TOPOLOGY PROCEEDINGS
Volume 41, 2013
Pages 271–310
http://topology.auburn.edu/tp/

Dynamics of Discontinuous Maps Via Closed Relations

by

Ethan Akin

Electronically published on October 3, 2012

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 41 (2013)
Pages 271-310

E-Published on October 3, 2012

DYNAMICS OF DISCONTINUOUS MAPS VIA CLOSED RELATIONS

ETHAN AKIN

Abstract. For the dynamics of a discontinuous map on a compact metric space, we describe an approach using suitable closed relations and connect it with the continuous dynamics on an invariant G_δ subset and with the continuous dynamics on the compact space of sample paths.

1. Introduction

There has been some interest in extending the theory of dynamical systems to discontinuous maps with special focus on quasi-continuous maps. See, for example, Crannell and Martelli (2000). A subset A of a metric space X is called *quasi-open* when $A \subset \overline{A^\circ}$. That is, A is an open set together with part of its topological boundary. A map $f : X_1 \to X_2$ between metric spaces is call *quasi-continuous* when the pre-image of every open subset of X_2 is at least quasi-open.

As might be expected this leads to some oddities which, in my opinion, are best handled by using the existing theory as extended to closed relations, e.g. as in Akin (1993), but with a bit of trimming. Let me illustrate with an example of maps on $I = [0, 1]$.

$$(1.1)\qquad \begin{aligned} f_0(x) &= \begin{cases} \frac{1}{2} - x & 0 \leq x \leq \frac{1}{2}, \\ \frac{3}{2} - x & \frac{1}{2} < x \leq 1. \end{cases} \\ f_1(x) &= \begin{cases} \frac{1}{2} - x & 0 \leq x < \frac{1}{2}, \\ \frac{3}{2} - x & \frac{1}{2} \leq x \leq 1. \end{cases} \end{aligned}$$

2010 *Mathematics Subject Classification.* 54H20, 54C60, 54C08, 37B99.

Key words and phrases. Discontinuous Map, Quasi-Continuous Map, Almost One-to-One Map, Irreducible Map, Closed Relation, Almost Open Map, Suitable Relation.

These maps describe the flipping of each of the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ about its mid-point. The maps disagree only in the choice of destination for the unique point of discontinuity $x = \frac{1}{2}$.

Iterating we obtain

$$(1.2)\qquad \begin{aligned} f_0 \circ f_0(x) \quad &= \quad f_0 \circ f_1(x) \quad = \quad \begin{cases} x & 0 \le x < 1, \\ 0 & x = 1. \end{cases} \\ f_1 \circ f_1(x) \quad &= \quad f_1 \circ f_0(x) \quad = \quad \begin{cases} 1 & x = 0, \\ x & 0 < x \le 1. \end{cases} \end{aligned}$$

Instead, define

$$(1.3)\qquad F_{01} \quad = \quad f_0 \cup f_1 \quad = \quad \overline{f_0} \quad = \quad \overline{f_1}.$$

That is, F_{01} is the common closure of f_0 and f_1, regarded as subsets of $I \times I$. If we iterate the closed relation F_{01} twice we obtain the identity map on all of I together with the two extra points $(0, 1)$ and $(1, 0)$. That is,

$$(1.4)\qquad F_{01} \circ F_{01} \quad = \quad 1_I \ \cup \ \{(0, 1), (1, 0)\}.$$

In my opinion the choice between f_0 and f_1 should be irrelevant to the dynamics, and the way to handle this is to use the closed relation F_{01} instead. Next, the anomalous points should be discarded from our description of the dynamics of the system. There are three related ways to look at this approach and the purpose of this paper is to describe and relate them.

A closed relation $F : X_1 \to X_2$ is a closed subset of the product $X_1 \times X_2$. We regard a continuous map as a special case of a closed relation, identifying the map with what is sometimes called the graph of the map. Our spaces are compact metric spaces so that the composition of closed relations is a closed relation. Thus, when F is a closed relation on X, i.e. $X_1 = X_2 = X$, we can iterate defining inductively $F^{n+1} = F^n \circ F = F \circ F^n$.

If $f : X_1 \to X_2$ is a continuous surjection of compact metric spaces then a closed subset $A \subset X_1$ is called *minimal* for f when it is minimal in the class of closed subsets of X_1 which are mapped onto X_2 by f. Equivalently, $f(A) = X_2$ and if B is a closed proper subset of A then $f(B)$ is a proper subset of X_2. When X_1 itself is minimal for f then the map is called *irreducible.* This is equivalent to the condition that f be an *almost one-to-one* map, i.e. the set

$$(1.5)\qquad IN_f \quad = \quad \{x \in X_1 : \ f^{-1}(f(x)) \ = \ \{x\} \ \}$$

is dense in X_1.

A continuous map $f : X_1 \to X_2$ is called *almost open* when for $A \subset X_1$, $A^\circ \neq \emptyset$ implies $f(A)^\circ \neq \emptyset$ where A° is the interior of A.

If $F : X_1 \to X_2$ is a closed relation then by restricting the coordinate projections to F we obtain the continuous maps $\pi_{1F} : F \to X_1$ and $\pi_{2F} : F \to X_2$. We will call F a *suitable relation* when it satisfies two conditions:

- π_{1F} is irreducible.
- π_{2F} is almost open.

The first condition says that F is a minimal subset for the projection $\pi_1 : X_1 \times X_2 \to X_1$. It says exactly that F is the closure in $X_1 \times X_2$ of a continuous map defined on a dense subset of X_1. The second condition says that for $A \subset X_1$, $A^\circ \neq \emptyset$ implies $F(A)^\circ \neq \emptyset$.

With $F : X_1 \to X_2$ and $G : X_2 \to X_3$ suitable, it need not be true that the composition $G \circ F : X_1 \to X_3$ is suitable. In our example above, F_{01} is a suitable relation but $F_{01} \circ F_{01}$ is not suitable.

In general, if F and G are suitable then $G \circ F$ contains a unique subset minimal for π_1. We denote this by $G \bullet F : X_1 \to X_3$. It clearly satisfies the first condition and in fact satisfies the second as well and so is a suitable relation. That is, $G \circ F$ contains a unique suitable relation from X_1 to X_3 and we call this the *suitable composition.*

The uniqueness requirement is why we impose the condition that π_{2F} be almost open. Without it an open set might be crushed to point and then under composition yield an open set of anomalous points. Suppose we define $\tilde{F}_{01}$ on $[-1.1]$ by extending the definition of F_{01} by letting $x \mapsto \frac{1}{2}$ for all $x \in [-1, 0]$. Then

$$\tilde{F}_{01} \circ \tilde{F}_{01} \quad = \quad 1_I \ \cup \ [-1,0] \times \{0,1\} \ \cup \ \{(1,0)\}. \tag{1.6}$$

This contains many minimal subsets. If U is any open subset of $[-1, 0]$ then $1_I \ \cup \ \overline{U} \times \{0\} \ \cup \ ([-1,0] \setminus U) \times \{1\}$ is a minimal subset for the projection π_1.

Suitable composition is associative and so we obtain a category with objects compact metric spaces and morphisms suitable relations with suitable composition. A suitable relation $F : X_1 \to X_2$ is an isomorphism in this category exactly when π_{2F} is an irreducible map. In particular, an irreducible map is an isomorphism, and any isomorphism is a composition of irreducible maps and the inverses of irreducible maps.

The relationship between quasi-continuous maps and closed relations has been described in an elegant paper by Crannell, Frantz and LeMasurier (2006). From their results we see that:

Theorem 1.1. *Let X_1 and X_2 be compact metric spaces.*

(a) *If $g : X_1 \to X_2$ is a quasi-continuous function and F is the closure of g in $X_1 \times X_2$ then the closed relation F is a minimal set for the projection $\pi_1 : X_1 \times X_2 \to X_1$, i.e. π_{1F} is irreducible. Furthermore, g is continuous at the points of a dense subset of X_1.*

(b) *If $F \subset X_1 \times X_2$ is a closed relation with $\pi_{1F} : F \to X_1$ irreducible and $g : X_1 \to X_2$ is a map with $g \subset F$, i.e.*

$$g(x) \in F(x) \quad \text{for all } x \in X_1, \tag{1.7}$$

then g is quasi-continuous and F is the closure of g.

Remark: A map $g : X_1 \to X_2$ which satisfies (1.7) is called a *selection function* for F.

Proof. Clearly if D is a dense subset of X_1 and a map $g_0 : D \to X_2$ is contained in a closed relation F then $\overline{g_0}$ is a subset of F which is mapped onto X_1 by π_1. If F is minimal for π_1 then $F = \overline{g_0}$. In particular, if F is minimal then it is the closure of any of its selection functions. Corollary 5 of Crannell, Frantz and LeMasurier (2006) says that these functions are quasi-continuous. This proves (b).

To say that a point $(x, y) \in F$ lies in $IN_{\pi_{1F}}$ says exactly that $F(x)$ is the singleton $\{y\}$. In particular, $g(x) = y$ for any selection function g. If π_{1F} is irreducible then $IN_{\pi 1F}$ is dense in F and so its projection D is dense in X_1. As we will show below (and it is easy to check) each selection function g is continuous at each point of this dense set D.

Now suppose that $g : X_1 \to X_2$ is quasi-continuous and that F is its closure. Let F_1 be a closed subset of F which projects onto X_1 and let h be a selection function for F_1. Clearly, $\overline{h} \subset F_1 \subset F$. By Theorem 2 (ii) of Crannell, Frantz and LeMasurier (2006) $\overline{h} = \overline{g}$. Hence, $F_1 = F$. Thus, F is minimal for π_1. This completes the proof of (a). ☐

Corollary 1.2. *If $f : X_2 \to X_1$ is an irreducible map then any selection function $g : X_1 \to X_2$ for the inverse relation $f^{-1} : X_1 \to X_2$ is a quasi-continuous map.*

Proof. The projection from the continuous map f to its domain X_2 is a homeomorphism. The projection of f to its range X_1 is thus homeomorphic to f itself and so is irreducible. For the inverse relation f^{-1} this means that the projection to the domain X_1 is irreducible. By the above theorem the selection functions are quasi-continuous. ☐

Closed relations are used in Crannell, Frantz and LeMasurier (2006) as a tool to study quasi-continuous functions. Instead, I suggest that we focus on the closed relations themselves and eliminate the use of quasi-continuous functions and the associated choices implicit in the use of selection functions.

Let F be a suitable relation on X, i.e. $F : X \to X$. We will see that there exists a dense, G_δ subset D^+_F of X and a continuous map $t_F : D^+_F \to D^+_F$ such that F is the closure in $X \times X$ of the map t_F. The three descriptions which I want to connect can be labeled

- Closing up the continuous map dynamics.
- Suitable composition dynamics.
- Sample path dynamics.

Closing up the continuous map dynamics: Since D^+_F is a dense G_δ subset of a compact metric space it is a Polish space, i.e., a space that admits a complete, separable metric. Quite a bit of the usual dynamical systems theory applies to t_F on D^+_F itself. On the other hand, without compactness there are undesirable holes in the theory. For example, orbit sequences might have empty limit point sets. Since D^+_F is sitting in the compact space X we can use the limit points which occur in X. In general, we focus on the dynamics of t_F on D^+_F and use the compactness of X to define various limit point sets.
Suitable composition dynamics: We define the iterates $F^{\bullet n+1} = F^{\bullet n} \bullet F = F \bullet F^{\bullet n}$. That is, we use the relation iterates F^n and discard the points which do not lie in the unique minimal subset for $\pi_{1 F^n}$.
Sample path dynamics: Let $\mathbb{N} = \{0, 1, ...\}$. For a closed relation F on X we define the sample path space X^+_F by

$$X^+_F \quad = \quad \{ \ z \in X^{\mathbb{N}} : (z_i, z_{i+1}) \in F \quad \text{for all} \ \ i \in \mathbb{N} \ \}. \tag{1.8}$$

This is a closed, subset of $X^{\mathbb{N}}$ which is +invariant for the shift map σ on $X^{\mathbb{N}}$ defined by

$$\sigma(z)_i \quad = \quad z_{i+1} \qquad \text{for all} \ \ i \in \mathbb{N}. \tag{1.9}$$

The map $\pi_0 \times \pi_n$ maps X^+_F onto the n-fold iterate F^n. Equivalently, π_0 maps the restriction of the map $\sigma^n | X^+_F$ onto the n-fold relation composite F^n.

When F is a suitable relation, there is a unique subset of X^+_F which is minimal for π_0. We label it S^+_F. Thus, the restriction $\pi_0 : S^+_F \to X$ is irreducible. This subset is +invariant for the shift and we denote by $s_F : S^+_F \to S^+_F$ the restriction of the shift map. It is an almost open, continuous map.

We will show that that three approaches are related in that the closure in $X \times X$ of the n-fold iterate $(t_F)^n \subset D_F^+ \times D_F^+$ is exactly $F^{\bullet n}$. Furthermore, the map $\pi_0 \times \pi_n$ maps S_F^+ onto $F^{\bullet n}$. Equivalently, π_0 maps $(s_F)^n$ on S_F^+ onto $F^{\bullet n}$.

When F is a suitable relations isomorphism, there is an irreducible lift to a homeomorphism.

Finally, recall that all measure spaces associated with nonatomic measures are isomorphic to the unit interval with Lebesgue measure. Similarly, every compact metric space without isolated points is isomorphic in the suitable relations category to the Cantor Set. Furthermore any suitable relation on such a space can be lifted, via an irreducible map to an almost open, continuous map on the Cantor Set and a suitable relations isomorphism can be similarly lifted to a homeomorphism on the Cantor Set.

2. Almost Open Maps and Closed Relations

All of our spaces are nonempty separable metric spaces. For a subset A we will use the usual notation $\overline{A}$ and A° for the closure and interior, respectively. Our main focus is upon compact metric spaces. We will reserve the letters X, Y with or without subscripts for such spaces and use other letters for more general metric spaces.

A space is Polish when it is separable and admits a complete metric. For example, a compact metric space is Polish.

In this section we review some foundational results about almost open maps and about closed relations between compacta.

Definition 2.1. Let $f : E_1 \to E_2$ be a continuous map.

- The map f is *open at* $x \in E_1$ if $x \in U$ implies $f(x) \in f(U)^\circ$ for all open $U \subset E_1$.
- The map f is *open* if $f(U)$ is open for all open $U \subset E_1$.
- The map f is *almost open* if $f(U)^\circ \neq \emptyset$ for all nonempty open $U \subset E_1$.
- The map f is *weakly open* if $(\overline{f(U)})^\circ \neq \emptyset$ for all nonempty open $U \subset E_1$.

Clearly, open implies almost open and almost open implies weakly open. We label by $OPEN_f$ the set of points of E_1 at which f is open. For an open $U \subset E_1$ we define the open $U_f \subset U$ by

$$U_f \quad =_{def} \quad U \cap f^{-1}[(f(U)^\circ]. \tag{2.1}$$

That is, U_f is the set of points of $x \in U$ such that $f(U)$ is a neighborhood of $f(x)$.

Proposition 2.2. *Let $f : E_1 \to E_2$ be a continuous map.*

(a) *The map f is open at x iff $x \in U_f$ for every open $U \subset E_1$ with $x \in U$.*
(b) *The map f is open iff $U = U_f$ for every open $U \subset E_1$.*
(c) *The map f is open iff it is open at every point of E_1, i.e. $OPEN_f = E_1$.*
(d) *The map f is almost open iff U_f is dense U for every open $U \subset E_1$.*
(e) *The map f is almost open if it is open at a dense set of points, i.e. if $OPEN_f$ is dense in E_1.*

Proof. Results (a), (b) and (c) are obvious. It is also clear that f is almost open iff $U_f \neq \emptyset$ whenever U is open and nonempty. This, together with (a) implies (e). Density of U_f in U certainly implies that U_f is nonempty when U is.

We are left with showing that U_f is dense in U whenever f is almost open. Let V be an arbitrary nonempty open subset of U. Because f is almost open, $V_f \subset U_f \cap V$ is nonempty. Hence, there exist points of U_f which lie in V. □

Proposition 2.3. *Let $f : E_1 \to E_2$ be a continuous map.*

The map f is almost open iff D dense in E_2 implies $f^{-1}(D)$ is dense in E_1.

The map f is weakly open iff D open and dense in E_2 implies $f^{-1}(D)$ is open and dense in E_1.

If E_1 is a Polish space and f is weakly open then D a dense G_δ subset of E_2 implies $f^{-1}(D)$ is a dense G_δ subset of E_1.

Proof. If f is not almost open then there exists a nonempty open U with $f(U)^\circ$ empty and so the complement D of $f(U)$ in E_2 is dense with $f^{-1}(D)$ disjoint from U. If f is not weakly open then there exists U with $(\ \overline{f(U)}\ )^\circ$ empty and so the complement D of $\overline{f(U)}$ is open and dense with $f^{-1}(D)$ disjoint from U.

For the converses, let D be a dense subset of E_2 and U be an arbitrary nonempty open subset of E_1. If f is almost open then $f(U)$ has a nonempty interior and so meets D. Hence, U meets $f^{-1}(D)$. If f is weakly open then $\overline{f(U)}$ has a nonempty interior and so meets D. If D is also open then $f(U)$ meets D and so, as before, U meets $f^{-1}(D)$.

If D is a countable intersection of a sequence $\{V_n\}$ of open subsets of E_2 then each V_n is a dense open set and so when f is weakly open each $f^{-1}(V_n)$ is a dense open subset of E_2. Thus, $f^{-1}(D) = \bigcap_n f^{-1}(V_n)$ is a G_δ which is dense by the Baire Category Theorem when E_1 is Polish. □

Corollary 2.4. *Let $f : E_1 \to E_2$ and $g : E_2 \to X_3$ be continuous maps.*

If both f and g are almost open (or weakly open) then $g \circ f$ is almost open (resp. weakly open).

If f is surjective and $g \circ f$ is almost open (or weakly open) then g is almost open (resp. weakly open).

Proof. Let D be dense in X_3. If f and g are almost open then $g^{-1}(D)$ is dense in E_2 and so $f^{-1}(g^{-1}(D)) = (g \circ f)^{-1}(D)$ is dense in E_1. So $g \circ f$ is almost open.

If $g \circ f$ is almost open then $f^{-1}(g^{-1}(D)) = (g \circ f)^{-1}(D)$ is dense in E_1 and since f is surjective $g^{-1}(D) = f(f^{-1}(g^{-1}(D)))$ is dense in E_2. Hence g is almost open.

For weakly open the proof is the same with dense replaced by open and dense throughout. □

Theorem 2.5. *Let $f : E_1 \to E_2$ be a continuous map with E_1 locally compact.*

The map f is almost open iff it is weakly open.

Proof. If f is weakly open, U is open and $x \in U$ then there exists a compact $K \subset U$ with $x \in K^\circ$. Since $f(K)$ is compact, it is closed. Because f is weakly open $[\overline{f(K^\circ)}]^\circ$ is nonempty. It is contained in $f(K)^\circ \subset f(U)^\circ$. As U was arbitrary, it follows that f is almost open. □

While our focus will be on almost open maps, the weaker notion is a useful tool because of the following result which we will use often and refer to as the *Variation of Domain Theorem.*

Theorem 2.6. *Let $f : E_1 \to E_2$ be a continuous map and for $i = 1, 2$ let D_i be a dense subset of E_i. Assume that $f(D_1) \subset D_2$ so that we can define the restricted map $f : D_1 \to D_2$.*

The map $f : E_1 \to E_2$ is weakly open iff the restriction $f : D_1 \to D_2$ is weakly open.

The proof requires the following:

Lemma 2.7. *Let D be a dense subset of E.*

(a) *If $U \subset E$ is open then $U \cap D$ is dense in U.*
(b) *If $A \subset D$ then $\overline{A} \cap D$ is the D-closure of A.*
(c) *If $B \subset E$ is closed then $B^\circ \cap D$ is the D-interior of $B \cap D$.*

Proof. (a) If $V \subset U$ is an arbitrary nonempty open subset then V meets the dense set D.

(b) $\overline{A} \cap D$ is D-closed and contains A and so contains the D-closure.

On the other hand, there is a closed set F such that $F \cap D$ is the D-closure of A. In particular, F contains A and so since F is closed it contains $\overline{A}$. Hence the D-closure of A which is $F \cap D$ contains $\overline{A} \cap D$.

(c) $B^\circ \cap D$ is D-open and is contained in $B \cap D$ and so is contained in the D-interior of $B \cap D$. On the other hand, there exists an open set U such that $U \cap D$ is the D-interior of $B \cap D$. Since $U \cap D \subset B$ and B is closed, $\overline{U \cap D} \subset B$. By (a) $U \cap D$ is dense in U and so $\overline{U} = \overline{U \cap D} \subset B$. Hence, the open set U is contained in B and so in B°. Hence the D-interior of $B \cap D$ which is $U \cap D$ is contained in $B^\circ \cap D$. □

Proof of Theorem 2.6: The open subsets of D_1 are of the form $U \cap D_1$. By Lemma 2.7(a) $U \cap D_1$ is dense in U and so $f(U \cap D_1)$ is dense in $f(U)$. Thus,

$$\overline{f(U)} \quad = \quad \overline{f(U \cap D_1)}. \tag{2.2}$$

Since $f(U \cap D_1) \subset D_2$, Lemma 2.7(b) implies that $\overline{f(U)} \cap D_2 = \overline{f(U \cap D_1)} \cap D_2$ is the D_2-closure of $f(U \cap D_1)$.

Finally Lemma 2.7 (c) implies that $[\overline{f(U)}]^\circ \cap D_2$ is the D_2-interior of $\overline{f(U \cap D_1)} \cap D_2$ which we have seen is the D_2-closure of $f(U \cap D_1)$.

Thus, $[\overline{f(U)}]^\circ \cap D_2$ is the D_2-interior of the D_2-closure of $f(U \cap D_1)$.

By Lemma 2.7(a) again $[\overline{f(U)}]^\circ \cap D_2$ is dense in $[\overline{f(U)}]^\circ$.

It follows that $[\overline{f(U)}]^\circ$ is nonempty iff the D_2-interior of the D_2 closure of $f(U \cap D_1)$ is nonempty. As U varies over all the nonempty open subsets of E_1, $U \cap D_1$ varies over all the nonempty D_1-open subsets of D_1. Thus, f is weakly open iff the restriction $f : D_1 \to D_2$ is weakly open. □

Remark: Note that the identity map on the unit interval I is open, but the restriction with D_1 the rationals and $D_2 = I$ is not almost open.

Proposition 2.8. *Let $f : E_1 \to E_2$ be a continuous map with E_1 a Polish space.*

The set $OPEN_f$, the set of points of E_1 at which f is open, is a G_δ subset of E_1. The map f is almost open iff $OPEN_f$ is dense in E_1.

Proof. Let $\mathcal{A}$ be the set of pairs (U_1, U_2) members of a countable basis with $\overline{U_2} \subset U_1$ so that $\{U_1, E_1 \setminus \overline{U_2}\}$ is an open cover of E_1. I claim that

$$OPEN_f \quad = \quad \bigcap_{(U_1,U_2) \in \mathcal{A}} (U_1)_f \cup (E_1 \setminus \overline{U_2})_f. \tag{2.3}$$

By Proposition 2.2(a) x is in the intersection when f is open at x. On the other hand, suppose that x is in the above intersection and U is a open set with $x \in U$. We can choose $(U_1, U_2) \in \mathcal{A}$ such that $x \in U_2$ and $U_1 \subset U$. Since x is in the intersection but not in $E_1 \setminus \overline{U_2}$ it follows that $x \in (U_1)_f \subset U_f$. As U was arbitrary, f is open at x.

Now if f is almost open then by Proposition 2.2(d) each of the open sets $(U_1)_f \cup (E_1 \setminus \overline{U_2})_f$ is dense in $(U_1) \cup (E_1 \setminus \overline{U_2}) = E_1$. As the countable intersection of dense open sets, $OPEN_f$ is dense by the Baire Category Theorem applied to the Polish space E_1.

The converse follows from Proposition 2.2(e). $\square$

We now review the theory of relations following Akin (1993).

As do the set theorists, we regard a map $f : E_1 \to E_2$ as a subset of $E_1 \times E_2$. For example, the identity map 1_E is the diagonal set $\{(x, x) : x \in E\}$. A relation $F : E_1 \to E_2$ is an arbitrary subset of $E_1 \times E_2$. For $x \in E_1$ and $A \subset E_1$ we let

$$\begin{aligned} F(x) &= \{y \in E_2 : (x, y) \in F\} \\ F(A) &= \{y : (x, y) \in F \quad \text{for some } x \in A\} \\ &= \bigcup_{a \in A} F(a) \quad = \quad \pi_1(F \cap (A \times X_2)), \end{aligned} \tag{2.4}$$

where $\pi_1 : X_1 \times X_2 \to X_1$ is the projection to the first coordinate.

Thus, F is a map exactly when each $F(x)$ is a singleton. In that case, we use $F(x)$ to stand both for the singleton set and for the point it contains, allowing context to determine the choice of meaning.

The *inverse relation* $F^{-1} : E_2 \to E_1$ is defined by

$$F^{-1} \quad = \quad \{(y, x) : (x, y) \in F\}. \tag{2.5}$$

Thus, for $B \subset E_2$ we see that

$$F^{-1}(B) \quad = \quad \{x \in E_1 : F(x) \cap B \neq \emptyset\}. \tag{2.6}$$

We call

$$F^{-1}(E_2) \quad = \quad \{x \in E_1 : F(x) \neq \emptyset\} \tag{2.7}$$

the *domain* of F. When $F^{-1}(E_2) = E_1$ we will say that F *has full domain* or call F a *full domain relation.*

For $A_1 \subset E_1$ and $A_2 \subset E_2$ the *restriction* is $F \cap (A_1 \times A_2)$ regarded as a relation from A_1 to A_2. Thus, we can always restrict to obtain a full domain relation.

There is an alternative version of the pre-inverse of B which we will label $F^*(B)$.

$$\begin{aligned} F^*(B) &= \{x \in E_1 : F(x) \subset B\} \\ &= E_1 \setminus F^{-1}(E_2 \setminus B). \end{aligned} \tag{2.8}$$

Both operators are monotone with respect to set inclusion. While $F^{-1}(\emptyset) = \emptyset$, $F^*(\emptyset)$ is the complement of the domain of F, i.e. the set of x such that $F(x) = \emptyset$. Clearly,

$$F^*(B) \quad \subset \quad F^{-1}(B) \cup F^*(\emptyset). \tag{2.9}$$

So $F^*(B) \subset F^{-1}(B)$ for all B exactly when F is a full domain relation.

If $\{B_\alpha\}$ is a family of subsets of E_2 then

$$\begin{aligned} F^{-1}(\bigcup_\alpha B_\alpha) &= \bigcup_\alpha F^{-1}(B_\alpha), \\ F^*(\bigcap_\alpha B_\alpha) &= \bigcap_\alpha F^*(B_\alpha). \end{aligned} \tag{2.10}$$

When F is a function these two operators agree and are the usual pre-image operator.

If $F : E_1 \to E_2$ and $G : E_2 \to E_3$ then we define

$$\begin{aligned} G \otimes F \quad &=_{def} \quad (F \times E_3) \cap (E_1 \times G) \quad = \\ &\{(x, y, z) \in E_1 \times E_2 \times E_3 : (x, y) \in F \text{ and } (y, z) \in G\}, \end{aligned} \tag{2.11}$$

and the *composition* $G \circ F$

$$\begin{aligned} G \circ F \quad &=_{def} \quad \pi_{13}(G \otimes F) \quad = \\ &\{(x, z) \in E_1 \times E_3 : \text{for some } y \in E_2 \ \ (x, y) \in F \text{ and } (y, z) \in G\} \end{aligned} \tag{2.12}$$

where π_{13} is the projection to the product of the first and third coordinates.

Clearly, for $A \subset E_1$

$$(G \circ F)(A) \quad = \quad G(F(A)) \tag{2.13}$$

and

$$(G \circ F)^{-1} \quad = \quad F^{-1} \circ G^{-1}. \tag{2.14}$$

Hence, for $B \subset E_3$

$$\begin{aligned} (G \circ F)^{-1}(B) \quad &= \quad F^{-1}(G^{-1}(B)), \\ (G \circ F)^*(B) \quad &= \quad F^*(G^*(B)). \end{aligned} \tag{2.15}$$

Composition of relations automatically takes care of domain problems. Suppose $D_1 \subset E_1$, $D_2 \subset E_2$ on which are defined maps $f : D_1 \to E_2$ and $g : D_2 \to E_3$. We can restrict f to $f^{-1}(D_2) \subset D_1$ to get a map $f : f^{-1}(D_2) \to D_2$ which we can compose with g to get a map $g \circ f : f^{-1}(D_2) \to E_3$. Instead we can regard $f : E_1 \to E_2$ and $g : E_2 \to E_3$ as relations with $f^{-1}(E_2) = D_1$ and $g^{-1}(E_3) = D_2$. The composed relation $g \circ f : E_1 \to E_3$ has $(g \circ f)^{-1}(E_3) = f^{-1}(D_2)$ and is just the composed function on that set.

Since a relation $F : E_1 \to E_2$ is a subset of the product, we can define the maps $\pi_{1F} : F \to E_1$ and $\pi_{2F} : F \to E_2$ to be the restrictions of the projection maps. Notice that F is a map iff π_{1F} is a bijection. In general, we have:

$$F \quad = \quad \pi_{2F} \circ (\pi_{1F})^{-1}. \tag{2.16}$$

For $A \times B \subset E_1 \times E_2$ we see that

$$\pi_{1F}((A\times B)\cap F) = A\cap F^{-1}(B) \text{ and } \pi_{2F}((A\times B)\cap F) = F(A)\cap B. \tag{2.17}$$

and so we have

$$F(A)\cap B \neq \emptyset \iff A\cap F^{-1}(B) \neq \emptyset \iff (A\times B)\cap F \neq \emptyset. \tag{2.18}$$

Furthermore,

$$F(A) \subset B \iff A \subset F^*(B) \iff \pi_{1F}^{-1}(A) \subset A \times B, \tag{2.19}$$

because each of these says

$$x \in A \text{ and } (x,y) \in F \implies y \in B. \tag{2.20}$$

Now we introduce topology.

Proposition 2.9. *(a) If $f : E_1 \to E_2$ is a continuous map then the relation f is a closed subset of $E_1 \times E_2$ and $\pi_{1f} : f \to E_1$ is a homeomorphism.*

(b) If $f : X_1 \to X_2$ is a map between compact metric spaces which is a closed subset of $X_1 \times X_2$ then it is a continuous map.

Proof. (a) If $(x,y) \notin f$ then there exist disjoint open $U, V \subset E_2$ with $f(x) \in U$ and $y \in V$. It follows that $f^{-1}(U) \times V$ is an open subset of $E_1 \times E_2$ which contains (x,y) and is disjoint from f. As the complement of f is open, f is closed. Clearly, π_{1f} is continuous and the continuous inverse is given by $x \mapsto (x, f(x))$ for all $x \in E_1$.

(b) Since X_1 and X_2 are compact the closed $f \subset X_1 \times X_2$ is also a compact set. Hence, the continuous bijection $\pi_{1f} : f \to X_1$ is a homeomorphism. So $f = \pi_{2f} \circ (\pi_{1f})^{-1}$ is a continuous map. □

For compact metric spaces X_1, X_2 a closed (or open) relation $F : X_1 \to X_2$ is a relation which is a closed (resp. open) subset of $X_1 \times X_2$. For example, if X is a compact metric space and $\epsilon \geq 0$ let

$$\begin{aligned} V_\epsilon &= \{(x,y) : d(x,y) < \epsilon \} \\ \bar{V}_\epsilon &= \{(x,y) : d(x,y) \leq \epsilon \} . \end{aligned} \tag{2.21}$$

Each V_ϵ is open and each $\bar{V}_\epsilon$ is closed. If $A \subset X$ then $V_\epsilon(A)$ is the ϵ neighborhood of A, i.e. the set of points each of which has distance less than ϵ from some point of A.

Proposition 2.10. *Let X_1, X_2, X_3 be compact metric spaces. Let $F : X_1 \to X_2$ and $G : X_2 \to X_3$ be relations.*

(a) *If F and G are both closed relations (or both open relations) then the inverse F^{-1} and the composition $G \circ F$ are both closed relations (resp. both open relations).*

(b) *Let $A \subset X_1$. If F is an open relation then $F(A)$ is an open subset of X_2. If A and F are both closed then $F(A)$ is closed.*
(c) *Let $B \subset X_2$ and let F be a closed relation. If B is closed then $F^{-1}(B)$ is closed. If B is open then $F^*(B)$ is open.*
(d) *Let F be a closed relation and A be a closed subset of X_1. For every $\epsilon > 0$ there exists $\delta > 0$ such that $F(V_\delta(A)) \subset V_\epsilon(F(A))$, equivalently, $V_\delta(A) \subset F^*(V_\epsilon(F(A)))$.*
(e) *If F is a closed relation and A is a closed subset of X_1 then the restriction $F \cap (A \times X_2)$ is a closed relation from A to X_2.*

Proof. (a) The results for F^{-1} are obvious since switching coordinates is a homeomorphism from $X_1 \times X_2$ to $X_2 \times X_1$. $G \otimes F$ is closed (or open) when both F and G are closed (resp. open). The projection map π_{13} is always an open map and by compactness it is a closed map. So the results for $G \circ F$ follow from (2.12).

(b) When F is open, each $F(x)$ is open and so $F(A)$ is open by the union representation in (2.4). Notice that no conditions on A are required. When A and F are closed then $F(A)$ is closed by the projection representation in (2.4).

(c) When B is closed, $F^{-1}(B)$ is closed by (b) applied to F^{-1}. Hence, when B is open $F^*(B) = X_1 \setminus F^{-1}(X_2 \setminus B)$ is open.

(d) Since $F(A) \subset V_\epsilon(F(A))$ it follows from (2.19) that the compact set A is contained in the open set $F^*(V_\epsilon(F(A)))$ and hence there exists $\delta > 0$ such that $V_\delta(A) \subset F^*(V_\epsilon(F(A)))$. By (2.19) again it follows that $F(V_\delta(A)) \subset V_\epsilon(F(A))$.

(e) Obvious. □

If $F : X_1 \to X_2$ and $G : X_1 \to X_3$ are closed relations we define the product relation $F \triangle G : X_1 \to X_2 \times X_3$ to be

$$\{(x, y, z) : (x, y) \in F, (x, z) \in G \}.$$

This is $G \otimes F^{-1}$ with the first and second coordinates switched. Hence, $F \triangle G$ is a closed relation.

For a closed relation $F : X_1 \to X_2$ we define

$$ONE_F \quad =_{def} \quad \{x \in X_1 : F(x) \text{ is a singleton set } \}. \tag{2.22}$$

The restriction

$$f_F \quad =_{def} \quad F \cap (ONE_F \times X_2) \tag{2.23}$$

is a map from ONE_F into X_2.

Proposition 2.11. *If $F : X_1 \to X_2$ is a closed relation between compact metric spaces then ONE_F is a G_δ subset of X_1 and $f_F : ONE_F \to X_2$ is a continuous map.*

Proof. For every $\epsilon > 0$ the set $(F \triangle F)^*(V_\epsilon)$ consists of the points $x \in X_1$ such that the diameter of $F(x)$ is less than ϵ. By Proposition 2.10(c) this is an open set. Intersecting over positive rationals we obtain the G_δ set which is the disjoint union of ONE_F and $F^*(\emptyset)$. The latter is open and so its complement is closed and hence G_δ. Intersecting with this complement we obtain ONE_F as a G_δ.

Now if $x \in ONE_F$, and $\{(x_n, y_n)\}$ is a sequence in F with $\{x_n\}$ converging to x, then for every limit point y of the sequence $\{y_n\}$ the point $(x, y) \in F$ because the relation is closed. By definition of ONE_F it must be that $(x, y) = (x, f_F(x))$. Since $f_F(x)$ is the only limit point of the sequence $\{y_n\}$ and the space X_2 is compact it follows that $\{y_n\}$ converges to $f_F(x)$. In particular, f_F is continuous on ONE_F. $\square$

Now let $f : X_1 \to X_2$ be a continuous map between compact metric spaces. We say that f is *injective at* $x \in X_1$ when $f^{-1}(f(x)) = \{x\}$, i.e. $f(x_1) = f(x)$ implies $x_1 = x$. We denote by IN_f the set of points at which f is injective so that

$$IN_f \ =_{def} \ f^{-1}(ONE_{f^{-1}}) \ = \ \{x \in X_1 : f^{-1}(f(x)) = \{x\}\ \}. \tag{2.24}$$

For an open $U \subset X_1$ we define the open subset U^f of U by

$$\begin{aligned} U^f \ &= \ X_1 \setminus f^{-1}(f(X_1 \setminus U)) \ = \ f^{-1}(X_2 \setminus f(X_1 \setminus U)) \\ &= \ f^{-1}((f^{-1})^*(U)) \ = \ \{x \in X_1 : f^{-1}(f(x)) \subset U\ \}. \end{aligned} \tag{2.25}$$

Proposition 2.12. *If $f : X_1 \to X_2$ is a continuous map between compact metric spaces then IN_f is a G_δ subset of X_1 and $x \in IN_f$ iff for every open $U \subset X_1$, $x \in U$ implies $x \in U^f$.*

The continuous map f restricts to a homeomorphism $f : IN_f \to ONE_{f^{-1}}$.

Proof. For a continuous map, f^{-1} is a closed relation and so $ONE_{f^{-1}}$ is a G_δ set by Proposition 2.11. Hence, the pre-image under f is G_δ.

Clearly, $f^{-1}(f(x)) = \{x\}$ implies that $f^{-1}(f(x)) \subset U$ when $x \in U$. On the other hand, if $f(x_1) = f(x)$ with $x_1 \neq x$ then $U = X_1 \setminus \{x_1\}$ is an open set containing x but not containing $f^{-1}(f(x))$.

Now it is clear that $f(IN_f) = ONE_{f^{-1}}$. The restriction of f to IN_f is of course continuous. By Proposition 2.11 applied to the closed relation $F = f^{-1}$ the restriction of f^{-1} to $ONE_{f^{-1}}$ defines a continuous map. These are clearly inverse homeomorphisms between the G_δ sets $IN_f \subset X_1$ and $ONE_{f^{-1}} \subset X_2$. $\square$

Clearly, f is *injective* or *one-to-one* exactly when $IN_f = X_1$.

Definition 2.13. A continuous map $f : X_1 \to X_2$ between compact metric spaces is called *almost one-to-one* when IN_f is a dense subset of X_1.

We saw in Proposition 2.9(a) that a closed relation $f : X_1 \to X_2$ between compact metric spaces is a continuous map iff the projection $\pi_{1f} : f \to X_1$ is a bijection. For F to be a suitable relation we will weaken this to demand that π_{1F} be a surjection which is almost one-to-one. So we pause to sketch the properties of such maps or, equivalently in this context, irreducible maps.

Proposition 2.14. *Let $f : X_1 \to X_2$ be a continuous map between compact metric spaces. The following are equivalent:*

(i) *The map f is almost one-to-one.*
(ii) *For every nonempty open $U \subset X_1$ there exists an open $W \subset X_2$ such that $W \cap f(X_1) \neq \emptyset$ and $f^{-1}(W) \subset U$.*
(iii) *U^f is nonempty for every nonempty open $U \subset X_1$.*
(iv) *U^f is dense in U for every open $U \subset X_1$.*

Proof. (ii) $\Leftrightarrow$ (iii) Clearly, $f^{-1}(W) \subset U$ implies $f^{-1}(W) \subset U^f$. $f^{-1}(W)$ is nonempty iff W meets $f(X_1)$. Thus, U^f is nonempty if $f^{-1}(W) \subset U$ for a set W which meets $f(X_1)$. Conversely, $W = (f^{-1})^*(U)$ is an open subset of X_2 with $f^{-1}(W) = U^f \subset U$. If U^f is nonempty then W meets $f(X_1)$.

(i) $\Rightarrow$ (iii) By Proposition 2.12 $IN_f \cap U \subset U^f$ for every open set U and so if IN_f is dense, U^f is nonempty whenever U is.

(iii) $\Rightarrow$ (iv) If V is any open subset of U then $V^f \subset V \cap U^f$. By (iii) V^f is nonempty whenever V is nonempty and so U^f meets every nonempty open subset of U and so is dense in U.

(iv) $\Rightarrow$ (i) Follow the proof of Proposition 2.8 with $\mathcal{A}$ the same set of pairs as was used there. We show with a similar argument that

$$IN_f = \bigcap_{(U_1,U_2)\in\mathcal{A}} (U_1)^f \cup (X_1 \setminus \overline{U_2})^f. \tag{2.26}$$

By Proposition 2.12 again IN_f is contained in the intersection. On the other hand if x is in the intersection and $x \in U$ then choose a pair with $x \in U_2$ and $U_1 \subset U$. Since $x \notin X_1 \setminus U_2$ it must be that $x \in U_1^f \subset U^f$. So by Proposition 2.12 $x \in IN_f$.

This expresses IN_f as the intersection of a countable family of dense open sets. Hence, IN_f is dense by the Baire Category Theorem. □

Definition 2.15. Let $f : X_1 \to X_2$ be a surjective continuous map between compact metric spaces. A closed subset $A \subset X_1$ is called *minimal for f* when it is minimal in the class of closed subsets of X_1 which are mapped onto X_2 by f. A continuous map $f : X_1 \to X_2$ is called *irreducible* when it is a surjection and X_1 is minimal for f.

So the continuous surjection f is irreducible when X_1 itself is the only closed subset of X_1 mapped by f onto X_2. Thus, the restriction of a continuous surjection to a minimal subset is an irreducible map.

Proposition 2.16. *Let $f : X_1 \to X_2$ be a surjective continuous map between compact metric spaces.*

(a) *X_1 contains a closed subset which is minimal for f.*

(b) *The following are equivalent:*

  (i) *f is irreducible, i.e. X_1 is minimal for f.*

  (ii) *f is almost one-to-one.*

  (iii) *$D \subset X_1$ is dense in X_1 iff $f(D)$ is dense in X_2.*

Proof. (a) By compactness of X_1 Zorn's Lemma applies to the collection of closed subsets of X_1 which are mapped onto X_2. Hence, the collection contains minimal elements.

(b)(ii) $\Rightarrow$ (i) If A maps onto X_2 then clearly $IN_f \subset A$. If A is closed and IN_f is dense in X_1 then $A = X_1$ and so X_1 is minimal.

(i) $\Leftrightarrow$ (iii) Since f is surjective, D dense implies $f(D)$ is dense. On the other hand, if $f(D)$ is dense then $\overline{D}$ maps onto $\overline{f(D)} = X_2$. If f is irreducible then $\overline{D}$ must equal X_1 and so D is dense. On the other hand, given (iii) $f(A) = X_2$ implies that A is dense in X_1. So if A is closed it equals X_1. Thus, f is irreducible.

(i) $\Rightarrow$ (ii) Let $U \subset X_1$ be open and nonempty. Since $X_1 \setminus U$ is a proper closed subset of X_1 and f is irreducible, the image $f(X_1 \setminus U)$ is a proper closed subset of X_2 and its complement, $X_2 \setminus f(X_1 \setminus U)$ is a nonempty open subset of X_2. Since f is surjective $U^f = f^{-1}(X_2 \setminus f(X_1 \setminus U))$ is nonempty. By Proposition 2.14 f is almost one-to-one. □

Proposition 2.17. *If $f : X_1 \to X_2$ is an irreducible map then f is almost open with $IN_f = OPEN_f$.*

Proof. If U is a nonempty open subset of X and $x \in U \cap IN_f$ then $x \in U^f = f^{-1}[(f^{-1})^*(U)]$. Because f is surjective, $f(U^f)$ is the open set $(f^{-1})^*(U)$ and it contains $f(x)$. Hence, $x \in OPEN_f$. That is, $IN_f \subset OPEN_f$ and since IN_f is dense f is almost open.

Now suppose that f is a continuous surjection and that there exists $x \in OPEN_f \setminus IN_f$. We will show that f is not irreducible. Since $x \notin IN_f$ there exists a point x_1 distinct from x with $f(x_1) = f(x)$. Let U, U_1 be disjoint open sets with $x \in U$ and $x_1 \in U_1$. Since $x \in OPEN_f$, $f(x) = f(x_1) \in f(U)^\circ$. Hence, $V = U_1 \cap f^{-1}[f(U)^\circ]$ is an open set containing x_1 and so $A = X_1 \setminus V$ is a proper closed subset of X_1. Note that $f(V) \subset f(U)$ and $U \subset A$. As f is surjective $X_2 = f(A \cup V) = f(A) \cup f(V) = f(A)$. It follows that X_1 is not minimal for f.

Thus, if f is irreducible $OPEN_f \subset IN_f$. □

Proposition 2.18. *Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be surjective continuous maps of compact metric spaces. The composition $g \circ f : X_1 \to X_3$ is irreducible iff both f and g are irreducible.*

Proof. The proof is an easy exercise using minimality. □

Proposition 2.19. *Let $f : X_1 \to X_2$ be a surjective continuous map between compact metric spaces. The following are equivalent:*

(i) *X_1 contains a unique closed subset which is minimal for f.*
(ii) *$f(IN_f) = ONE_{f^{-1}}$ is dense in X_2.*

When these conditions hold the unique minimal set is the closure $\overline{IN_f}$.

Proof. If $A \subset X_1$ maps onto X_2 then A contains IN_f and so if A is closed it contains $\overline{IN_f}$. On the other hand, $f(IN_f)$ is dense then $\overline{IN_f}$ maps onto X_2. Thus, condition (ii) implies that $\overline{IN_f}$ is the unique closed subset of X_1 which is minimal for f.

To complete the proof we need to use some results from Akin (1993).

Lemma 2.20. *If $f : X_1 \to X_2$ is a surjective continuous map between compact metric spaces then $(f^{-1})^*(OPEN_f) = \{y \in X_2 : f^{-1}(y) \subset OPEN_f\}$ is a dense G_δ subset of X_2.*

Proof. The closed relation $f^{-1} : X_2 \to X_1$ is lower semicontinuous at $y \in X_2$ when $U \subset X_1$ is open and $f^{-1}(y) \cap U \neq \emptyset$ implies $\{y_1 : f^{-1}(y_1) \cap U \neq \emptyset\}$ is a neighborhood of y. This says that for all $x \in f^{-1}(y)$ and U open with $x \in U$, $f(U)$ is a neighborhood of $y = f(x)$. That is, f is open at every point of $f^{-1}(y)$. See Akin (1993) Proposition 7.11. By Theorem 7.19 of Akin (1993) the set of points $y \in X_2$ at which f^{-1} is lower semicontinuous is a dense G_δ subset of X_2. This is $(f^{-1})^*(OPEN_f)$. □

Now assume that $f(IN_f)$ is not dense in X_2. By Lemma 2.20 we can choose a point y in the complement of the closure of $f(IN_f)$ such that $f^{-1}(y) \subset OPEN_f$. Since $y \notin f(IN_f)$ we can choose open $U_1, U_2 \subset X_1$ with disjoint closures such that each meets $f^{-1}(y)$. Let $V = f(U_1)^\circ \cap f(U_2)^\circ$ which is an open set containing y because f is open at the points of $f^{-1}(y)$. For $i = 1, 2$ let $A_i = (X_1 \setminus f^{-1}(V)) \cup \overline{U_i}$. Each of these is a proper closed subset of X_1. For example, A_1 is disjoint from $U_2 \cap f^{-1}(V)$. On the other hand, since f is surjective and $V \subset f(U_1) \cap f(U_2)$ we have $X_2 = f(A_i \cup f^{-1}(V)) = f(A_i) \cup V = f(A_i)$. By Zorn's Lemma each A_i contains a minimal set M_i for f. They cannot be the same set because $M_1 \cap M_2 \subset A_1 \cap A_2 = X_1 \setminus f^{-1}(V)$ whose image under f does not contain the points of V. Thus, there are at least two distinct subsets minimal for f.

This completes the proof that (i) implies (ii). □

Recall that a point x is called *isolated* when it is open as well as closed.

Lemma 2.21. *Let $f : E_1 \to E_2$ be a continuous map and let $x \in E_1$*

(a) *If x is isolated in E_1 and f is weakly open then $f(x)$ is is isolated in E_2.*

(b) *If $f(x)$ is isolated in E_2 and IN_f is dense in E_1 then x is isolated in E_1 and $x \in IN_f$.*

(c) *If f is an irreducible map between compact metric spaces then x is isolated iff $f(x)$ is isolated and in that case, $x \in IN_f$.*

Proof. (a) If x is open and f is weakly open then the interior of $\overline{f(x)} = f(x)$ is nonempty which implies that $f(x)$ is open.

(b) If $f(x)$ is open and IN_f is dense then the nonempty open set $f^{-1}(f(x))$ meets IN_f. If x_1 is in the intersection then $\{x_1\} = f^{-1}(f(x_1)) = f^{-1}(f(x))$. As x is in this set, we have $x_1 = x$. Since $f(x)$ is open, $\{x\} = f^{-1}(f(x))$ is open.

(c) An irreducible map is both almost open and almost one-to-one by Propositions 2.16 and 2.17. □

Recall the Uniqueness of Cantor Theorem which says that any compact, zero-dimensional metric space without isolated points is homeomorphic to the Cantor Set.

Theorem 2.22. *If X is a compact metric space without isolated points then there exists an irreducible map $f : C \to X$ with C the Cantor Set. If, in addition, X is connected then IN_f has empty interior. That is, $C \setminus IN_f$ is dense in C.*

Proof. It is well known that there exists a continuous map F from a Cantor set C onto any compact metric space X.

By Proposition 2.16(a) there is a closed subset Z of C which is minimal for F. Since Z is a closed subset of C it is a compact, zero-dimensional metric space. By Lemma 2.21 it has no isolated points since X does not and since $F : Z \to X$ is irreducible. Hence, there is a homeomorphism $h : C \to Z$. Let $f = F \circ h$.

If $(IN_f)^\circ$ is nonempty then because C is zero-dimensional there is a clopen $A \subset IN_f$ which is neither empty nor all of C. Since f is irreducible $f(A)$ is a nonempty, proper, closed subset of X. By Proposition 2.17 f is open at every point of $A \subset IN_f$. Hence, $f(A)$ is open and so X is not connected. □

From this we notice a problem that might arise with selection functions.

Theorem 2.23. *Let $f : X_2 \to X_1$ be an irreducible map. If $g : X_1 \to X_2$ is a selection function for the relation $f^{-1} : X_1 \to X_2$, i.e. g is a map and $g \subset f^{-1}$, then $(IN_f)^\circ$ is a dense subset of $g(X_1)^\circ$. In particular, if $(IN_f)^\circ = \emptyset$ then $g(X_1)^\circ = \emptyset$.*

Proof. If $f^{-1}(f(x)) = \{x\}$ then $g(f(x)) = x$. Hence, $IN_f \subset g(X_1)$ and so we have inclusion of the interiors as well.

Now let U be any nonempty open subset disjoint from $(IN_f)^\circ$. We will show that $U \setminus g(X_1)$ is nonempty. This will show that $g(X_1)^\circ$ is contained in the closure of $(IN_f)^\circ$.

Because f is irreducible, it is almost one-to-one by Proposition 2.16 and so by Proposition 2.14 there is a nonempty open set $V \subset X_1$ such that $f^{-1}(V) \subset U$. Since U is disjoint from $(IN_f)^\circ$ the open set $f^{-1}(V)$ meets the complement of IN_f. That is, there exist distinct points x_1, x_2 with $x_1 \in f^{-1}(V)$ and with a common value y. That is, $y = f(x_1) = f(x_2)$. Because $x_1 \in f^{-1}(V)$, $y \in V$ and so $x_2 \in f^{-1}(V)$ as well. So either $x_1 \neq g(y)$ or $x_2 \neq g(y)$. Hence, one or the other is a point of $U \setminus g(X_2)$. $\square$

Remark: $(IN_f)^\circ$ might be a proper subset of $g(X_1)^\circ$. For example, let $X_2 = [-\frac{1}{2}, 0] \cup [1, \frac{3}{2}]$, $X_1 = [0, 1]$ and define $f : X_2 \to X_1$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{for} \quad -\frac{1}{2} \leq x \leq 0, \\ x - \frac{1}{2} & \text{for} \quad 1 \leq x \leq \frac{3}{2}. \end{cases} \tag{2.27}$$

Clearly, f is irreducible with IN_f the open set $[\frac{1}{2}, 0) \cup (1, \frac{3}{2}]$ and if g is either of the two selection functions for f^{-1} then $g(X_1)$ is an open set containing the endpoint $g(\frac{1}{2})$ as well.

Thus, if X is a compact connected metric space with more than one point, and so without isolated points, then there exists an irreducible map $f : C \to X$ and if $g : X \to C$ is any selection function for f^{-1} then $g(X)$ has empty interior in C. By Theorem 1.1 the map g is quasi-continuous but it is as far as possible from being an almost open map.

Furthermore, if $f_1 : C \to X$ and $f_2 : C \to X$ are irreducible maps then $F = f_2 \circ (f_1)^{-1} : X \to X$ is a closed relation with π_{1F} irreducible. In fact F is a suitable relation. If $g_1 : X \to C$ is a selection function for $(f_1)^{-1}$ then $f_2 \circ g_1 : X \to X$ is a selection function for F and so is quasi-continuous. If $IN_{f_1} \cup IN_{f_2} = C$, or, equivalently the complements of IN_{f_1} and IN_{f_2} are disjoint, then $((f_2 \circ g_1)(X))^\circ = \emptyset$. To prove this let $U \subset X$ be an arbitrary nonempty open set. Then $(f_2)^{-1}(U)$ is a nonempty open subset of C and so contains an $(f_1)^{-1}(V)$ for some nonempty open $V \subset X$. As in the above proof there exist distinct $x_1, x_2 \in C$ with $y = f_1(x_1) = f_1(x_2) \in V$. Suppose $g_1(y) \neq x_1$. Since $x_1 \notin IN_{f_1}$ it follows that $x_1 \in IN_{f_2}$ and so there does not exist any $x \neq x_1$ with $f_2(x) = f_2(x_1)$. In particular, there does not equal any $z \in X$ such that $f_2(g(z)) = f_2(x_1)$. Hence, $f_2(x_1) \in U \setminus f_2(g_1(X))$.

Example: Let $f_1 : C = \{0,1\}^{\mathbb{N}} \to [0,1]$ be the map given by $z \mapsto \Sigma_i \frac{z_i}{2^{i+1}}$ then the complement of IN_{f_1} is contained in the countable set of sequences z such either z_i eventually 0 or is eventually 1. Let $\tilde{0} = 1$ and $\tilde{1} = 0$. Define the homeomorphism h on C by

$$h(z)_i \quad = \quad \begin{cases} z_i & \text{for } i \text{ even} \\ \tilde{z}_i & \text{for } i \text{ odd.} \end{cases} \tag{2.28}$$

Let $f_2 = f_1 \circ h$. The complement of IN_{f_2} is contained in the countable set of sequences which are eventually 010101.... This is disjoint from the complement of IN_{f_1}.

3. Suitable Relations

Let $F : X_1 \to X_2$ be a closed relation between compact metric spaces. For $i = 1, 2$ we denote by $\pi_{iF} : F \to X_i$ the restrictions of the projections. We have

$$F \quad = \quad \pi_{2F} \circ (\pi_{1F})^{-1}. \tag{3.1}$$

Lemma 3.1. *The following are equivalent:*

(i) *F has full domain, i.e. $F^{-1}(X_2) = X_1$.*
(ii) *For every $x \in X_1$, $F(x)$ is nonempty.*
(iii) *π_{1F} is surjective.*
(iv) *$F^*(\emptyset) = \emptyset$.*
(v) *For some $B \subset X_2$, $F^*(B) \subset F^{-1}(B)$.*
(vi) *For all $B \subset X_2$, $F^*(B) \subset F^{-1}(B)$.*

Proof. An easy exercise applying the various definitions. □

For $(x, y) \in F$ it is clear that π_{1F} is injective at (x, y), i.e. $(x, y) \in IN_{\pi_{1F}}$, precisely when $F(x) = \{y\}$ and so when $x \in ONE_F$.

Proposition 3.2. *π_{1F} restricts to a homeomorphism from $IN_{\pi_{1F}} \subset F$ onto $ONE_F \subset X_1$. With $f_F : ONE_F \to X_2$ the continuous map obtained by restricting the closed relation F to ONE_F we see that*

$$f_F \quad = \quad IN_{\pi_{1F}} \quad \subset \quad ONE_F \times X_2. \tag{3.2}$$

Proof. That π_{1F} restricts to a homeomorphism follows from Proposition 2.12 applied to the map π_{1F}. Notice that $ONE_F = ONE_{(\pi_1 F)^{-1}}$. The set $IN_{\pi_{1F}}$ is just the restriction of F to ONE_F and this is the map f_F which is continuous by Proposition 2.11. □

Corollary 3.3. *Let $F : X_1 \to X_2$ be a closed relation between compact metric spaces. The following are equivalent:*

(i) *The map $\pi_{1F} : F \to X_1$ is surjective and F contains a unique closed subset which is minimal for π_{1F}.*
(ii) *ONE_F is dense in X_1.*

When these conditions hold the unique minimal set is the closure in $X_1 \times X_2$ *of the map* $f_F : ONE_F \to X_2$.

Proof. The closed set $F^{-1}(X_2)$ contains the set ONE_F and so it equals X_1 when the latter is dense. So by Lemma 3.1 π_{1F} is surjective when ONE_F is dense. The result then follows from Proposition 2.19 and Proposition 3.2. □

By Proposition 2.16(a) any closed full domain relation contains at least one minimal closed full domain relation.

Theorem 3.4. *Let* $F : X_1 \to X_2$ *be a closed relation between compact metric spaces and* $\pi_{1F} : F \to X_1$ *be the restriction to* $F \subset X_1 \times X_2$ *of the first coordinate projection map.*

(a) *The following conditions are equivalent.*
 (i) F *is a minimal closed full domain relation from* X_1 *to* X_2.
 (ii) *The map* π_{1F} *is irreducible, i.e.* F *is minimal for* π_{1F}.
 (iii) *The map* π_{1F} *is surjective and almost one-to-one.*
 (iv) F *has full domain and for every open* $U \times V \subset X_1 \times X_2$ *which meets* F *there exists* $x \in U$ *such that* $F(x) \subset V$. *That is,* $U \cap F^*(V)$ *is nonempty.*
 (v) *For every open* $V \subset X_2$ $F^*(V)$ *is a dense subset of* $F^{-1}(V)$. *That is,*

$$F^*(V) \quad \subset \quad F^{-1}(V) \quad \subset \quad \overline{F^*(V)}. \tag{3.3}$$

(b) *If* F *is a minimal full domain closed relation then* ONE_F *is dense in* X_1 *and* F *is the closure in* $X_1 \times X_2$ *of the map* $f_F : ONE_F \to X_2$. *That is,* f_F *is dense in* F.

(c) *If there exists* D *a dense subset of* X_1 *and a continuous map* $f : D \to X_2$ *such that* f *is dense in* F *then* π_{1F} *is irreducible,* $D \subset ONE_F$ *and* $f = f_F$ *on* D.

Proof. (a)(i) ⇔ (ii) This is the definition of an irreducible map.

(ii) ⇔ (iii) Apply Proposition 2.16.

(iii) ⇒ (iv) $X_1 = F^{-1}(X_2)$ because π_{1F} is surjective. If an open $U \times V$ meets F then $F \cap (U \times V)$ is a nonempty open subset of F. Because π_{1F} is almost one-to-one Proposition 2.14 implies there exists $x \in X_1$ such that $(\pi_{1F})^{-1}(x) \subset U \times V$ or, equivalently, $x \in U$ and $F(x) \subset V$.

(iv) ⇒ (v) Because F has full domain Lemma 3.1 implies $F^*(V) \subset F^{-1}(V)$. If $x \in F^{-1}(V)$ then there exists $y \in V$ with $(x, y) \in F$. For any open set U with $x \in U$, (iv) implies there exists $x_1 \in U$ such that $F(x_1) \subset V$. Thus, U meets $F^*(V)$ and so the latter is dense in $F^{-1}(V)$.

(v) ⇒ (iii) By Lemma 3.1 π_{1F} is surjective because $F^*(V) \subset F^{-1}(V)$.

Any open subset of F contains a nonempty open set of the form $F \cap (U \times V)$, i.e. U meets $F^{-1}(V)$. Since U is open (v) implies that U meets $F^*(V)$. There is a point $x \in U$ such that $F(x) \subset V$ and so $\pi_{1F}^{-1}(x) \subset F \cap (U \times V)$. It follows from Proposition 2.14 that π_{1F} is almost one-to-one.

(b) Since π_{1F} is almost one-to-one, $IN_{\pi_{1F}}$ is dense in F. By (3.2) this says f_F is dense in F. Since π_{1F} is surjective, it follows that $ONE_F = \pi_{1F}(IN_{\pi_{1F}})$ is dense in X_1.

(c) If $(x, y) \in F$ with $x \in D$ then since f is dense in F there is a sequence $\{x_n\}$ in D such that $\{(x_n, f(x_n))\}$ converges to (x, y). By continuity of f on D $\{f(x_n)\}$ converges to $f(x)$ and so $y = f(x)$. Thus, for $x \in D$, $F(x) = \{f(x)\}$. It follows that $D \subset ONE_F$ and that on D the map f_F restricts to f. Finally, $IN_{\pi_{1F}} = f_F$ contains f and so is dense in F. Hence, π_{1F} is almost one-to-one. Since D is dense in X_1, $\pi_{1F}(F)$ contains the closure of D which is X_1. That is, π_{1F} is surjective. □

Remarks: In part (a) it does not suffice that $F^*(V)$ be nonempty for all open $V \subset X_2$ which meet $F(X_1)$. Let X_1 be the disjoint union of two nontrivial spaces X and Y and let $X_2 = X$. For $F = 1_X \cup (Y \times X)$ it follows that for every proper open subset $V \subset X_2 = X$ that $F^*(V) = V \subset X \subset X_1$ but for no point $x \in Y$ is $F(x) \subset V$.

Corollary 3.5. *Let $F : X_1 \to X_2$ be a closed relation between compact metric spaces and let $f : D \to X_2$ be a continuous map in with $D \subset X_1$. The following are equivalent:*

(i) *D is dense in X_1 and F is the closure in $X_1 \times X_2$ of f.*
(ii) *F has full domain and F is the closure in $X_1 \times X_2$ of f.*
(iii) *F is a minimal closed full domain relation, D is a dense subset of ONE_F and f is the restriction of f_F to D.*

When these conditions hold we will say the map f is dense in F .

Proof. (i) ⇔ (ii) If f is dense in F then $D = \pi_{1F}(f)$ is dense in $\pi_{1F}(F)$. Then D is dense in X_1 iff $F^{-1}(X_2) = \pi_{1F}(F) = X_1$.

(i) ⇒ (iii) Apply Theorem 3.4(c).

(iii) ⇒ (i) Assuming (iii) Theorem 3.4(b) says that $f_F : ONE_F \to X_2$ is dense in F. If $x \in ONE_F$ then it is the limit of some sequence $\{x_n\}$ in D because D is dense on ONE_F. By continuity of f_F, $f_F(x)$ is the limit of the sequence $\{f(x_n) = f_F(x_n)\}$. Thus, f is dense in f_F and so is dense in F. D is dense in X_1 because it is dense in ONE_F and the latter is dense in X_1. □

While proper containment between closed relations can occur, we have:

Lemma 3.6. *Let $\tilde{F}, F : X_1 \to X_2$ be closed relations such that $\tilde{F}$ has full domain and π_{1F} is irreducible. If $\tilde{F} \subset F$ then $\tilde{F} = F$.*

Proof. $\tilde{F}$ is a closed full domain relation contained in F and F is a minimal closed full domain relation. Hence, $\tilde{F} = F$. □

Theorem 3.7. *Let $F : X_1 \to X_2$ be a closed relation between compact metric spaces with $\pi_{1F} : F \to X_1$ irreducible. The following conditions are equivalent.*

(i) *$\pi_{2F} : F \to X_2$ is an almost open continuous map.*
(ii) *$f_F : ONE_F \to X_2$ is a weakly open continuous map.*
(iii) *If $U \subset X_1$ is open and nonempty then $F(U)^\circ \neq \emptyset$.*
(iv) *If $U \subset X_1$ is open then the open subset*

$$U_F \quad =_{def} \quad U \cap F^*(F(U)^\circ\ ) \tag{3.4}$$

is dense in U.
(v) *If D is a dense subset of X_2 then $F^{-1}(D)$ is dense in X_1.*
(vi) *If D is a dense open subset of X_2 then $F^*(D)$ is a dense open subset of X_1.*
(vii) *If D is a dense G_δ subset of X_2 then $F^*(D)$ is a dense G_δ subset of X_1.*
(viii) *If B is a closed nowhere dense subset of X_2 then $F^{-1}(B)$ is a closed nowhere dense subset of X_1.*

Proof. (i) $\Leftrightarrow$ (ii) By Theorem 2.5 π_{2F} is almost open iff it is weakly open and by the Variation of Domain Theorem 2.6 it is weakly open iff its restriction to a dense set is weakly open. Since π_{1F} is irreducible, f_F is a dense subset of F. Furthermore, π_{1F} restricts to a homeomorphism of f_F onto its domain ONE_F. Thus, the restriction of π_{2F} to f_F is weakly open iff $f_F = (\pi_{2F}|f_F) \circ (\pi_{1F})^{-1}|ONE_F$ is weakly open.

(i) $\Rightarrow$ (iii) If $U \subset X_1$ is open and nonempty then the open set $U \times X_2$ meets F because π_{1F} is surjective. Since π_{2F} is almost open $F(U) = \pi_{2F}(F \cap (U \times X_2))$ has a nonempty interior.

(iii) $\Rightarrow$ (iv) It suffices as usual to show that U_F is nonempty when U is because we then apply the result to arbitrary open subsets V of U and note that $V_F \subset V \cap U_F$. By (iii) $F(U)^\circ$ is nonempty and by (3.3) $F^*(F(U)^\circ)$ is dense in $F^{-1}(F(U)^\circ)$. Since U is open it suffices to show that U meets $F^{-1}(F(U)^\circ)$. But if $y \in F(U)^\circ$ then since $y \in F(U)$ there exists $x \in U$ such that $y \in F(x)$. Hence, $x \in F^{-1}(F(U)^\circ)$.

(iv) $\Rightarrow$ (iii) Obvious because $F^*(\emptyset) = \emptyset$ by Lemma 3.1.

(iii) $\Rightarrow$ (v) If $U \subset X_1$ is open an nonempty then $F(U)$ has a nonempty interior by (iii) and so it meets the dense set D. Hence, U meets $F^{-1}(D)$ by (2.18). As U was arbitrary $F^{-1}(D)$ is dense.

(v) ⇒ (vi) D open implies $F^*(D)$ is open and by (3.3) it is dense in $F^{-1}(D)$ which is dense in X_1 by (v).

(vi) ⇔ (vii) If D is open the $F^*(D)$ is open and so by (2.10) if D is G_δ then $F^*(D)$ is G_δ. So (vii) obviously implies (vi) and (vi) implies (vii) by the Baire Category Theorem.

(vi) ⇔ (viii) If D is the complement of B then $F^*(D)$ is the complement of $F^{-1}(B)$. An open set is dense iff its complement is nowhere dense.

(vi) ⇒ (ii) Let D be an open dense subset of X_2. By (vi), $F^*(D)$ is an open dense subset of X_1. Clearly, $ONE_F \cap F^*(D) = (f_F)^{-1}(D)$. Since ONE_F is dense in X_1 it follows from Lemma 2.7 (a) that $ONE_F \cap F^*(D)$ is dense in $F^*(D)$ and hence in X_1. So as a subset of ONE_F it is dense in ONE_F. Thus, D open and dense in X_2 implies $(f_F)^{-1}(D)$ is open and dense in ONE_F. It follows from Proposition 2.3 that f_F is weakly open. □

Notice that if $F \subset [0,1] \times [0,1]$ is the closed full domain relation defined by:

$$F \quad = \quad [0,1] \times \{0\} \cup \{(\frac{m}{n}, y) : y \leq \frac{1}{n} \quad \text{for all} \quad m \leq n\}, \tag{3.5}$$

where m and n vary over the positive integers, then π_{2F} is almost open. However, the unique minimal full domain relation contained in F is the constant function $[0,1] \times \{0\}$ which is not almost open.

Definition 3.8. A *suitable relation* is a closed relation $F : X_1 \to X_2$ between compact metric spaces such that $\pi_{1F} : F \to X_1$ is irreducible and $\pi_{2F} : F \to X_2$ is almost open.

Proposition 3.9. *(a)Let $f : X_1 \to X_2$ be a continuous map between compact spaces. f is a suitable relation iff it is an almost open map.*

(b) Assume X_1 and X_2 are compact metric spaces and that $f : D \to X_2$ is a continuous map with D dense in X_1. Let F be the closure of f in $X_1 \times X_2$. F is a suitable relation iff f is a weakly open map.

Proof. (a) $\pi_{1f} : f \to X_1$ is a homeomorphism and so is irreducible. Since $f = \pi_{2f} \circ (\pi_1 f)^{-1}$ the map f is almost open iff the map π_{2f} is almost open.

(b) By Theorem 3.4(c) π_{1F} is irreducible and f is the restriction to $D \subset ONE_F$ of f_F. π_{2F} is almost open iff f_F is weakly open by Theorem 3.7 and so iff f is weakly open by the Variation of Domain Theorem 2.6. □

Now we consider composition of suitable relations.

Lemma 3.10. *If $F : X_1 \to X_2$ and $G : X_2 \to X_3$ are closed relations between compact metric spaces such that $X_2 = G^{-1}(X_3)$ and $X_1 = F^{-1}(X_2)$, then*

$$ONE_{G\circ F} \quad \subset \quad F^*(ONE_G) \quad \subset \quad F^{-1}(ONE_G), \tag{3.6}$$

and

$$\begin{aligned} ONE_F \cap ONE_{G\circ F} \quad &= \quad ONE_F \cap F^*(ONE_G) \\ &= \quad ONE_F \cap F^{-1}(ONE_G) \quad = \quad (f_F)^{-1}(ONE_G). \end{aligned} \tag{3.7}$$

Proof. If $y \in F(x)$ then $G(y) \subset (G \circ F)(x)$ with equality if $F(x) = \{y\}$. In particular, if $F(x)$ contains some point y not in ONE_G then x is not in $ONE_{G\circ F}$ and conversely if $x \in ONE_F$. Thus, $ONE_{G\circ F} \subset F^*(ONE_G)$ with equality after intersection by ONE_F. Since $X_1 = F^{-1}(X_2)$ the second inclusion in (3.6) follows from Lemma 3.1. For any $B \subset X_2$ we clearly have

$$ONE_F \cap F^*(B) \quad = \quad ONE_F \cap F^{-1}(B) \quad = \quad (f_F)^{-1}(B). \tag{3.8}$$

From this follow the remaining equalities in (3.7). □

We can restrict the map $f_F : ONE_F \to X_2$ to $(f_F)^{-1}(ONE_G) \subset ONE_F$ and obtain a map from $(f_F)^{-1}(ONE_G)$ to ONE_G which can be composed with $f_G : ONE_G \to X_3$. That is, we have the composed map $f_G \circ f_F : (f_F)^{-1}(ONE_G) \to X_3$.

Theorem 3.11. *Assume that $F : X_1 \to X_2$ and $G : X_2 \to X_3$ are suitable relations. The set $(f_F)^{-1}(ONE_G)$ is a G_δ subset of X_1 which is dense in X_1. It is contained in $ONE_{G\circ F}$ and on it the functions $f_G \circ f_F$ and $f_{G\circ F}$ agree. The function $f_{G\circ F} : ONE_{G\circ F} \to X_3$ and its restriction $f_G \circ f_F : (f_F)^{-1}(ONE_G) \to X_3$ are both weakly open maps.*

Proof. Because G is suitable, ONE_G is a dense G_δ subset of X_2. Since F is suitable f_F is a weakly open map and so by Proposition 2.3 $(f_F)^{-1}(ONE_G)$ is a dense G_δ subset of ONE_F. Because F is suitable the latter is a dense G_δ in X_1 it follows that $(f_F)^{-1}(ONE_G)$ is a dense G_δ in X_1. By (3.7) $(f_F)^{-1}(ONE_G) \subset ONE_{G\circ F}$ and for $x \in (f_F)^{-1}(ONE_G)$ it is clear that the single point in $G \circ F(x)$ is the point$f_G \circ f_F(x)$. That is, $f_{G\circ F}$ agrees with $f_G \circ f_F$ on $(f_F)^{-1}(ONE_G)$.

Because F and G are suitable the maps $f_F : ONE_F \to X_2$ and $f_G : ONE_G \to X_3$ are weakly open. By the Variation of Domain Theorem 2.6 the restriction $f_F : (f_F)^{-1}(ONE_G) \to ONE_G$ is weakly open. By Corollary 2.4 the composition $f_G \circ f_F : (f_F)^{-1}(ONE_G) \to X_3$ is weakly open and by the Variation of Domain Theorem 2.6 again the extension to $f_{G\circ F} : ONE_{G\circ F} \to X_3$ is weakly open as well. □

Lemma 3.12. *If $F : X_1 \to X_2$ and $G : X_2 \to X_3$ are suitable relations then $f_G \circ f_F \subset (f_F)^{-1}(ONE_G) \times X_3$ is a dense subset of*

$$\pi_{13}((X_1 \times ONE_G \times X_3) \cap G \otimes F)$$

where $\pi_{13} : X_1 \times X_2 \times X_3 \to X_1 \times X_3$ is the projection to the first and third coordinates.

Proof. Recall that $G \otimes F = \{(x, y, z) : (x, y) \in F \text{ and } (y, z) \in G\}$. For any $(x, y, z) \in G \otimes F$ we can choose a sequence $\{(x_n, y_n) \in (f_F) \cap (f_F)^{-1}(ONE_G) \times X_2$ converging to (x, y) because the map f_F is dense in F and $(f_F)^{-1}(ONE_G)$ is dense in ONE_F. See Corollary 3.5. Since y_n must equal $f_F(x_n)$ and $x_n \in (f_F)^{-1}(ONE_G)$ it follows that the sequence $\{y_n\}$ is in ONE_G. Let $z_n \in X_3$ be the unique point in $G(y_n)$. By going to a subsequence if necessary we can assume that $\{z_n\}$ converges to a point $\tilde{z} \in X_3$. Since $G \otimes F$ is closed we have that $(x, y, \tilde{z}) \in G \otimes F$. Now if $y \in ONE_G$ then $z, \tilde{z} \in G(y)$ implies $z = \tilde{z}$. Thus, when $(x, y, z) \in G \otimes F$ with $y \in ONE_G$ we obtain a sequence $\{(x_n, z_n)\}$ which converges to (x, z) with $x_n \in (f_F)^{-1}(ONE_G)$ and $z_n = f_G \circ f_F(x_n)$. □

Definition 3.13. *For suitable relations $F : X_1 \to X_2$ and $G : X_2 \to X_3$ define the* suitable composition *$G \bullet F : X_1 \to X_3$ by*

$$G \bullet F \quad =_{def} \quad \pi_{13}[\ \overline{(G \otimes F) \cap (X_1 \times ONE_G \times X_3)}\ ]. \tag{3.9}$$

The name is justified by the following:

Theorem 3.14. *For suitable relations $F : X_1 \to X_2$ and $G : X_2 \to X_3$ the suitable composition $G \bullet F : X_1 \to X_3$ is a suitable relation with*

$$(f_F)^{-1}(ONE_G) \quad \subset \quad ONE_{G \circ F} \quad \subset \quad ONE_{G \bullet F}. \tag{3.10}$$

$G \bullet F$ is the unique closed subset of $G \circ F$ which is minimal for $\pi_{1G \circ F}$. If D is any dense subset of $ONE_{G \circ F}$ then $G \bullet F$ is the closure in $X_1 \times X_3$ of the restriction of $f_{G \circ F}$ to D.

Proof. The first inclusion of (3.10) follows from (3.7). On the other hand, $G \bullet F \subset G \circ F$ and the projection of $G \bullet F$ to X_1 is closed and contains the dense set $(f_F)^{-1}(ONE_G)$. Hence, for every $x \in X_1$, $\emptyset \neq G \bullet F(x) \subset G \circ F(x)$ which implies the second inclusion of (3.10).

Because $ONE_{G \circ F}$ is dense in X_1 it follows from Corollary 3.3 that $G \circ F$ contains a unique minimal subset and that it is the closure of the map $f_{G \circ F}$ on $ONE_{G \circ F}$ which contains the map $f_G \circ f_F$ on $(f_F)^{-1}(ONE_G)$. So by Lemma 3.12 it contains $G \bullet F$. π_1 maps $f_G \circ f_F$ onto the dense set $(f_F)^{-1}(ONE_G)$ and so $G \bullet F$ is mapped onto X_1 by π_1. It follows that the minimal set equals $G \bullet F$. Thus, $\pi_{1G \bullet F}$ is irreducible because it is the restriction of $\pi_{1G \circ F}$ to the minimal subset.

By Theorem 3.11 $f_{G\circ F}$ is weakly open and so by the Variation of Domain Theorem 2.6 the extension $f_{G\bullet F}$ is weakly open. By Theorem 3.7 $\pi_{2G\bullet F}$ is almost open. It follows that $G \bullet F$ is a suitable relation.

The result for general D follows from Corollary 3.5. □

Proposition 3.15. *Let $F : X_1 \to X_2$ and $G : X_2 \to X_3$ be suitable relations and let $f : D_1 \to X_2$ and $g : D_2 \to X_3$ be continuous maps which are dense in F and G respectively.*

(a) If $f(D_1) \subset D_2$ then the continuous map $g \circ f : D_1 \to X_3$ is dense in $G \bullet F$.

(b) If D_2 is a G_δ subset of X_2 then $f^{-1}(D_2)$ is a dense subset of D_1 and the continuous map $g \circ f : f^{-1}(D_2) \to X_3$ is dense in $G \bullet F$.

Proof. By Corollary 3.5 D_1 is a dense subset of ONE_F and f is the restriction of f_F to D_1. Since f_F is weakly open, f is weakly open by the Variation of Domain Theorem 2.6. Similarly for g. It follows that $f^{-1}(D_2) \subset (f_F)^{-1}(ONE_G)$ and $g \circ f : f^{-1}(D_2) \to X_3$ is a restriction to $f^{-1}(D_2)$ of $f_G \circ f_F : (f_F)^{-1}(ONE_G) \to X_3$ which is in turn dense in $G \bullet F$. In order to apply Corollary 3.5 the other way it suffices to show that $f^{-1}(D_2)$ is dense in D_1 and so is dense in X_1.

If $f(D_1) \subset D_2$ then $f^{-1}(D_2) = D_1$ and so is dense. This proves (a).

If D_2 is a dense G_δ subset of X_2 then Proposition 2.3 implies $f^{-1}(D_2)$ is dense in D_1 because f is weakly open. This proves (b). □

Theorem 3.16. *(a) If $F : X_1 \to X_2$ is a suitable relation and $g : X_2 \to X_3$ is an almost open continuous map then $g \bullet F = g \circ F$.*

(b) If $G : X_2 \to X_3$ is a suitable relation and $f : X_1 \to X_2$ is an open map then $G \bullet f = G \circ f$.

Proof. (a) Since g is a map, $ONE_g = X_2$ and so $(g \otimes F) \cap (X_1 \times ONE_g \times X_3) = g \otimes F$ which projects to $g \circ F$.

(b) It suffices to show that $(G \otimes f) \cap (X_1 \times ONE_G \times X_3)$ is dense in $G \otimes f$.

Let $(x, y, z) \in G \otimes f$ and let $U \times V \times W \subset X_1 \times X_2 \times X_3$ be an open set containing (x, y, z). Since $f(x) = y$ and f is an open map $(V \cap f(U)) \times W$ is an open subset of $X_2 \times X_3$ which contains $(y, z) \in G$. Since G is suitable the map f_G is dense in G and so there exists $(y_1, z_1) \in (f_G) \cap [(V \cap f(U)) \times W]$. That is, $y_1 \in ONE_G$ and $(y_1, z_1) \in G$. Since $y_1 \in f(U)$ there exists $x_1 \in U$ with $f(x_1) = y_1$. Thus, $(x_1, y_1, z_1) \in (G \otimes f) \cap (X_1 \times ONE_G \times X_3) \cap (U \times V \times W)$. □

In (b) it is necessary that the map be open. Almost open will not suffice. Let $X_2 = X_3$ be the unit interval and let G be the closed relation F_{01} defined by (1.3). Now on $X_1 = [-\frac{1}{2}, 0] \cup [1, \frac{3}{2}]$ define $f : X_1 \to X_2$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{for} \quad -\frac{1}{2} \leq x \leq 0, \\ x - \frac{1}{2} & \text{for} \quad 1 \leq x \leq \frac{3}{2}. \end{cases} \tag{3.11}$$

Clearly, f is an irreducible map and $G \bullet f$ is the irreducible map given by

$$G \bullet f(x) = \begin{cases} -x & \text{for} \quad -\frac{1}{2} \leq x \leq 0, \\ 2 - x & \text{for} \quad 1 \leq x \leq \frac{3}{2}. \end{cases} \tag{3.12}$$

while

$$G \circ f \quad = \quad G \bullet f \cup \{(0,1),(1,0)\}. \tag{3.13}$$

Note also that $ONE_{G \circ f} = [-\frac{1}{2}, 0) \cup (1, \frac{3}{2}]$ is a proper subset of $ONE_{G \bullet f} = X_1$.

We will call a relation $F : X_1 \to X_2$ *surjective* when $F(X_1) = X_2$ and $F^{-1}(X_2) = X_1$. That is, F and F^{-1} both have full domain.

Proposition 3.17. *(a) For a suitable relation $F : X_1 \to X_2$ the following are equivalent*

(i) *F is surjective.*
(ii) *$\pi_{2F} : F \to X_2$ is a surjective map.*
(iii) *$f_F(ONE_F)$ is a dense subset of X_2.*
(iv) *If D is a dense subset of ONE_F then $f_F(D)$ is a dense subset of X_2.*

(b) If $F : X_1 \to X_2$ and $G : X_2 \to X_3$ are surjective suitable relations then the suitable relation $G \bullet F : X_1 \to X_3$ is surjective.

Proof. Since F is suitable, π_{1F} is surjective. So (i) $\Leftrightarrow$ (ii) is obvious. If D is dense in ONE_F then $f_F(D)$ is dense in $f_F(ONE_F)$ because f_F is continuous. So (iii) $\Leftrightarrow$ (iv) is obvious.

(iii) $\Rightarrow$ (ii) $F(X_1)$ is closed and contains $f_F(ONE_F)$. If the latter is dense then $F(X_1) = X_2$ which implies (ii).

(ii) $\Rightarrow$ (iii) f_F is dense in F and $\pi_{2F}(f_F) = f_F(ONE_F)$. So $f_F(ONE_F)$ is dense in $F(X_1)$ in any case.

(b) $D = (f_F)^{-1}(ONE_G)$ is dense in ONE_F and so if F is surjective, $f_F(D) = ONE_G \cap f_F(ONE_F)$ is dense in X_2 by (a). If G is surjective then $f_G(f_F(D)) = f_{G \bullet F}(D)$ is dense in X_2. This is a subset of $f_{G \bullet F}(ONE_{G \bullet F})$ and so the latter is dense in X_2. By (a) again $G \bullet F$ is surjective. □

Theorem 3.18. *If $F : X_1 \to X_2$, $G : X_2 \to X_3$ and $H : X_3 \to X_4$ are suitable relations then $(H \bullet G) \bullet F \ = \ H \bullet (G \bullet F) : X_1 \to X_4$. That is, suitable composition is associative.*

Proof. Using Proposition 3.15 it is easy to check that both are the closure of $f_H \circ f_G \circ f_F$ defined on the dense G_δ subset of X_1 obtained by pulling back the dense G_δ subset $(f_G)^{-1}(ONE_H)$ of ONE_G via $f_F : ONE_F \to X_2$. □

It follows that we can define a category whose objects are compact metric spaces and morphisms are suitable relations. Associativity follows from Theorem 3.18. By Theorem 3.16 $1_{X_2} \bullet F = 1_{X_2} \circ F = F = F \circ 1_{X_1} = F \bullet 1_{X_1}$.

The definition of isomorphism in the category says that a suitable relation $F : X_1 \to X_2$ is an isomorphism iff there exists a suitable relation $G : X_2 \to X_1$ such that $G \bullet F = 1_{X_1}$ and $F \bullet G = 1_{X_2}$. G is the inverse relation in the category. When it exists it is unique.

Theorem 3.19. *For a suitable relation $F : X_1 \to X_3$ the following are equivalent.*

(i) *F is an isomorphism in the suitable relations category.*
(ii) *$\pi_{2F} : F \to X_2$ is an irreducible map.*
(iii) *F^{-1} is a suitable relation.*
(iv) *There exist subsets $D_1 \subset X_1$ and $D_2 \subset X_2$ both dense and a homeomorphism $f : D_1 \to D_2$ such that $f : D_1 \to X_2$ is dense in F.*

When these conditions hold, the suitable relation F^{-1} is the inverse of F in the suitable relations category. Furthermore, if $f : D_1 \to D_2$ satisfies the conditions of (iv) then $f^{-1} : D_2 \to X_1$ is dense in F^{-1}. The sets D_1, D_2 in (iv) can be chosen to be dense G_δ sets.

Proof. (i) ⇒ (iii) Let G be the inverse in the category. $(f_G)^{-1}(ONE_F) \subset ONE_G$ is dense and $f_F \circ f_G : (f_G)^{-1}(ONE_F) \to X_2$ is dense in $F \bullet G = 1_{X_2}$. If $y \in (f_G)^{-1}(ONE_F)$ then $f_F(f_G(y)) = y$. Hence, $f_F(ONE_F)$ contains the dense set $(f_G)^{-1}(ONE_F)$ and is contained in the closed set $F(X_1)$. Thus, $F(X_1) = X_2$ and so π_{2F} is surjective, or, equivalently, $\pi_{1F^{-1}}$ is surjective

The restriction of f_F to $(f_F)^{-1}(ONE_G)$ is dense in F. Hence, its inverse is dense in F^{-1}.

$f_G \circ f_F : (f_F)^{-1}(ONE_G) \to X_1$ is dense in $G \bullet F = 1_{X_1}$. Thus, if $x \in (f_F)^{-1}(ONE_G)$ and $y = f_F(x)$ then $f_G(y) = x$. This implies that the inverse of $f_F : (f_F)^{-1}(ONE_G) \to X_2$ is contained in the closed relation G. Since this inverse is dense in F^{-1}. It follows that $F^{-1} \subset G$. But G is suitable and $\pi_{1F^{-1}}$ is surjective. From Lemma 3.6 it follows that $F^{-1} = G$.

This proves (iii) and also shows that the inverse G, when it exists, is equal to F^{-1}.

(ii) $\Leftrightarrow$ (iii) Let $T : X_1 \times X_2 \to X_2 \times X_1$ be the homeomorphism which switches coordinates.

$$\pi_{1F^{-1}} \quad = \quad \pi_{2F} \circ T \qquad \text{and} \qquad \pi_{2F^{-1}} \quad = \quad \pi_{1F} \circ T. \tag{3.14}$$

Whenever F is suitable $\pi_{2F^{-1}}$ is irreducible and so is almost open. Hence, F^{-1} is suitable iff the almost open map $\pi_{1F^{-1}}$ is irreducible.

(iii) $\Rightarrow$ (iv) For any closed relation $F : X_1 \to X_2$ it is always true that:

$$(ONE_F \times ONE_{F^{-1}}) \cap F \quad = \quad (f_F)^{-1}(ONE_{F^{-1}}) \times (f_{F^{-1}})^{-1}(ONE_F). \tag{3.15}$$

That is, if $(x, y) \in F$ with x and y in the domains of f_F and $f_{F^{-1}}$ respectively then $y = f_F(x)$ and $x = f_{F^{-1}}(y)$.

If F and F^{-1} are suitable then $D_1 = (f_F)^{-1}(ONE_{F^{-1}}) \subset X_1$ and $D_2 = (f_{F^{-1}})^{-1}(ONE_F) \subset X_2$ are dense G_δ sets and f_F and $f_{F^{-1}}$ are inverse homeomorphisms between them.

This proves (iv) and also shows that the sets D_1 and D_2 can be chosen to be G_δ's.

(iv) $\Rightarrow$ (i) Since the map f is a dense subset of F, we can apply the coordinate switching homeomorphism and see that f^{-1} is dense in F^{-1}. Because $f^{-1} : D_2 \to D_1$ is a homeomorphism, $f^{-1} : D_2 \to X_1$ is a weakly open continuous map by Variation of Domain. By Proposition 3.9 F^{-1} is a suitable relation because it is the closure of a densely defined, weakly open continuous map.

By Proposition 3.15(a) $1_{D_1} = f^{-1} \circ f$ is dense in $F^{-1} \bullet F$. As the latter is closed and D_1 is dense in X_1 we have $1_{X_1} = \overline{1_{D_1}} = F^{-1} \bullet F$. Similarly, $F \circ F^{-1} = 1_{X_2}$. Thus, F and F^{-1} are inverse isomorphisms in the suitable relation category. ☐

Corollary 3.20. *(a) An almost open continuous map $f : X_1 \to X_2$ between compact metric spaces is an isomorphism in the suitable relations category iff it is an irreducible map. In that case*

$$1_{X_2} \quad = \quad f \bullet f^{-1} \quad = \quad f \circ f^{-1} \qquad \textit{and} \qquad 1_{X_1} \quad = \quad f^{-1} \bullet f. \tag{3.16}$$

(b) If $F : X_1 \to X_2$ is a suitable relation then the almost open continuous map $\pi_{2F} : F \to X_2$ is a suitable relation and the irreducible map $\pi_{1F} : F \to X_1$ is a suitable relation isomorphism with inverse $(\pi_{1F})^{-1}$. Furthermore,

$$F \quad = \quad \pi_{2F} \circ (\pi_{1F})^{-1} \quad = \quad \pi_{2F} \bullet (\pi_{1F})^{-1}. \tag{3.17}$$

If F is a suitable relation isomorphism this expresses F as a suitable composition of an irreducible map and the inverse of an irreducible map.

Proof. (a) For a continuous map $f : X_1 \to X_2$, $\pi_{1f} : f \to X_1$ is a homeomorphism. Hence, f is irreducible iff $\pi_{2f} : f \to X_2$ is irreducible.

The second equations in (3.16) and (3.17) follow from Theorem 3.16(a).

The rest should by now be obvious. □

Remark: For a surjective map $f : X_1 \to X_2$ $1_{X_1} = f^{-1} \circ f$ iff f is injective and so is a homeomorphism when it is continuous. So whenever f is an almost one-to-one surjection which is not a homeomorphism we can let $G = f^{-1}$ and $F = f$ to get an example where $G \bullet F$ is a proper subset of $G \circ F$.

4. Suitable Relation Dynamics

When the domain and range of a map f or relation F are the same set E we will say f is a map on E or F is a relation on E. Thus, $F : E \to E$ is a *relation on* E and we can iterate just as we would with a map on E. Define $F^0 = 1_E, F^1 = F$ and, inductively, for positive integers n

$$F^{n+1} =_{def} F \circ F^n. \tag{4.1}$$

The inverse relation F^{-1} is a relation on E and we let

$$F^{-n} =_{def} (F^{-1})^n = (F^n)^{-1}. \tag{4.2}$$

Because composition is associative we have

$$F^{m+n} = F^m \circ F^n \tag{4.3}$$

provided that the integers m and n have the same sign. With opposite signs the result is not true in general. For example, $h : E_1 \to E_2$ is a relation then it is easy to check that h is a map iff

$$h \circ h^{-1} \subset 1_{E_2} \quad \text{and} \quad h^{-1} \circ h \supset 1_{X_1} \tag{4.4}$$

and the inclusions are equalities iff h is a bijection.

If A is a subset of E we call A a + *invariant subset* for F when $F(A) \subset A$ and an *invariant subset* when $F(A) = A$.

Now suppose that F is a suitable relation on a compact metric space X. We define the suitable iterates by $F^{\bullet 0} = 1_E, F^{\bullet 1} = F$ and, inductively, for positive integers n

$$F^{\bullet n+1} =_{def} F \bullet F^{\bullet n}. \tag{4.5}$$

Since suitable composition is associative we have

$$F^{\bullet m+n} = F^{\bullet m} \bullet F^{\bullet n} \tag{4.6}$$

for all nonnegative integers m and n. By Theorem 3.19 F^{-1} is a suitable relation iff F is an isomorphism in the suitable relations category. In that case, with

$$F^{\bullet -n} =_{def} (F^{-1})^{\bullet n} = (F^{\bullet n})^{-1}, \tag{4.7}$$

equation (4.6) holds for all integers m and n.

Recall that the continuous, weakly open map $f_F : ONE_F \to X$ is the restriction of the relation F to the dense G_δ set of points x at which $F(x)$ is a singleton. With $D_0 = X$ define, inductively, for positive integers n

$$D_n \quad = \quad (f_F)^{-1}(D_{n-1}) \quad = \quad ((f_F)^n)^{-1}(X). \tag{4.8}$$

That is, D_n is the domain of the iterated relation $(f_F)^n$. Since f_F is weakly open with a dense domain, each D_n is a dense G_δ subset of X by Proposition 2.3. With $n \in \mathbb{N} = \{0, 1, ...\}$ the first equation implies, by induction, that the sequence $\{D_n\}$ is monotone decreasing and we intersect to define

$$D_F^+ \quad =_{def} \quad \bigcap_{n \in \mathbb{N}} ((f_F)^n)^{-1}(X). \tag{4.9}$$

Theorem 4.1. *Let F be a suitable relation on X. $D_F^+ = \bigcap_n ((f_F)^n)^{-1}(X)$ is a dense G_δ subset of X. It is the set of points $x \in ONE_F$ such that $\{(f_F)^n(x) \in ONE_F\}$ is defined as an infinite sequence of points. The map f_F, or equivalently the relation F, restricts to define a map which we denote $t_F : D_F^+ \to D_F^+$.*

F is a surjective relation iff $t_F(D_F^+)$ is a dense subset of D_F^+ or, equivalently, a dense subset of X.

Let $D \subset X$ and $t : D \to D$ be a continuous map. The following are equivalent:

(i) *Regarded as a map from D to X, t is dense in F.*
(ii) *$D \subset D_F^+$ dense in X and t is the restriction of t_F to the set D.*

When these conditions hold, D is $+$invariant for t_F and t^n is dense in $F^{\bullet n}$ for every positive integer n. In particular, t_F^n is dense in $F^{\bullet n}$ for every positive integer n.

Proof. The G_δ set D_F^+ is dense in X by the Baire Category Theorem. It is clear that if $x \in D_F^+$ then $f_F(x) \in D_F^+$ and so f_F restricts to define the map t_F from D_F^+ to itself.

Proposition 3.17 (a) implies that F is surjective iff $t_F(D_F^+) = f_F(D_F^+)$ is dense.

By Corollary 3.5 (i) is equivalent to (ii) since π_{1F} is irreducible and hence surjective. Then t^n is dense in $F^{\bullet n}$ for every positive integer n by induction using Proposition 3.15(a). □

Thus, we see that the suitable iterations $F^{\bullet n}$ provide the natural closure of the dynamics given by the iterations t_F^n of the continuous map t_F on the Polish space $D_F^+ \subset ONE_F \subset X$.

Now we consider continuous maps between closed relations.

Let G be a closed relation on a compact metric space Y and $H : Y \to X$ be a closed relation between compact metric spaces. Define $H \times H : Y \times Y \to X \times X$ by $\{(x_1, x_2, x_3, x_4) : (x_1, x_3), (x_2, x_4) \in H\}$. The closed relation $H \times H$ is just the set product of the two relations with the second and third coordinates switched. The image set $(H \times H)(G) \subset X \times X$ is a closed relation on X and it is easy to check that the image can be written as a composition of closed relations:

$$(H \times H)(G) \quad = \quad H \circ G \circ H^{-1}. \tag{4.10}$$

If F is a closed relation on X then we can apply this to H^{-1} to get

$$(H \times H)^{-1}(F) \quad = \quad H^{-1} \circ F \circ H. \tag{4.11}$$

If G and F are a closed relations on Y and X and $h : Y \to X$ is a continuous map, then we will say that h *maps* G *to* F when $(h \times h)(G) \subset F$.

Proposition 4.2. *Let G, F be closed relations on the compact metric spaces Y and X respectively. Let $h : Y \to X$ be a continuous map. h maps G to F iff*

$$h \circ G \quad \subset \quad F \circ h. \tag{4.12}$$

If h maps G to F then it maps G^n to F^n for every integer n. In particular, h maps G^{-1} to F^{-1}.

If h maps G to F and h is surjective, then F has full domain (or is surjective) when G has full domain (resp. is surjective).

Assume h and π_{1G} are surjective and that π_{1F} is irreducible. If h maps G to F then $(h \times h)(G) = F$.

Proof. If $h \circ G \circ h^{-1} \subset F$ then by (4.4)

$$h \circ G \quad \subset \quad h \circ G \circ h^{-1} \circ h \quad \subset \quad F \circ h, \tag{4.13}$$

while if $h \circ G \subset F \circ h$ then by (4.4) again

$$h \circ G \circ h^{-1} \quad \subset \quad F \circ h \circ h^{-1} \quad \subset \quad F. \tag{4.14}$$

If $h \circ G \subset F \circ h$ then inductively, we have

$$h \circ G^n \ = \ h \circ G \circ G^{n-1} \ \subset \ F \circ h \circ G^{n-1} \ \subset \ F \circ F^{n-1} \circ h \ = \ F^n \circ h. \tag{4.15}$$

Alternatively beginning with $h \circ G \circ h^{-1} \subset F$, we can use (4.4) and induction

$$\begin{gathered} h \circ G^n \circ h^{-1} \ = \ h \circ G \circ G^{n-1} \circ h^{-1} \ \subset \\ h \circ G \circ h^{-1} \circ h \circ G^{n-1} \circ h^{-1} \ \subset \ F \circ F^{n-1} \ = \ F^n. \end{gathered} \tag{4.16}$$

So h maps G^n to F^n for all positive integers n. Also

$$h \circ G^{-1} \circ h^{-1} \ = \ (h \circ G \circ h^{-1})^{-1} \ \subset \ F^{-1}. \tag{4.17}$$

That is, h maps G^{-1} to F^{-1}. Hence, h maps $(G^{-1})^n$ to $(F^{-1})^n$ for all positive integers n.

If h is surjective and $G(Y) = Y$ then $X = h(G(h^{-1}(X))) \subset F(X)$ and if $G^{-1}(Y) = Y$ then $X = h(G^{-1}(h^{-1}(X))) \subset F^{-1}(X)$. Hence F is surjective when h and G are and F has full domain when h is surjective and G has full domain.

If h and π_{1G} are surjective then $(h \circ G \circ h^{-1})^{-1}(X) = X$ and so with $\tilde{F} = h \circ G \circ h^{-1}$ we have $\pi_{1\tilde{F}}$ is surjective. $\tilde{F} \subset F$ since h maps G to F. If π_{1F} is irreducible then $\tilde{F} = F$ by Lemma 3.6 $\tilde{F} = F$. □

Theorem 4.3. *Let G, F be suitable relations on the compact metric spaces Y and X respectively. Let $h : Y \to X$ be an almost open, continuous map. h maps G to F iff*

$$h \bullet G \quad = \quad F \bullet h. \tag{4.18}$$

If h maps G to F then it maps $G^{\bullet n}$ to $F^{\bullet n}$ for every nonnegative integer n.

If h maps G to F then $D^+_G \cap h^{-1}(D^+_F)$ is a G_δ dense in Y and $+$invariant for t_G. On this set, $h \circ t_G = t_F \circ h$.

Proof. By Theorem 3.16 (a) $h \circ G = h \bullet G$. Hence, given (4.18) we have

$$h \circ G \ = \ h \bullet G \ = \ F \bullet h \ \subset \ F \circ h. \tag{4.19}$$

On the other hand, if $h \circ G \subset F \circ h$ then $h \bullet G$ and $F \bullet h$ are both suitable relations from Y to X which are contained in $F \circ h$. So each is minimal for the surjection $\pi_{1F \circ h}$. By Theorem 3.14, $F \bullet h$ is the unique subset minimal for $\pi_{1F \circ h}$ and hence $h \bullet G = F \bullet h$.

From (4.18) we obtain, inductively, for all positive integers n

$$h \bullet G^{\bullet n} \ = \ h \bullet G \bullet G^{\bullet n-1} \ = \ F \bullet h \bullet G^{\bullet n-1} \ = \ F \bullet F^{\bullet n-1} \bullet h \ = \ F^{\bullet n} \bullet h. \tag{4.20}$$

Since h is almost open, the G_δ set $D^+_G \cap h^{-1}(D^+_F)$ is dense in Y. If $x \in D^+_G \cap h^{-1}(D^+_F)$ then $t_G(x) \in D^+_G$ since the latter is t_G +invariant, and $h \circ G(x) \subset F \circ h(x)$ which consists of the single point $t_F(h(x))$ because $x \in h^{-1}(D^+_F)$. Thus, $t_G(x) \in D^+_G \cap h^{-1}(D^+_F)$ and $h(t_G(x)) = t_F(h(x))$. □

For a closed relation F on a compact metric space X we define the *sample path space* X^+_F to be

$$X^+_F \quad =_{def} \quad \{z \in X^{\mathbb{N}} : (z_n, z_{n+1}) \in F \quad \text{for all} \quad n \in \mathbb{N}\ \}. \tag{4.21}$$

If the domain of F is X then we clearly have:

$$\pi_0(X^+_F) \ = \ X \qquad \text{and} \qquad \pi_0 \bigtriangleup \pi_n(X^+_F) \ = \ F^n, \tag{4.22}$$

for $n = 1, 2, ...,$ where $\pi_0 \bigtriangleup \pi_n$ is defined by $z \mapsto (z_0, z_n)$.

On $X^{\mathbb{N}}$ we define the *shift map* σ by

$$\sigma(z)_n \quad = \quad z_{n+1} \qquad \text{for} \quad n \in \mathbb{N}. \tag{4.23}$$

It is clear that X_F^+ is a closed subset which is +invariant for the shift. So σ restricts to define a continuous map σ_F on X_F^+. The continuous surjection $\pi_0 : X_F^+ \to X$ maps σ_F on X_F^+ to F by (4.22). It is easy to check that the map σ_F is surjective on X_F^+ iff the relation F is surjective.

The construction is functorial. If $h : Y \to X$ is a continuous map then the continuous map $h^{\mathbb{N}} : Y^{\mathbb{N}} \to X^{\mathbb{N}}$ is defined by using h on each coordinate. If G is a closed relation on Y and h maps G to F then it is clear that $h^{\mathbb{N}}(Y_G^+) \subset X_F^+$ and $h^{\mathbb{N}}$ maps σ_G to σ_F.

If F is a suitable relation then we define the *orbit map* $o_F^+ : D_F^+ \to X_F^+$ by

$$o_F^+(x)_n \quad = \quad (t_F)^n(x) \qquad \text{for} \quad n \in \mathbb{N}. \tag{4.24}$$

Theorem 4.4. *Let F be a suitable relation on X.*

(a) The map o_F^+ is a homeomorphism from D_F^+ onto $X_F^+ \cap (\pi_0)^{-1}(D_F^+)$. In particular, for $x \in D_F^+$ $o_F^+(x)$ is the unique point of X_F^+ with $x = o_F^+(x)_0$.

(b) The closed set $S_F^+ \subset X_F^+$ defined by

$$S_F^+ \quad = \quad \overline{o_F^+(D_F^+)} \quad = \quad \overline{X_F^+ \cap (\pi_0)^{-1}(D_F^+)} \tag{4.25}$$

is the unique subset of X_F^+ which is minimal for the restriction of π_0 to X_F^+. For $n \in \mathbb{N}$ let $\rho_n : S_F^+ \to X$ denote the restriction of π_n to S_F^+. The map $\rho_0 : S_F^+ \to X$ is irreducible and for $n = 1, 2, ...$

$$\rho_0 \triangle \rho_n(S_F^+) \ = \ F^{\bullet n}. \tag{4.26}$$

(c) S_F^+ is a closed +invariant subset of X_F^+. Let s_F denote the restriction of σ_F to S_F^+. The map s_F is almost open and ρ_0 maps s_F to F.

(d) The relation F is surjective iff s_F is a surjective map on S_F^+.

Proof. (a) If $x \in D_F^+$ then $(f_F)^n(x) \in ONE_F$ for all $n \in \mathbb{N}$ and so it is clear that $o_F^+(x)$ is the unique point of X_F^+ with $x = o_F^+(x)_0$. By definition of the product topology, continuity of f_F implies continuity of o_F^+. The continuous map π_0 on $X_F^+ \cap (\pi_0)^{-1}(D_F^+)$ is the inverse map and so o_F^+ is a homeomorphism onto its image.

(b) By part (a) the image $o_F^+(D_F^+)$ is contained in $IN_{\pi_0|X_F^+}$. Since D_F^+ is dense in X it is dense in $\pi_0(IN_{\pi_0|X_F^+})$ and the latter is dense in X. By Proposition 2.19 the closure of $IN_{\pi_0|X_F^+}$ is the unique subset of X_F^+ which is minimal for π_0 on X_F^+. Since $(\pi_0|X_F^+)^{-1}$ is continuous on $\pi_0|X_F^+(IN_{\pi_0|X_F^+})$ and this continuous map restricts to o_F^+ on the dense set D_F^+ it follows that the minimal set is the closure of the image of o_F^+. The restriction ρ_0 is irreducible because S_F^+ is minimal.

(c) The set $o_F^+(D_F^+)$ is clearly +invariant for the shift map and so its closure is as well. The homeomorphism $\rho_0 : o_F^+(D_F^+) \to D_F^+$ maps the restriction of the shift to the weakly open map t_F on D_F^+. Hence, the restriction of the shift to $o_F^+(D_F^+)$ is weakly open as well. By the Variation of Domain Theorem 2.6 s_F is weakly open on S_F^+. By Theorem 2.5 it is almost open. Since s_F is a continuous map and $o_F^+(D_F^+)$ is dense in S_F^+ it follows that the restriction of $s_F|o_F^+(D_F^+)$ is dense in s_F. Clearly, ρ_F maps $s_F|o_F^+(D_F^+)$ to t_F on D_F^+. Since t_F is a subset of the closed set F and $s_F|o_F^+(D_F^+)$ is dense in s_F it follows that $\rho_0 \times \rho_0(s_F^+) \subset F$.

(d) If F is surjective then by Theorem 4.1 the set $t_F(D_F^+)$ is dense in D_F^+ and hence $o_F^+(t_F(D_F^+)) = s_F(o_F^+(D_F^+))$ is dense in $o_F^+(D_F^+)$ which is dense in S_F^+. Since $s_F(S_F^+)$ is closed and contains $s_F(o_F^+(D_F^+))$ it equals S_F^+.

If s_F is a surjective map on S_F^+ then it is a surjective relation and ρ_0 is surjective and maps s_F to F. Hence, F is a surjective relation by Proposition 4.2. □

Theorem 4.5. *Let G and F be suitable relations on Y and X, respectively and let $h : Y \to X$ be an almost open continuous map which maps G to F. $h^{\mathbb{N}} : Y_G^+ \to X_F^+$ restricts to an almost open map from $S_G^+ \to S_F^+$. The restriction maps s_G to s_F.*

Proof. By Theorem 4.3 the restriction $h : D_G^+ \cap h^{-1}(D_F^+) \to D_F^+$ maps the restriction of t_G to t_F. These are dense sets in Y and X and h is almost open and so weakly open. By the Variation of Domain Theorem the restriction $h : D_G^+ \cap h^{-1}(D_F^+) \to D_F^+$ is weakly open. Since o_G^+ and o_F^+ are homeomorphisms, the restriction $h^{\mathbb{N}} : o_G^+(D_G^+ \cap h^{-1}(D_F^+)) \to o_F^+(D_F^+)$ is weakly open and maps the restriction of s_G to the restriction of s_F. By Variation of Domain again $h^{\mathbb{N}} : S_G^+ \to S_F^+$ is weakly open and hence almost open by Theorem 2.5. Since $o_G^+(D_G^+ \cap h^{-1}(D_F^+))$ is dense in S_G^+ and s_F is closed it follows that $h^{\mathbb{N}}$ maps s_G to s_F. □

Corollary 4.6. *Let F be a suitable relation on X and g be an almost open, continuous map on a compact metric space Y. If $h : Y \to X$ is an almost open continuous map which maps g to F then there exists a unique continuous map $h_1 : Y \to S_F^+$ which maps g to s_F. The map h_1 is almost open and satisfies $\rho_0 \circ h_1 = h$.*

Proof. $h^{\mathbb{N}} : S_g^+ \to S_F^+$ is an almost open map taking s_g to s_F and $\rho_0 : S_g^+ \to Y$ maps s_g to g. Because g is a map $D_g^+ = ONE_g = Y$ and the orbit map $o_g^+ : Y \to S_g^+$ is a homeomorphism inverse to ρ_0 on S_g^+. Hence, $h_1 = h^{\mathbb{N}} \circ o_g^+$ is the required almost open continuous map. Uniqueness is obvious on the dense set $h^{-1}(D_F^+)$ and so by continuity uniqueness follows on all of Y. □

There is a category theory description of what is happening here. The *suitable dynamics category* has as objects suitable relations on compact metrizable spaces and has as morphisms almost open maps between them. The *almost open dynamics category* is the full subcategory whose objects are almost open maps on compact metrizable spaces. The association from F on X to s_F on S_F^+ is a functor from the former category to the latter which is adjoint to the inclusion functor of the latter into the former.

Let F be a suitable relation on X and g be a continuous map on a compact metric space Y. We will say that a continuous map $h : Y \to X$ *resolves the discontinuities of* F *via* g if h maps g to F and there exists a dense $D \subset Y$ which is $+$invariant for g such that h restricts to a homeomorphism $h : D \to h(D)$ where $h(D)$ is a dense subset of ONE_F.

Corollary 4.7. *Let F be a suitable relation on X. If $h : Y \to X$ resolves the discontinuities of F via the continuous map g on Y then h is an irreducible map with $h(D) \subset D_F^+$. h maps g on D to t_F on $h(D)$. The map g is almost open. Furthermore, there exists a unique continuous map $h_1 : Y \to S_F^+$ which maps g to s_F. The map h_1 is irreducible and satisfies $\rho_0 \circ h_1 = h$.*

Proof. Because $h : D \to h(D)$ is a homeomorphism both it and its inverse are weakly open. By the usual Variation of Domain argument, density of D implies that h is almost open and that it is the closure of the restriction of h to D. Since $h(D)$ is dense, h is surjective and the relation h^{-1} is the closure of $h^{-1} : h(D) \to D$. By Proposition 3.9 the inverse h^{-1} as well as h are suitable relations. By Corollary 3.20 h is irreducible.

Clearly, h maps the orbit sequence of a point $y \in D$ to the orbit sequence of $h(y)$. That is, $h(g^n(y)) = (f_F)^n(h(y))$ for every positive integer n and so $h(y) \in D_F^+$ and h maps g on D to t_F on D_F^+.

By the Variation of Domain Theorem again, the restriction of t_F to $h(D)$ is weakly open and so composing with the homeomorphism $h|D$ and its inverse we see that g is weakly open on D. As usual, g is almost open on Y.

The existence of h_1 now follows from Corollary 4.6. It is irreducible by Proposition 2.18 h_1 is irreducible because h is. □

Thus, ρ_0 resolves the discontinuities of F via s_F and Corollary 3.20 says that every other resolution factors through this one.

In resolving the discontinuities the space X is replaced by S_F^+ which is usually more topologically complicated as the points of X are "split" the relation F in S_F^+. However, when F is simple, the space Y can be simple as well. Consider from the Introduction the suitable relation F_{01} on the unit interval. Define $Y = ([0, \frac{1}{2}] \times \{0\}) \cup ([\frac{1}{2}, 1] \times \{1\})$.

Define the homeomorphism g on Y by

$$(4.27) \qquad g(x,i) \quad = \quad (i + \frac{1}{2} - x, i) \qquad \text{for} \quad i = 0, 1.$$

That is, g flips each interval about its midpoint. The projection $\pi_1 : Y \to [0.1]$ maps g into F and resolves the discontinuities of the quasi-continuous maps f_0 and f_1.

We can sharpen the above results when F is an isomorphism in the suitable relations category. If F is a suitable relations isomorphism on X then (3.16) says

(4.28)

$$(ONE_F \times ONE_{F^{-1}}) \cap F \quad = \quad (f_F)^{-1}(ONE_{F^{-1}}) \times (f_{F^{-1}})^{-1}(ONE_F).$$

That is, if $(x,y) \in F$ with x and y in the domains of f_F and $f_{F^{-1}}$ respectively then $y = f_F(x)$ and $x = f_{F^{-1}}(y)$ and these sets are dense.

The definition of the dense invariant set D_F on which f_F is a homeomorphism will require a bit of delicacy. With $\tilde{D}_0 = X$ define, inductively, for positive integers n the monotone decreasing sequence of sets

$$(4.29) \qquad \tilde{D}_n \quad = \quad (f_F)^{-1}(\tilde{D}_{n-1}) \cap (f_{F^{-1}})^{-1}(\tilde{D}_{n-1}).$$

Since f_F and $f_{F^{-1}}$ are weakly open with a dense domain, each $\tilde{D}_n$ is a dense G_δ subset of X by Proposition 2.3. With $n \in \mathbb{N} = \{0, 1, ...\}$ we intersect to define

$$(4.30) \qquad D_F \quad =_{def} \quad \bigcap_{n \in \mathbb{N}} \tilde{D}_n \quad \subset \quad D^+_F \cap D^+_{F^{-1}}.$$

The final inclusion is obvious but it might be proper. For example, if $F = f$ is an irreducible map then $D^+_F = X$ but $D^+_{f^{-1}}$ need not be +invariant for f. A point x is in D_F when the sequences $\{(f_F)^n(x)\}$ and $\{(f_{F^{-1}})^n(x)\}$ remain in both the domain of f_F and $f_{F^{-1}}$ for every positive integer n. The following is clear and we omit the proof.

Theorem 4.8. *Let F be a suitable relations isomorphism on X. D_F is a dense G_δ subset of X. The map t_F, or equivalently the relation F, restricts to define a homeomorphism on D_F whose inverse is the restriction of $t_{F^{-1}}$.*

For a closed relation F on a compact metric space X we define the *sample path space* X_F when F is surjective, i.e. $F(X) = F^{-1}(X) = X$.

$$(4.31) \qquad X_F \quad =_{def} \quad \{z \in X^{\mathbb{Z}} : (z_n, z_{n+1}) \in F \quad \text{for all} \quad n \in \mathbb{Z} \}.$$

Since F is surjective we have:

$$(4.32) \qquad \pi_0(X_F) \; = \; X \qquad \text{and} \qquad \pi_0 \bigtriangleup \pi_n(X_F) \; = \; F^n,$$

for $n \in \mathbb{Z}$, where $\pi_0 \bigtriangleup \pi_n$ is defined by $z \mapsto (z_0, z_n)$.

On $X^{\mathbb{Z}}$ we define the *shift homeomorphism* σ by

$$\sigma(z)_n \quad = \quad z_{n+1} \qquad \text{for} \quad n \in \mathbb{Z}. \tag{4.33}$$

It is clear that X_F is a closed subset which is invariant for the shift. So σ restricts to define a homeomorphism σ_F on X_F. The continuous surjection $\pi_0 : X_F \to X$ maps σ_F on X_F to F and its inverse to F^{-1} by (4.32).

If F is a suitable relations isomorphism then we define the *orbit map* $o_F : D_F \to X_F$ by

$$o_F(x)_n \quad = \quad (t_F)^n(x) \qquad \text{for} \quad n \in \mathbb{Z}. \tag{4.34}$$

Theorem 4.9. *Let F be a suitable relations isomorphism on X.*

(a) The map o_F is a homeomorphism from D_F onto $X_F \cap (\pi_0)^{-1}(D_F)$. In particular, for $x \in D_F$ $o_F(x)$ is the unique point of X_F with $x = o_F(x)_0$.

(b) The closed set $S_F \subset X_F$ defined by

$$S_F \quad = \quad \overline{o_F(D_F)} \quad = \quad \overline{X_F \cap (\pi_0)^{-1}(D_F)} \tag{4.35}$$

is the unique subset of X_F which is minimal for the restriction of π_0 to X_F. For $n \in \mathbb{Z}$ let $\rho_n : S_F \to X$ denote the restriction of π_n to S_F. The map $\rho_0 : S_F \to X$ is irreducible and for $n \in \mathbb{Z}$

$$\rho_0 \triangle \rho_n(S_F) \;=\; F^{\bullet n}. \tag{4.36}$$

(c) S_F is a closed invariant subset of X_F. Let s_F denote the restriction of σ_F to S_F. The map s_F is a homeomorphism and ρ_0 maps s_F to F.

Proof. As the proof is completely analogous to that of Theorem 4.4 we will omit it, leaving the adjustments to the reader. □

From Theorem 2.22 it follows that every space without isolated point is isomorphic in the suitable relations category to the Cantor Set. We show that every suitable relation is isomorphic to a continuous almost open map on the Cantor Set.

Theorem 4.10. *Let F be a suitable relation on X. If X has no isolated points then there exists an almost open continuous map g on the Cantor Set C and an irreducible map $h : C \to X$ which maps g to F. Furthermore, if F is a suitable relations isomorphism then g can be chosen to be a homeomorphism on C.*

Proof. By Theorem 2.22 there is an irreducible map $h_1 : C \to X$. By Corollary 3.20 h_1 is a suitable relations isomorphism and so we can define a suitable relation F_1 on C by $F_1 = (h_1)^{-1} \bullet F \bullet h_1$ and so $h_1 \bullet F_1 = F \bullet h$. By Theorem 4.3 h_1 maps F_1 to F.

Apply Theorem 4.4 to F_1. We obtain an almost open map s_{F_1} on $S^+_{F_1}$ and a irreducible map $\rho_0 : S^+_{F_1} \to C$ which maps s_{F_1} to F_1. Now $S^+_{F_1}$ is a closed subset of $C^{\mathbb{N}}$ and so is a compact, zero-dimensional, metrizable space. Since C has no isolated points and ρ_0 is irreducible, Lemma 2.21 implies that $S^+_{F_1}$ has no isolated points. Hence, there is a homeomorphism $h_3 : C \to S^+_{F_1}$. Let $g = (h_3)^{-1} \circ s_{F_1} \circ h_3$. Clearly g is an almost open map on C. Furthermore, $h_1 \circ \rho_0 \circ h_3$ is irreducible and maps g to F.

If F is a suitable relations isomorphism then so is F_1. We apply Theorem 4.9 to F_1 instead of Theorem 4.4. This replaces the almost open map s_{F_1} on $S^+_{F_1}$ by the homeomorphism s_{F_1} on S_{F_1}. Then proceed as before to obtain the homeomorphism g. □

References

[1] E. Akin, *The general topology of dynamical systems*, Amer. Math. Soc., Providence, R.I. (1993).

[2] A. Crannell and M. Martelli , *Dynamics of quasicontinuous systems*, J. of Difference Equations and Applications **6:** (2000), 351–361.

[3] A. Crannell, M. Frantz and M. LeMasurier, *Closed relations and equivalence classes of quasicontinuous functions*, Real Analysis Exchange, **31.2:** (2006), 409–424.

E-mail address: ethanakin@earthlink.net

Mathematics Department, The City College, 137 Street and Convent Avenue, New York City, NY 10031, USA