(p) \in \mathrm{succ}(t)\}.$$

We shall show that $D \in \mathcal{U}_s$. Otherwise $E = \mathrm{succ}(s) \setminus D \in \mathcal{U}_s$ and for every $p \in E$ we can choose an open neighborhood $U_p$ such that $f[U_p] \subseteq [f(p), \to)$. Then the set

$$V = \{s\} \cup \bigcup \{U_p\colon p \in E\}$$

is an open neighborhood of $s$. On the other hand $f(p) \notin \mathrm{succ}(t)$ for every $p \in E$. Therefore $f[V] \cap \mathrm{succ}(t) \notin \mathcal{U}_t$, which means that $t = f(s) \notin \mathrm{Int}\, f[V]$; we get a contradiction since $f$ is an open mapping. Therefore, $D \in \mathcal{U}_s$ and in consequence $f[D] \in \mathcal{U}_t$. Otherwise the set

$$W_t = \{t\} \cup \bigcup \{[p, \to)\colon p \in \mathrm{succ}(t) \setminus f[D]\}$$

is an open neighborhood of $f(s)$. Since the function $f$ is continuous, there exists an open neighborhood $W_s$ of the point $s$ such that $f[W_s] \subseteq W_t$. But this is impossible since $W_s \cap D \neq \emptyset$ and $f[D] \cap W_t = \emptyset$. Hence $f[D] \in \mathcal{U}_t$ and, by the the Lemma 3.2, we get a contradiction which proves the Claim.

Using the Claim and the fact that $\mathrm{ht}(T) = \omega$, we can inductively construct a basic neighborhood $U_s$ of the point $s$ such that $f \restriction U_s$ is constant. But this is impossible since $T$ is dense in itself and $f$ is open. □

**2. Mappings of the Stone–Čech compactification of $\mathfrak{F}$-tries.**
Obviously, properties of $\mathfrak{F}$–tries depends on the type of filters in $\mathfrak{F}$. In particular, for some families $\mathfrak{F}$ an $\mathfrak{F}$-tree is homogeneous whereas for some other it can be rigid. Let us recall that a space $T$ is rigid if there are no other homeomorphism of $T$ onto itself except the identity.

The $\mathfrak{F}$–topologies on the tree Seq were used by A. Dow, A. Gubi and A. Szymański [10] for a very elegant construction of a rigid extremally disconnected compact space. The existence of extremally disconnected compact rigid spaces was first proved, using a forcing argument, by K. McAloon [23] in 1971. Later S. Shelah [27] and also P. Štěpánek and B. Balcar [28] obtained another, but still complicated constructions. Using a result of K. Kunen [19], that there exist $2^{2^{\aleph_0}}$ incomparable weak P–ultrafilters there was proved the following theorem:

**Theorem 3.5** ( Dow–Gubi–Szymański). *Assume $T = \mathrm{Seq}$ is an $\mathfrak{F}$–tree for $\mathfrak{F} = (\mathcal{U}_s\colon s \in T)$ consisting of pairwise incomparable weak P–ultrafilters. Then the space $\beta T$ is an extremally disconnected compact rigid space.*

An ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\omega)$ is called a weak P–ultrafilter if it is a weak P–point in the reminder of the Stone–Čech compactification of $\omega$. Let us recall that a (non isolated) point $x \in X$ is a weak P-point in $X$ if it is not in the closure of any countable set contained in $X \setminus \{x\}$.

Let us recall that by the Theorem 1.10 and the Theorem 1.9, if $T$ is a tree of height $\omega$ and $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ is an indexed family of ultrafilters, then the $\mathfrak{F}$–tree $T$ is an extremally disconnected normal space. Therefore $\beta T$ is an extremally disconnected compact space. Then $T$ is a dense subspace of an extremally disconnected compact space $\beta T$. On the other hand, by Theorem 1.6(2), also $\beta T \setminus T$ is a dense subset of $\beta T$. Some properties of $\beta\,\mathrm{Seq} \setminus \mathrm{Seq}$ were investigated by several authors. In particular, in [6] there was proved the following:

**Theorem 3.6** (Błaszczyk–Szymański)**.** *Assume $T = \mathrm{Seq}$ is an $\mathfrak{F}$–tree for $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ and $F \subseteq \beta T \setminus T$ is a countable union of compact sets. Then $\mathrm{cl}\, F \subseteq \beta T \setminus T$ iff the following condition holds true:*

$$(\forall s \in T)(\exists A_s \in \mathcal{F}_s)(F \cap \mathrm{cl}\, A_s = \emptyset). \qquad (*)$$

There was observed in [6] that the condition $(*)$ is true whenever $\mathfrak{F}$ consists of $P$–ultrafilters. This follows that Seq is a $P$–set in $\beta\,\mathrm{Seq}$, i.e., it is contained in the interior of any $G_\delta$–set containing Seq; see [6], Corollary 13. Subsequently P. Simon [29] has proved that Seq is a P–set in $\beta\,\mathrm{Seq}$ if and only if in the collection $(\mathcal{F}_s \colon s \in \mathrm{Seq})$ every $\mathcal{F}_s$ is a P–filter (not necessary ultrafilter). Another result of this type was obtained by I. Juhász and A. Szymański [14]: if $\mathcal{F}_s = \mathcal{F}$ for all $s \in \mathrm{Seq}$ and $\lambda > \omega$, then Seq is a $P_\lambda$-subset of $\beta\,\mathrm{Seq}$ if and only if $\mathcal{F}$ is a $P_\lambda$-ultrafilter and $\lambda \leq \mathfrak{b}$, where $\mathfrak{b}$ is the minimal cardinality of an unbounded subset of ${}^\omega\omega$ ordered by the relation $\leq^*$. A set $A \subseteq X$ is a $P_\lambda$–subset of $X$ whenever $A$ is contained in the interior of the intersection of every family $\mathcal{R}$ of open sets such that $A \subseteq \bigcap \mathcal{R}$ and $|\mathcal{R}| < \lambda$. Recently A. Brzeska [8] proved the following extension of these results:

**Theorem 3.7** (Brzeska)**.** *If $\lambda > \omega$, $T = \mathrm{Seq}(\omega)$ and $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ is such that for some filter (not necessary ultrafilter) $\mathcal{F}$ there is $\mathcal{F}_t = \mathcal{F}$ for $t \in T$, then the $\mathfrak{F}$–tree $T$ is a $P_\lambda$–set in $\beta T$ iff $\mathcal{F}$ is a $P_\lambda$–filter and $\lambda < \mathfrak{b}$.*

It is not difficult to check that if the collection $\mathfrak{F} = (\mathcal{F}_s \colon s \in \mathrm{Seq})$ consists of weak P–ultrafilters and $F \subseteq \beta\,\mathrm{Seq} \setminus \mathrm{Seq}$ is a countable set, then for every $s \in \mathrm{Seq}$ there exists $A_s \in \mathcal{F}_s$ such that $F \cap \mathrm{cl}\, A_s = \emptyset$. Hence, from the Theorem 3.6 we obtain the following corollary:

**Corollary 3.8.** *Assume $T = \mathrm{Seq}$ is an $\mathfrak{F}$–tree where $\mathfrak{F} = (\mathcal{F}_s \colon s \in \mathrm{Seq})$ is a collection of pairwise incomparable weak P–ultrafilters. If $F \subseteq \beta T \setminus T$ is countable, then $\mathrm{cl}\, F \subseteq \beta T \setminus T$.*

As a consequence of the Corollary 3.8 and the Theorem 3.4 we obtain the following:

**Theorem 3.9.** *Assume* $T = \mathrm{Seq}$ *is an* $\mathfrak{F}$*–tree, where* $\mathfrak{F} = (\mathcal{F}_s \colon s \in \mathrm{Seq})$ *is a collection of pairwise incomparable weak P–ultrafilters. If* $f \colon \beta T \to \beta T$ *is an open surjection, then* $f$ *is the identity.*

*Proof.* Since $f[T]$ is countable, by the Corollary 3.8 we have

$$F = \mathrm{cl}(f[T] \cap (\beta T \setminus T)) \subseteq \beta T \setminus T.$$

Hence $F$ is closed and nowhere dense. Then the set $U = T \cap f^{-1}[\beta T \setminus F]$ is an open and dense subset of $T$. Moreover,

$$f[U] = f[T \cap f^{-1}[\beta T \setminus F]] = f[T] \cap (\beta T \setminus F)$$

is an open subset of T because $f[T] \cap (\beta T \setminus T) \subseteq F$. Therefore $f \restriction U$ is an open mapping of an open subset of $T$ onto an open subset of $T$. Then, by the Theorem 3.4, $f$ is the identity on $U$. Therefore, $f$ is an identity on $\beta T$ since $U$ is open in $\beta T$. □

Theorem 3.4 implies that if $\mathfrak{F} = (\mathcal{U}_s \colon s \in \mathrm{Seq})$ is a collection of pairwise incomparable ultrafilters, then the $\mathfrak{F}$–tree Seq is rigid. The Stone–Čech compactification of a rigid space need not be rigid. In fact, homeomorphism of $\beta T$ onto itself need not be an extension of a homeomorphism of the space $T$ onto itself. Hence, the Theorem 3.4 cannot be used in a direct way to a construction of a rigid extremally disconnected compact space. However, one can prove the following extension of the Theorem 3.5; see also [3].

**Theorem 3.10.** *Assume* $T = \mathrm{Seq}$ *is an* $\mathfrak{F}$*–tree where* $\mathfrak{F} = (\mathcal{F}_s \colon s \in \mathrm{Seq})$ *is a collection of pairwise incomparable weak P–ultrafilters. Then for every continuous injection* $f \colon \beta T \to \beta T$ *there exists a clopen set* $U \subseteq \beta T$ *such that* $f \restriction U$ *is the identity and* $f[\beta T \setminus U]$ *is a nowhere dense subset of* $\beta T$*. In particular, if* $f$ *is a homeomorphism of* $\beta T$ *onto itself, then it is the identity.*

*Proof.* Let us denote $U = \mathrm{cl}\{s \in T \colon f(s) = s\} \subseteq \beta T$ and $T^* = \beta T \setminus T$. The Frolík's Theorem [15] says that for every injection of an extremally disconnected compact space into itself the set of all fixed points is clopen (see also S. Koppelberg [18]). Hence, by the Theorem 1.10, $U$ is a clopen subset of $\beta T$ and, $f(t) \neq t$ for all $t \in T \setminus U$. It remains to show that $\mathrm{Int}\, f[\beta T \setminus U] = \emptyset$.

Suppose $\emptyset \neq W \subseteq \beta T$ is clopen and $W \subseteq f[\beta T \setminus U]$. Then the set $V = f^{-1}[W]$ is a clopen subset of $\beta T$ and $f \restriction V \to W$ is a homeomorphism. Since $T$ is countable and $f$ is one–to–one, by the Corollary 3.8, the sets

$$F = \mathrm{cl}(f[T] \cap T^*) \text{ and } G = \mathrm{cl}(f^{-1}[T] \cap T^*)$$

are nowhere dense because they are closed and contained in $T^*$.

Let us consider open sets

$$V' = V \setminus (G \cup f^{-1}[F]) \text{ and } W' = W \setminus (f[G] \cup F).$$

We shall show that

$$(*) \qquad f[V' \cap T] = W' \cap T.$$

Since $f$ is one–to–one, we have $f[V'] = W'$. Then the inclusion "$\subseteq$" follows from the equality $f[V' \cap T] = W' \cap f[T]$. In fact, $W' \cap f[T] \cap T^* = \emptyset$ because $f[T] \cap T^* \subseteq F$ and $W' \cap F = \emptyset$. Remaining inclusion follows from the fact that $W' \cap T \subseteq f[T]$. In fact, it suffices to show that $W' \cap T \cap f[T^*] = \emptyset$. We have $W' \cap f[G] = \emptyset$ and $T \cap f[T^*] \subseteq f[G]$ because $f^{-1}[T] \cap T^* \subseteq G$.

Now, by the condition $(*)$ and the Theorem 3.4, $V' \cap T = W' \cap T$ and $f \restriction V' \cap T$ is the identity. We get a contradiction since $V' \cap U = \emptyset$ and $U = \text{cl}\{s \in T \colon f(s) = s\}$. □

One can happen that the set $U$ in the Theorem 3.10 is empty or equals $\beta T$. In fact, the Balcar– Franěk Theorem asserts that every complete infinite Boolean algebra $\mathbb{B}$ contains a free Boolean algebra of cardinality $|\mathbb{B}|$; see B. Balcar and F. Franěk [2] and also S. Koppelberg [18]. It follows that every extremally disconnected compact space $X$ is homeomorphic to a closed nowhere dense subspace of $X$; see [3]. In that case the set $U$ is empty.

In [6] there is presented yet another application of the Theorem 3.6. Let us recall that a continuous mapping $f$ is semi–open if the image under $f$ of any nonempty open set has non-empty interior. There was proved that if each ultrafilter $\mathcal{F}_s$, for $s \in T = \text{Seq}$ is a nowhere dense ultrafilter, then the Stone-Čech compactification of the $\mathfrak{F}$–tree $T$ has no semi–open mappings onto any compact metric space; see [6] and M. R. Burke [9] for a similar result. An ultrafilter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called nowhere dense whenever for every function $f \colon \omega \to {}^{\omega}2$ there exists a set $A \in \mathcal{F}$ such that $f[A]$ is a nowhere dense subset of the Cantor cube. Each P–point is a nowhere dense ultrafilter but not vice versa. The existence of nowhere dense ultrafilters is not provable in ZFC; see S. Shelah [26]. In the common paper of S. Shelah and the author there is proved that a nowhere dense ultrafilter exists iff there exists a complete, atomless $\sigma$–centered Boolean algebra which does not contain any regular, atomless countable subalgebra, see [5]. In the topological language the result reads as follows: a nowhere dense ultrafilter exists iff there exists a dense in itself separable extremally disconnected compact space that has no semi–open mappings onto any dense in itself compact metric space.

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