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APPLICATIONS OF CONVERGENCE SPACES TO VECTOR LATTICE THEORY

JAN HARM VAN DER WALT

Abstract. We introduce the concept of a locally solid convergence vector lattice as a generalization of locally solid Riesz spaces. This notion provides an appropriate context for a number of natural modes of convergence that cannot be described in terms of the usual Hausdorff-Kuratowski-Bourbaki (HKB) notion of topology. We discuss the dual and completion of a locally solid convergence vector lattice. As applications of this new concept we present a Closed Graph Theorem for linear operators on a class of vector lattices, and a duality result for locally convex, locally solid Riesz spaces.

1. Introduction

Many of the most important spaces that arise in analysis are vector lattices. Recall [1, page 3] that a subset A of a vector lattice L is called *solid* whenever

$$\begin{array}{ll} \forall & f \in A,\ g \in L\ : \\ & |g| \leq |f| \Rightarrow g \in A. \end{array} \tag{1.1}$$

A vector space topology on L is called *locally solid* if it has a basis at 0 consisting of solid sets [1, Definition 5.1].

As a first example, we may note that the space $\mathcal{C}(X)$ of continuous, real valued functions on a topological space is a vector lattice with respect to the pointwise ordering, and comes equipped with a host of useful topologies, such as the topology of pointwise convergence. Recall that this topology has a basis at 0 consisting of sets of the form

$$B(\epsilon, F) = \{f \in \mathcal{C}(X)\ :\ |f(x)| < \epsilon,\ x \in F\}, \tag{1.2}$$

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where F is any finite subset of X. Note that the sets $B(\epsilon, F)$ all satisfy

$$\begin{array}{l} \forall \quad f \in B(\epsilon, F), \ g \in \mathcal{C}(X) \ : \\ \qquad |g| \leq |f| \Rightarrow g \in B(\epsilon, F). \end{array} \tag{1.3}$$

Therefore the topology of pointwise convergence is an example of a locally solid topology on $\mathcal{C}(X)$. Similarly, the spaces $L^p(\Omega)$, with Ω a σ-finite measure space and $1 \leq p \leq \infty$, are locally solid Riesz spaces with respect to their natural norm topologies. More generally, any Banach lattice is a locally solid Riesz space. On the other hand, a number of fundamental topological processes on a vector lattice cannot be described in terms of HKB topology, see Examples 1.2 and 1.4 in the following.

Recall [8] that topological convergence of nets is characterized by the so called Moore-Smith Axioms (MS1) to (MS4) given below in terms of sequences.

Consider a mapping $\sigma : \mathcal{S} \ni s \mapsto \sigma(s) \in \mathcal{P}(X)$ from the set $\mathcal{S}$ of sequences in X to the powerset $\mathcal{P}(X)$ of X. Here $\sigma(s)$ is interpreted as the set of limits of s. Note that if $\sigma(s) = \emptyset$, then s does not converge to any member of X. If there is a topology τ on X so that $\sigma(s)$ is the set of τ-limits of s for all $s \in \mathcal{S}$, then the following conditions are satisfied:

(MS1) If s is the sequence with all terms equal to x, then $x \in \sigma(s)$.
(MS2) If $x \in \sigma(s)$, then $x \in \sigma(s')$ for all subsequences of s' of s.
(MS3) If every subsequence s' of s contains a further subsequence s'' so that $x \in \sigma(s'')$, then $x \in \sigma(s)$.
(MS4) Assume that $s^n = (x^n_m)$, $x_n \in \sigma(s^n)$ for every $n \in \mathbb{N}$, and $x \in \sigma(s)$ where $s = (x_n)$. Then there exists a strictly increasing mapping $\delta : \mathbb{N} \to \mathbb{N}$ so that $x \in s'$ where $s' = (x^n_{\delta(n)})$.

Condition (MS3) is usually referred to as the Urysohn property, while (MS4) is, for obvious reasons, called the Diagonal property.

A more general notion of topology, which may describe notions of convergence that do not satisfy the Moore-Smith Axioms, is that of a convergence structure and a convergence space, see for instance [3].

Definition 1.1. A convergence structure on a set X is a mapping λ from X into powerset of the set of all filters on X that satisfies the following:

(i) For all $x \in X$, the filter $[x] = \{F \subseteq X \ : \ x \in F\}$ belongs to $\lambda(x)$.
(ii) If $\mathcal{F} \in \lambda(x)$ and $\mathcal{G} \supseteq \mathcal{F}$, then $\mathcal{G} \in \lambda(x)$.
(iii) If $\mathcal{F}, \mathcal{G} \in \lambda(x)$, then $\mathcal{F} \cap \mathcal{G} \in \lambda(x)$.

If λ is a convergence structure on X, then the pair (X, λ) is called a convergence space. If there is no ambiguity, we will denote the pair (X, λ) by X alone. If $\mathcal{F} \in \lambda(x)$ we say that $\mathcal{F}$ converges to x.

Every topology τ on a set X is identified with a convergence structure λ_τ on X given by

$$\mathcal{F} \in \lambda_\tau(x) \Leftrightarrow \mathcal{F} \text{ contains the } \tau\text{−neighborhood filter at } x.$$

However, not all convergence structures can be defined in this way.

In connection with the aforementioned non-topological modes of convergence on a vector lattice L, we recall the following well known topological notion of convergence.

Example 1.2. Let X be a compact topological space. A sequence (f_n) in $\mathcal{C}(X)$ converges to $f \in \mathcal{C}(X)$ in the uniform topology whenever

$$\begin{array}{ll} \forall & \epsilon > 0 \ : \\ \exists & N_\epsilon \in \mathbb{N} \ : \\ \forall & x \in X,\ n \in \mathbb{N},\ n \geq N_\epsilon \ : \\ & |f(x) - f_n(x)| < \epsilon. \end{array} \tag{1.4}$$

Note that the condition (1.4) in Example 1.5 is equivalent to

$$\begin{array}{ll} \exists & u \in \mathcal{C}(X),\ u \geq 0 \ : \\ \forall & \epsilon > 0 \ : \\ \exists & N_\epsilon \in \mathbb{N} \ : \\ \forall & n \in \mathbb{N},\ n \geq N_\epsilon \ : \\ & -\epsilon u \leq f - f_n \leq \epsilon u, \end{array}$$

which is formulated entirely in terms of the vector lattice structure of $\mathcal{C}(X)$. Thus one may generalize the notion of uniform convergence in $\mathcal{C}(X)$ to an arbitrary vector lattice L, see for instance [6, Theorem 16.2].

Definition 1.3. A sequence (f_n) in L converges *relatively uniformly* to $f \in L$ whenever

$$\begin{array}{ll} \exists & u \in L,\ u \geq 0 \ : \\ \forall & \epsilon > 0 \ : \\ \exists & N_\epsilon \in \mathbb{N} \ : \\ \forall & n \in \mathbb{N},\ n \geq N_\epsilon \ : \\ & |f - f_n| \leq \epsilon u. \end{array} \tag{1.5}$$

In contradistinction with uniform convergence on $\mathcal{C}(X)$, relatively uniform convergence on an arbitrary vector lattice is, in general, not induced by a topology. In this regard, we denote by $\mathcal{C}_0(\mathbb{R})$ the space of continuous, real valued functions defined on $\mathbb{R}$ with compact carrier, that is, each $f \in \mathcal{C}_0(\mathbb{R})$ satisfies $f(x) = 0$ for all x outside some compact set $K \subset \mathbb{R}$. The space $\mathcal{C}_0(\mathbb{R})$ is a vector lattice with respect to the usual pointwise order. In this space, relatively uniform convergence does not satisfy the Diagonal Property, see [6, Excercise 16.9].

Example 1.4. Let $(\Omega, \mathcal{M}, m)$ be a σ-finite measure space and $1 \leq p \leq \infty$. A sequence (f_n) in $L^p(\Omega)$ converges *boundedly almost everywhere* to $f \in L^p(\Omega)$ whenever

$$
\begin{array}{ll}
\exists & u \in L^p(\Omega),\ E \subset \Omega,\ m(E) = 0\ : \\
\forall & x \in \Omega \setminus E\ : \\
 & \quad 1)\quad |f_n(x)| \leq u(x),\ n \in \mathbb{N} \\
 & \quad 2)\quad (f_n(x)) \textit{ converges to } f(x) \textit{ in } \mathbb{R}.
\end{array}
\tag{1.6}
$$

The relevance of the concept of boundedly almost everywhere convergence is demonstrated by its role in the Lebesgue Dominated Convergence Theorem: If a sequence (f_n) converges boundedly almost everywhere to f in $L^1(\Omega)$, then $\displaystyle\int f_n(x)dx$ converges to $\displaystyle\int f(x)dx$.

We may note that (1.6) is equivalent to the condition

$$
\begin{array}{ll}
\exists & (u_n) \subset L^p(\Omega)\ : \\
 & \quad 1)\quad -u_n \leq f - f_n \leq u_n,\ n \in \mathbb{N} \\
 & \quad 2)\quad u_{n+1} \leq u_n,\ n \in \mathbb{N} \\
 & \quad 3)\quad \inf\{u_n\ :\ n \in \mathbb{N}\} = 0.
\end{array}
\tag{1.7}
$$

Since (1.7) is formulated only in terms of the vector lattice structure of $L^p(\Omega)$, we may generalize the notion of boundedly almost everywhere convergence to an arbitrary vector lattice L, see for instance [6, Theorem 16.1].

Definition 1.5. A sequence (f_n) in L *order converges* to $f \in L$ whenever

$$
\begin{array}{ll}
\exists & (u_n) \subset L\ : \\
 & \quad 1)\quad -u_n \leq f - f_n \leq u_n,\ n \in \mathbb{N} \\
 & \quad 2)\quad u_{n+1} \leq u_n,\ n \in \mathbb{N} \\
 & \quad 3)\quad \inf\{u_n\ :\ n \in \mathbb{N}\} = 0.
\end{array}
\tag{1.8}
$$

In general there is no topology on L that induces order convergence of sequences. Indeed, even on $L^1(\Omega)$ order convergence is not induced by a topology. In fact, while order convergence in this space satisfies the Diagonal Property [6, Theorem 71.8], the Urysohn property may fail, see [9]: There exists a sequence (f_n) in L, which does not order converge to 0, with the property that every subsequence of (f_n) contains a further subsequence that order converges to 0. Thus order convergence in $L^1(\Omega)$ is not induced by a topology, since convergent sequences in a topological space satisfy the Urysohn Property. Note that, in general, order convergence satisfies neither the Urysohn property, nor the Diagonal property, see for instance [2].

As indicated in Examples 1.2 and 1.4, relatively uniform convergence and order convergence are both natural generalizations of important modes of convergence. Furthermore, the aforementioned concepts of convergence play a fundamental role in vector lattice theory. However, neither relatively uniform convergence, nor order convergence can be completely described in terms of HKB topology. Thus a more general notion of topology is needed to provide a suitable context for these and other non-topological modes of convergence that appear in vector lattice theory. In this regard, we introduce the concept of a locally solid convergence vector lattice as a generalization of that of a locally solid Riesz space [1]. This notion is shown to provide a suitable context for both relatively uniform convergence and order convergence. Moreover, the full power and utility of the theory of convergence vector spaces is now at one's disposal. In this regard, we apply such methods to obtain a Closed Graph Theorem, as well as a duality result for locally convex locally solid Riesz spaces.

For notation and results related to vector lattices we refer the reader to [1, 6]. All definitions and results concerning convergence spaces may be found in [3, 4].

2. Locally solid convergence vector lattices

We now introduce the core concept to which this paper is devoted. Namely, a locally solid convergence vector lattice.

Definition 2.1. Let L be a vector lattice. A vector space convergence structure λ on L is called *locally solid*, and the pair (L, λ) a *locally solid convergence vector lattice*, if for every $\mathcal{F} \in \lambda(0)$ there is a coarser filter $\mathcal{G} \in \lambda(0)$ that has a basis consisting of solid sets.

Clearly every locally solid Riesz space L is a locally solid convergence vector lattice. Indeed, a filter $\mathcal{F}$ converges to 0 in L if and only if $\mathcal{V}(0) \subseteq \mathcal{F}$, where $\mathcal{V}(0)$ denotes the neighborhood filter at 0 in L. Since L is locally solid, $\mathcal{V}(0)$ has a basis consisting of solid sets. Two more examples are given in the following.

Example 2.2. Let L be an Archimedean vector lattice. A filter $\mathcal{F}$ on L converges to $f \in L$ with respect to the order convergence structure λ_o, see [2, 11], if and only if

$$\begin{array}{ll} & \exists \quad (l_n), (u_n) \subset L \ : \\ (2.1) & \quad (1) \quad \sup\{l_n \ : \ n \in \mathbb{N}\} = f = \inf\{u_n \ : \ n \in \mathbb{N}\} \\ & \quad (2) \quad l_n \leq l_{n+1} \leq u_{n+1} \leq u_n, \ [l_n, u_n] \in \mathcal{F}, \ n \in \mathbb{N}. \end{array}$$

The convergence structure λ_o is a Hausdorff vector space convergence structure, and is clearly locally solid. Furthermore, a sequence (f_n) in L converges to $f \in L$ with respect to λ_o if and only if (f_n) order converges to f.

Example 2.3. Let L be an Archimedean vector lattice and denote by λ_r the Mackey modification of λ_o, that is, a filter $\mathcal{F}$ converges to 0 with respect to λ_r if and only if there is a set $U \subseteq L$ which is bounded with respect to λ_o so that $\mathcal{N}U \subseteq \mathcal{F}$, where $\mathcal{N}$ denotes the neighbourhood filter at 0 in $\mathbb{R}$. According to [12, Proposition 2.1], the bounded sets with respect to λ_o are precisely the order bounded sets in L. Thus the convergence structure λ_r may be characterized as follows [12, Corollary 2.2]:

$$\mathcal{F} \in \lambda_r(0) \Leftrightarrow \left(\begin{array}{ll} \exists & u \in L^+ \ : \\ \forall & n \in \mathbb{N} \ : \\ & [-\frac{1}{n}u, \frac{1}{n}u] \in \mathcal{F}. \end{array} \right. \tag{2.2}$$

It is immediate that λ_r is a locally solid vector space convergence structure on L. Moreover, a sequence (f_n) in L converges to $f \in L$ with respect to λ_r if and only if (f_n) converges relatively uniformly to f.

The relatively uniform convergence structure λ_r satisfies the following minimality property.

Proposition 2.4. *Let L be a a locally solid convergence vector lattice. For any $u \in L^+$, the filter $\mathcal{F} = [\{[-\frac{1}{n}u, \frac{1}{n}u] \ : \ n \in \mathbb{N}\}]$ converges to 0 in L. In particular, if L is an Archimedean vector lattice, then the relatively uniform convergence structure λ_r is the finest locally solid convergence structure on L.*

Proof. Let λ be a locally solid convergence structure on L. Consider any $u > 0$ in L. We show that the filter

$$\mathcal{F} = [\{[-\frac{1}{n}u, \frac{1}{n}u] \ : \ n \in \mathbb{N}\}]$$

converges to 0 with respect to λ. In this regard, we recall that, due to the joint continuity of scalar multiplication, the filter $\mathcal{N}u$ converges to 0 in L with respect to any vector space convergence structure, thus in particular with respect to λ. Since λ is locally solid, it follows that there is some filter $\mathcal{G}$ with a basis $\mathcal{B}$ of solid sets so that $\mathcal{G}$ converges to 0 and $\mathcal{G} \subseteq \mathcal{N}u$. Thus for each $B \in \mathcal{B}$ there exists $n_B \in \mathbb{N}$ such that $\{\alpha u \ : \ |\alpha| \leq \frac{1}{n_B}\} \subseteq B$. Since each $B \in \mathcal{B}$ is solid, it follows that $[-\frac{1}{n}u, \frac{1}{n}u] \subseteq B$ for all $n \in \mathbb{N}, \ n \geq n_B$, which implies that $\mathcal{G} \subseteq \mathcal{F}$. Thus $\mathcal{F} \in \lambda(0)$.

If L is Archimedean, it now follows from Example 2.3 that λ_r is the finest locally solid convergence structure on L. □

The remainder of this section is devoted to some basic results on locally solid convergence vector lattices. These are mainly straightforward generalizations of results that hold for locally solid Riesz spaces, which can be found in [1, Section 5]. Therefore we will not give complete proofs of all results.

Proposition 2.5. *Let λ be a vector space convergence structure on a vector lattice L. Consider the following statements:*

(i) *λ is a locally solid convergence structure.*
(ii) *The mapping $\bigvee : L \times L \ni (f,g) \mapsto f \vee g \in L$ is uniformly continuous.*
(iii) *The mapping $\bigwedge : L \times L \ni (f,g) \mapsto f \wedge g \in L$ is uniformly continuous.*
(iv) *The mapping $L \ni u \mapsto |u| \in L$ is uniformly continuous.*
(v) *The mapping $L \ni u \mapsto u^+ \in L$ is uniformly continuous.*
(vi) *The mapping $L \ni u \mapsto u^- \in L$ is uniformly continuous.*

The statement (i) implies (ii) to (vi).

Proof. We show that (i) implies (ii). In this regard, consider filters $\mathcal{F}, \mathcal{G} \in \lambda(0)$. Without loss of generality, we may assume that $\mathcal{F}$ and $\mathcal{G}$ each have a basis $\{F_i \ : \ i \in I\}$, respectively $\{G_j \ : \ j \in J\}$, consisting of solid sets. It must be shown that there is a filter $\mathcal{H} \in \lambda(0)$ so that $\Delta(\mathcal{H}) \subseteq (\bigvee \times \bigvee)(\Delta(\mathcal{F} \times \mathcal{G}))$ where $\Delta(\mathcal{H}) = [\{\Delta(H) \ : \ H \in \mathcal{H}\}]$, with $\Delta(H) = \{(u,v) \in L \times L \ : \ u - v \in H\}$. The filter $\Delta(\mathcal{F} \times \mathcal{G})$ is defined in the same way. We note that $(\bigvee \times \bigvee)(\Delta(\mathcal{F} \times \mathcal{G}))$ is based on sets of the form $H_{i,j} = \{(u \vee v, f \vee g) \ : \ u - f \in F_i, \ v - g \in G_j\}$, with $i \in I$ and $j \in J$. For all $f,g,u,v \in L$ the inequality $|u \vee v - f \vee g| \leq |f-u| + |g-v|$ holds. Therefore

$$H_{i,j} \subseteq \left\{ (u,v) \, \middle| \, \begin{array}{ll} \exists & f \in F_i, \ g \in G_j \ : \\ & |u-v| \leq |f| + |g| \end{array} \right\}$$

for all $i \in I$ and $j \in J$. Since F_i and G_j are solid, it therefore follows that $H_{i,j} \subseteq \{(u,v) \ : \ u - v \in F_i + G_j\} = \Delta(F_i + G_j)$. Hence we have $\Delta(\mathcal{F} + \mathcal{G}) \subseteq (\bigvee \times \bigvee)(\Delta(\mathcal{F} \times \mathcal{G}))$, which verifies that (i) implies (ii).
It is easily seen that (ii) implies (iii), (v) and (vi), while (iv) follows from (v) and (vi). Thus the proof is complete. □

Remark 2.6. We note that conditions (ii) to (vi) are in fact equivalent. Furthermore, in the topological case (i) is also equivalent to each of (ii) to (vi). However, this is not the case for vector space convergence structures in general, as will be shown in Section 5.

Proposition 2.7. *Let L be a locally solid convergence vector lattice, with convergence structure λ. The following statements are true.*

(i) *The order bounded subsets of L are bounded with respect to λ.*
(ii) *The adherence of a solid subset of L is solid.*
(iii) *The adherence of a Riesz subspace of L is a Riesz subspace of L. In particular, the adherence of an ideal in L is an ideal in L.*

Proof of (i). This follows immediately from Proposition 2.4.
(ii). Let $A \subseteq L$ be a solid set and $f \in a(A)$. Then there is a filter $\mathcal{F} \in \lambda(f)$ so that $A \in \mathcal{F}$. Consider $g \in L$ with $|g| \leq |f|$ and the filter $\mathcal{G} = \bigvee(\bigwedge(g, |\mathcal{F}|), -|\mathcal{F}|)$. According to Proposition 2.5, $\mathcal{G}$ converges to $(g \wedge |f|) \vee (-|f|) = g$. For every $h \in A$ we have $(g \wedge |h|) \vee (-|h|) \leq |h|$. Since A is solid it follows that $\bigvee(\bigwedge(g, |A|), -|A|) \subseteq A$ so that $A \in \mathcal{G}$, thus $g \in a(A)$.
(iii). Let K be a Riesz subspace of L and consider $f \in a(K)$. It follows from Proposition 2.5 and [3, Lemma 1.3.5] that $f^+ \in a(K)$. Since $a(K)$ is a linear subspace of L the result follows from [6, Theorem 11.8]. If K is also an ideal in L, then it follows from (ii) above that $a(K)$ is an ideal in L. □

Proposition 2.8. *Let L be a Hausdorff locally solid convergence vector lattice with convergence structure λ. Then the following are true:*

(i) *L is Archimedean.*
(ii) *The cone L^+ of L is closed.*
(iii) *If an increasing net (f_α) converges to f in L, then (f_α) increases to f in L. Similarly, if (f_α) is decreasing and converges to f in L, then (f_α) decreases to f in L.*
(iv) *Every band of L is closed in L.*

Proof of (i). Suppose that there exists $f, g \in L$ such that $0 \leq nf \leq g$ for all $n \in \mathbb{N}$. Then $\mathcal{G} = [\{[0, \frac{1}{n}g] \ : \ n \in \mathbb{N}\}] \subseteq [f]$. But according to Proposition 2.4 the filter $\mathcal{G}$ converges to 0 in L so that $[f]$ converges to 0 in L. Since L is Hausdorff it follows that $f = 0$.

(ii). Consider $f \in a(L^+)$. Then there is a filter $\mathcal{F} \in \lambda(f)$ so that $L^+ \in \mathcal{F}$. Without loss of generality we may assume that $\mathcal{F}$ has a basis consisting of sets contained in L^+. Hence $|\mathcal{F}| = \mathcal{F}$. But $|\mathcal{F}| \in \lambda(|f|)$ according to Proposition 2.5. Since L is Hausdorff it therefore follows that $f = |f| \in L^+$, thus $a(L^+) = L^+$ so that L^+ is closed.

(iii). The proof follows in precisely the same manner as in the topological case, see for instance [1, Theorem 5.6].

(iv). We claim that $D^d = \{g \in L \ : \ |f| \vee |g| = 0, \ f \in D\}$ is closed in L for all nonempty $D \subseteq L$. In this regard, let $\mathcal{F}$ be a filter converging to some $h \in L$ and containing D^d. Without loss of generality we may assume that $\mathcal{F}$ has a basis consisting of subsets of D^d. Then $[0] = \bigwedge(|f|, |\mathcal{F}|)$ for all $f \in D$. But $\bigwedge(|f|, |\mathcal{F}|)$ converges to $|f| \wedge |h|$ by Proposition 2.5. Since L is Hausdorff it follows that $|f| \wedge |h| = 0$ for all $f \in D$. Hence $h \in D^d$ so that $a(D^d) = D^d$, where $a(D^d)$ denotes the adherence of D^d in L, see [3, Definition 1.3.1 (iii)]. Thus D^d is closed. Now recall [6, Theorem 22.3] that, since L is Archimedean by (i), $A = A^{dd}$ for each band A in L, so that $a(A) = A$. □

3. Permanence properties

In this section we consider some permanence properties of the class of locally solid convergence vector lattices. In particular, we consider initial and final vector space convergence structures. In general, neither initial nor final constructions preserve local solidity, as the following examples show.

Example 3.1. Let $L = \mathbb{R}^2$, with the usual coordinate-wise order, that is, $(u_1, u_2) \leq (v_1, v_2)$ if and only if $u_1 \leq v_1$ and $u_2 \leq v_2$. Let $K = \mathbb{R}^2$ equipped with the lexicographical order, that is, $(u_1, u_2) \leq (v_1, v_2)$ if and only if $u_1 < v_1$ or $u_1 = v_1$ and $u_2 \leq v_2$. Note that L is an Archimedean vector lattice, while K is non-Archimedean, see for instance [6, Example 18.6]. Let $I : K \to L$ denote the identity mapping. Consider on L the usual metric topology, which is locally solid, and on K the initial convergence structure with respect to I. Since I is injective, the initial convergence structure on L is Hausdorff. Thus, according to Proposition 2.8, this convergence structure is not locally solid, as K is non-Archimedean.

In fact, there is no locally solid convergence structure on K making I continuous. To see that this is so, suppose that λ is a locally solid convergence structure on K with respect to which I is continuous. Note that for $f = (1, 1) \in K$, the interval $A = [-f, f]$ is bounded in K by Proposition 2.7. Since I is continuous, $I(A) = A$ is bounded in L. But

$$\begin{aligned} A &= \{(g_1, g_2) \ : \ -1 < g_1 < 1\} \cup \{(-1, g_2) \ : \ g_2 \geq -1\} \\ &\quad \cup \{(1, g_2) \ : \ g_2 \leq 1\} \end{aligned} \tag{3.1}$$

is clearly not a bounded set in L.

Example 3.2. Let $K = \mathcal{C}(\mathbb{R})$, and let L be the vector lattice subspace of K consisting of all functions that vanish identically outside $(0, 1)$. Consider any locally solid convergence structure on L, and the final vector space convergence structure on K with respect to the inclusion mapping $\iota_L : L \ni f \mapsto f \in K$. Suppose that K is locally solid. According to Example 2.2 and Proposition 2.4, the filter

$$\mathcal{F} = [\{[-\frac{1}{n}, \frac{1}{n}] \ : \ n \in \mathbb{N}\}] \tag{3.2}$$

converges to 0 in K. Therefore, according to [3, Proposition 3.3.6] there exists a filter $\mathcal{G}$ converging to 0 in L, and $f_1, ..., f_k \in K$ such that

$$\mathcal{G} + \mathcal{N} f_1 + ... + \mathcal{N} f_k \subseteq \mathcal{F}. \tag{3.3}$$

This implies that, for each $G \in \mathcal{G}$ and $n_1, ..., n_k \in \mathbb{N}$, there exists $F \in \mathcal{F}$ such that

$$F \subseteq \left\{ g + \sum_{i=1}^{k} \alpha_i f_i \;\middle|\; \begin{array}{ll} (1) & g \in G \\ (2) & |\alpha_i| < \frac{1}{n_i},\ i = 1, ..., k \end{array} \right\}.$$

As $G \subseteq L$, this is clearly impossible, so that $\mathcal{F}$ does not converge to 0 in L. Thus K is not locally solid.

As Examples 3.1 and 3.2 show, neither initial nor final constructions preserve local solidity. However, it is possible to form initial and final convergence structures in the class of locally solid convergence vector lattices. These will not, in general, coincide with the initial and final vector space convergence structures.

Recall [6, Definition 18.1 & Theorem 18.2] that a linear mapping $T : K \to L$ from a vector lattice K into another vector lattice L is a Riesz homomorphism whenever

$$\begin{array}{ll} \forall & f, g \in K \ : \\ & T(f \wedge g) = T(f) \wedge T(g). \end{array} \tag{3.4}$$

Also recall [1, page 3] that for a subset F of a vector lattice L, the *solid hull* of F is the set

$$s(F) = \left\{ g \in L \;\middle|\; \begin{array}{ll} \exists & f \in F \ : \\ & 0 \leq |g| \leq |f| \end{array} \right\}. \tag{3.5}$$

Clearly $s(F)$ is a solid subset of L. In fact, it is the smallest solid subset of L containing F. Note that for $F, G \subseteq L$ we have

$$s(F + G) \subseteq s(F) + s(G). \tag{3.6}$$

In particular, if F and G are solid sets, then so is $F + G$. Property (3.6) follows from the Dominated Decomposition Property [6, Corollary 15.6 (i)]. The following also holds. For all $F, G \subseteq L$,

$$s(F \cup G) = s(F) \cup s(G). \tag{3.7}$$

More generally, if $\{F_i \ : \ i \in I\}$ is a family of subsets of L, then

$$s\left(\bigcup_{i \in I} F_i\right) = \bigcup_{i \in I} s(F_i). \tag{3.8}$$

For $\alpha \in \mathbb{R}$ we have

$$s(\alpha F) = \alpha s(F). \tag{3.9}$$

For a filter $\mathcal{F}$ on L we define the solid hull of $\mathcal{F}$ as

$$s(\mathcal{F}) = [\{s(F) \ : \ F \in \mathcal{F}\}].$$

Since $F \subseteq s(F)$, it follows that

$$s(\mathcal{F}) \subseteq \mathcal{F}. \tag{3.10}$$

Theorem 3.3. *Let $\{L_i \,:\, i \in I\}$ be a family of locally solid convergence vector lattices, L a vector lattice and $\{T_i : L_i \to L\}$ a family of linear mappings. There exists a finest locally solid convergence structure on L making all the T_i continuous.*

Proof. First we show that there exists a locally solid convergence structure λ on L such that every $T_i : L_i \to L$ is continuous. In this regard, denote by λ_0 the final vector space convergence structure on L with respect to the T_i. Now define λ as follows:

$$\mathcal{F} \in \lambda(0) \Leftrightarrow \left(\begin{array}{cc} \exists & \mathcal{G}_1, ..., \mathcal{G}_n \in \lambda_0(0) \;: \\ & s(\mathcal{G}_1) + ... + s(\mathcal{G}_n) \subseteq \mathcal{F} \end{array} \right). \tag{3.11}$$

For nonzero $f \in L$, let $\mathcal{F} \in \lambda(f)$ if and only if $\mathcal{F} - [f] \in \lambda(0)$. To see that λ is a vector space convergence structure on L we verify the conditions of [3, Proposition 3.2.3]. Since λ_0 is a vector space convergence structure on L, condition (i) in [3, Proposition 3.2.3] follows from (3.7), while (v) follows from (3.9) and (vi) follows from (3.10). Conditions (ii) and (iii) are trivially satisfied. It remains to verify that $\mathcal{N}\mathcal{F} \in \lambda(0)$ whenever $\mathcal{F} \in \lambda(0)$, where $\mathcal{N}$ denotes the neighborhood filter at 0 in $\mathbb{R}$. Note that $\{(\frac{1}{n}, \frac{1}{n}) \;:\; n \in \mathbb{N}\}$ is a basis for $\mathcal{N}$. Furthermore, since solid sets are balanced [1, page 3], it follows that $(-\frac{1}{n}, \frac{1}{n})s(G) \subseteq s(G)$ for every $G \subseteq L$. Thus $s(\mathcal{G}) \subseteq \mathcal{N}\mathcal{G}$ for every filter $\mathcal{G}$ on L. Now suppose that $\mathcal{F} \in \lambda(0)$, and let $\mathcal{G}_1, ..., \mathcal{G}_n \in \lambda_0(0)$ be the filters associated with $\mathcal{F}$ through (3.11). Then it follows that

$$\begin{array}{rcl} s(\mathcal{G}_1) + ... + s(\mathcal{G}_n) & \subseteq & \mathcal{N}s(\mathcal{G}_1) + ... + \mathcal{N}s(\mathcal{G}_n) \\ & \subseteq & \mathcal{N}\left(s(\mathcal{G}_1) + ... + s(\mathcal{G}_n)\right) \\ & \subseteq & \mathcal{N}\mathcal{F} \end{array}$$

so that $\mathcal{N}\mathcal{F} \in \lambda(0)$. Hence λ is a vector space convergence structure on L. Since the sum of solid sets is again a solid set, it follows from (3.11) that λ is locally solid. Since λ_0 is finer than λ, it follows that the mappings $T_i : L_i \to L$ are all continuous with respect to λ.

We now show that λ is the finest locally solid convergence structure on L making the T_i continuous. In this regard, suppose that λ_1 is a locally solid convergence structure on L with respect to which the T_i are continuous. Then, λ_0 being the final vector space convergence structure on L with respect to the T_i, it follows that λ_1 is coarser than λ_0. Thus for each $\mathcal{F} \in \lambda_0(0)$ there exists a coarser filter $\mathcal{G} \in \lambda_1(0)$ with a basis consisting of solid sets. Then $\mathcal{G} = s(\mathcal{G}) \subseteq s(\mathcal{F})$ so that $s(\mathcal{F}) \in \lambda_1(0)$. Since λ_1 is a vector space convergence structure, it follows from (3.11) that λ is finer than λ_1. $\square$

Theorem 3.4. *Let L be a vector lattice, $\{L_i \ : \ i \in I\}$ a family of locally solid convergence vector lattices and $\{T_i : L \to L_i \ : \ i \in I\}$ a family of order bounded linear mappings. There exists a coarsest locally solid convergence structure on L making all the T_i continuous.*

Proof. Denote by λ_0 the initial convergence structure on L with respect to the family of mappings $\{T_i : L \to L_i \ : \ i \in I\}$. Let $\mathcal{F} \in \lambda(0)$ if and only if $s(\mathcal{F}) \in \lambda_0(0)$, and for $f \neq 0$, let $\mathcal{F} \in \lambda(f)$ if and only if $\mathcal{F} - [f] \in \lambda(0)$. We show that λ is a vector space convergence structure on L by verifying the conditions of [3, Proposition 3.2.3]. Condition (i) follows from (3.7) and the fact that λ_0 is a convergence structure on L, while (ii) is trivially satisfied. Condition (iii) follows from (3.6), as λ_0 is a vector space convergence structure on L. To see that property (iv) holds, consider $\mathcal{F} \in \lambda(0)$, that is, $s(\mathcal{F}) \in \lambda_0(0)$. For $n \in \mathbb{N}$ and $F \in \mathcal{F}$ we have

$$\left(-\frac{1}{n}, \frac{1}{n}\right) F = \bigcup_{|\alpha| < \frac{1}{n}} \alpha F.$$

Thus (3.9) and (3.8) imply that

$$s\left(\left(-\frac{1}{n}, \frac{1}{n}\right) F\right) = \bigcup_{|\alpha| < \frac{1}{n}} \alpha s(F).$$

Since solid sets are balanced [1, page 3] it therefore follows that

$$s\left(\left(-\frac{1}{n}, \frac{1}{n}\right) F\right) \subseteq s(F).$$

Hence $s(\mathcal{F}) \subseteq s(\mathcal{N}\mathcal{F})$ so that $s(\mathcal{N}\mathcal{F}) \in \lambda_0(0)$, and consequently $\mathcal{N}\mathcal{F} \in \lambda(0)$. Condition (v) follows from (3.9). It remains to verity that $\mathcal{N}f \in \lambda(0)$ for all $f \in L$. In this regard, note that $\left(-\frac{1}{n}, \frac{1}{n}\right) f \subseteq \left[-\frac{1}{n}|f|, \frac{1}{n}|f|\right]$. Therefore $s\left(\left(-\frac{1}{n}, \frac{1}{n}\right) f\right) \subseteq \left[-\frac{1}{n}|f|, \frac{1}{n}|f|\right]$ so that $\mathcal{G} \subseteq s(\mathcal{N}f)$, with $\mathcal{G} = [\{\left[-\frac{1}{n}|f|, \frac{1}{n}|f|\right] \ : \ n \in \mathbb{N}\}]$. Since each T_i is order bounded, it follows that there exists $g \in L_i^+$ such that $T_i([-\frac{1}{n}|f|, \frac{1}{n}|f|]) \subseteq [-\frac{1}{n}g, \frac{1}{n}g]$ for all $n \in \mathbb{N}$. Hence $T_i(\mathcal{G})$ converges to 0 in L_i for each $i \in I$ by Proposition 2.4. Hence $\mathcal{G} \in \lambda_0(0)$ so that $s(\mathcal{N}f) \in \lambda_0(0)$. Consequently, $\mathcal{N}f \in \lambda(0)$. Since all the conditions of [3, Proposition 3.2.3] are satisfied, λ is a vector space convergence structure on L.

Furthermore, λ is finer than λ_0 so that all the T_i are continuous with respect to λ. Moreover, λ is locally solid. Indeed, $s(\mathcal{F})$ has a basis consisting of solid sets for all $\mathcal{F} \in \lambda(0)$. But $s(\mathcal{F}) \subseteq \mathcal{F}$ and $s(s(\mathcal{F})) = s(\mathcal{F}) \in \lambda_0(0)$. Hence $s(\mathcal{F}) \in \lambda(0)$.

Lastly we verify that λ is the coarsest locally solid convergence structure on L making the T_i continuous. In this regard, consider a locally solid convergence structure λ_1 on L such that all the T_i are continuous.

Consider a filter $\mathcal{F} \in \lambda_1(0)$. Without loss of generality, we may assume that $\mathcal{F}$ has a basis consisting of solid sets, in which case $s(\mathcal{F}) = \mathcal{F} \in \lambda_0(0)$. Thus $\mathcal{F} \in \lambda(0)$, and the proof is complete. $\square$

The following result gives sufficient conditions for the initial convergence structure to be locally solid.

Theorem 3.5. *Let $\{L_i \ : \ i \in I\}$ be a family of locally solid convergence vector lattices, L a vector lattice and $\{T_i : L \to L_i \ : \ i \in I\}$ a family of Riesz homomorphisms. Then the initial convergence structure on L with respect to the family of mappings $\{T_i : L \to L_i\}$ is locally solid.*

Proof. Let $\mathcal{F}$ converge to 0 in L with respect to the initial convergence structure. Fix $i \in I$. Since $T_i(\mathcal{F})$ converges to 0 in L_i and L_i is locally solid, there exists a filter $\mathcal{G}_i$ with a basis consisting of solid sets such that $\mathcal{G}_i$ converges to 0 in L_i and $\mathcal{G}_i \subseteq T_i(\mathcal{F})$. That is,

$$\begin{array}{ll} \forall & G \in \mathcal{G}_i \ : \\ \exists & F_i \in \mathcal{F} \ : \\ & T(F_i) \subseteq G. \end{array} \tag{3.12}$$

Consider any $f \in s(F_i)$. Thus there exists $g \in F_i$ such that $0 \leq |f| \leq |g|$. But T_i is a Riesz homomorphism so that $0 \leq |T_i(f)| = T_i(|f|) \leq T_i(|g|) = |T_i(g)|$. Since G is solid (3.12) implies that $T_i(f) \in G$. Thus $\mathcal{G}_i \subseteq T_i(s(\mathcal{F}))$ so that $T_i(s(\mathcal{F}))$ converges to 0 in L_i. Since this holds for all $i \in I$ it follows that $s(\mathcal{F})$ converges to 0 in L. But $s(\mathcal{F}) \subseteq \mathcal{F}$ has basis consisting of solid sets. Thus the initial convergence structure on L with respect to the family of mappings $\{T_i : L \to L \ : \ i \in I\}$ is locally solid. $\square$

Corollary 3.6. *Products and subspaces of locally solid convergence vector lattices are locally solid.*

Proof. Let K denote the Cartesian product of a family $\{L_i \ : \ i \in I\}$ of locally solid convergence vector lattices. Note that K is a vector lattice with respect to the coordinate-wise order

$$f = (f_i) \leq g = (g_i) \Leftrightarrow \left(\begin{array}{ll} \forall & i \in I \ : \\ & f_i \leq g_i \end{array} \right).$$

Since $(f_i) \wedge (g_i) = (f_i \wedge g_i)$, it follows that the projections $\pi_i : K \to L_i$ are Riesz homomorphisms. Therefore the product convergence structure on K is locally solid by Theorem 3.5.

Furthermore, any vector lattice subspace K of a locally solid convergence vector lattice L is locally solid, since the inclusion mapping $K \ni f \mapsto f \in L$ is a Riesz homomorphism. The result follows from Theorem 3.5. $\square$

4. The dual of a locally solid convergence vector lattice

This section is devoted to the study of the dual of a locally solid convergence vector lattice. In this regard, we recall that for a locally solid convergence vector lattice L there are two dual structures for L, namely, the topological dual L' of L consisting of all continuous linear functionals on L, and the order dual $L^{\sim}$ of L consisting of all order bounded linear functionals on L [13, Section 85]. The following result relates these two structures.

Proposition 4.1. *Let L be a locally solid convergence vector lattice with convergence structure λ. The topological dual L' of L is an ideal in the order dual $L^{\sim}$ of L.*

Proof. Since every continuous functional maps bounded sets in L into bounded sets in $\mathbb{R}$, it follows by Proposition 2.7 (i) that L' is a linear subspace of $L^{\sim}$. To see that L' is a Riesz subspace of $L^{\sim}$, consider $\varphi \in L'$ and a filter $\mathcal{F} \in \lambda(0)$. Note that, for all $F \subseteq L$ we have $F \subseteq F^+ - F^-$ so that $\mathcal{F}^+ - \mathcal{F}^- \subseteq \mathcal{F}$. Thus we may assume that $\mathcal{F}$ has a basis $\mathcal{B}$ consisting of positive sets, that is, sets contained in L^+. Furthermore, we may assume that $[0,g] \subseteq B$ for each $B \in \mathcal{B}$ and $g \in B$. According to [13, Theorem 83.6] we have $\varphi^+(f) = \sup\{\varphi(g) \ : \ 0 \leq g \leq f\}$ and $\varphi^-(f) = \sup\{-\varphi(g) \ : \ 0 \leq g \leq f\}$ for all $f \in L^+$. Therefore $\varphi^+(B) \subseteq \varphi(B) + (-\epsilon, \epsilon)$ for all $B \in \mathcal{B}$, $\epsilon > 0$. Thus $\varphi(\mathcal{F}) + \mathcal{N} \subseteq \varphi^+(\mathcal{F})$ so that $\varphi^+(\mathcal{F})$ converges to 0 in $\mathbb{R}$. Hence φ^+ is continuous. In the same way, it follows that φ^- is continuous. The result now follows from [6, Theorem 11.8].

Now consider $\psi \in L^{\sim}$ and $\varphi \in L'$ such that $|\psi| \leq |\varphi|$. In particular, this means that $\psi^+ \leq |\varphi|$ and $\psi^- \leq |\varphi|$. Again consider a filter $\mathcal{F}$ with a basis $\mathcal{B}$ consisting of positive sets such that $[0,g] \subseteq B$ for each $B \in \mathcal{B}$ and $g \in B$. It follows that $\alpha \in |\varphi|(B)$ for all $B \in \mathcal{B}$, $f \in B$ and $\alpha \in \mathbb{R}$ such that $0 \leq \alpha \leq |\varphi|(f)$. Thus $0 \leq \psi^+(f) \leq |\varphi|(f)$, $f \in L^+$, implies that $\psi^+(B) \subseteq |\varphi|(B)$ for all $B \in \mathcal{B}$. Hence $|\varphi|(\mathcal{F}) \subseteq \psi^+(\mathcal{F})$ so that $\psi^+(\mathcal{F})$ converges to 0 in $\mathbb{R}$. Therefore $\psi^+ \in L'$ and, in the same way, $\psi^- \in L'$ so that $\psi = \psi^+ - \psi^- \in L'$. $\square$

In the setting of convergence vector spaces, the natural convergence structure for the dual of a convergence vector space is the continuous convergence structure [3, Definition 1.1.5].

Definition 4.2. Let L be a (real) convergence vector space. Let

$$\omega : L' \times L \ni (\varphi, f) \mapsto \varphi(f) \in \mathbb{R}$$

denote the evaluation mapping. A filter $\mathcal{F}$ in L' converges to $\varphi \in L'$ with respect to the continuous convergence structure if

$$\begin{aligned} \forall \quad & f \in L,\ \mathcal{G} \text{ converging to } f \text{ in } L \ : \\ & \omega(\mathcal{F} \times \mathcal{G}) \text{ converges to } \varphi(f) \text{ in } \mathbb{R}. \end{aligned}$$

The distinguishing feature of the continuous convergence structure is that it makes the evaluation mapping continuous. In fact, it is the largest convergence structure on L' with this property. One should note that if L is a Hausdorff locally convex space, then there is no topology on L' making ω continuous, unless L is a normed space.

We will denote the topological dual L' of L, equipped with the continuous convergence structure, by L'_c. Since the dual L' of a locally solid convergence vector lattice L is an ideal in the order dual of L, it is a vector lattice in its own right. In particular, the order on L' is defined as

$$\varphi \leq \psi \Leftrightarrow \left(\begin{array}{ll} \forall & f \in L^+ : \\ & \varphi(f) \leq \psi(f) \end{array} \right). \tag{4.1}$$

Theorem 4.3. *Let L be a locally solid convergence vector lattice with convergence structure λ. Then L'_c is a locally solid convergence vector lattice.*

Proof. Let $\mathcal{G}$ converge to 0 in L'_c, that is, for every $f \in L$ and every $\mathcal{F}$ converging to f in L, the filter

$$\omega(\mathcal{G} \times \mathcal{F}) = [\{\{\varphi(g) \ : \ \varphi \in G,\ g \in F\} \ : \ G \in \mathcal{G},\ F \in \mathcal{F}\}]$$

converges to 0 in $\mathbb{R}$. Consider the filter $s(\mathcal{G}) = [\{s(G) \ : \ G \in \mathcal{G}\}]$. Clearly $s(\mathcal{G}) \subseteq \mathcal{G}$. We claim that $s(\mathcal{G})$ converges to 0 in L'_c. In this regard, consider $f \in L$ and a filter $\mathcal{F} \in \lambda(f)$ of the form $\mathcal{F} = \mathcal{H} + f$ where $\mathcal{H} \in \lambda(0)$ has a basis consisting of solid sets. According to Proposition 2.5 the filter $\mathcal{F}^+ = [\{\{f^+ + h^+ \ : \ h \in H\} \ : \ H \in \mathcal{H}\}]$ converges to f^+. Hence $\omega(\mathcal{G} \times \mathcal{F}^+)$ converges to 0 in $\mathbb{R}$. Hence, for all $\epsilon > 0$, there exists $G_\epsilon \in \mathcal{G}$, $H_\epsilon \in \mathcal{H}$ such that $\{\varphi(f^+ + h^+) \ : \ \varphi \in G_\epsilon,\ h \in H_\epsilon\} \subseteq (-\epsilon, \epsilon)$. Since $f^+ + h^+ \geq 0$ it follows that $\psi(f^+ + h^+) \in (-\epsilon, \epsilon)$ for all $\psi \in s(G_\epsilon)$, $h \in H_\epsilon$, so that $\omega(s(\mathcal{G}) \times \mathcal{F}^+)$ converges to 0 in $\mathbb{R}$. In the same way, $\omega(s(\mathcal{G} \times \mathcal{F}^-)$ converges to 0 in $\mathbb{R}$, where $\mathcal{F}^- = [\{\{f^- + h^- \ : \ h \in H\} \ : \ H \in \mathcal{H}\}]$. Since $\mathcal{F}^+ - \mathcal{F}^- \subseteq \mathcal{F}$, it follows that $\omega(s(\mathcal{G}) \times \mathcal{F}) \supseteq \omega(s(\mathcal{G}) \times \mathcal{F}^+) - \omega(s(\mathcal{G}) \times \mathcal{F}^-)$. Therefore $\omega(s(\mathcal{G}))$ converges to 0 in $\mathbb{R}$, so that $s(\mathcal{G})$ converges to 0 in L'_c. □

Theorem 4.3 has important implications for locally solid convergence vector lattices, and locally convex locally solid Riesz spaces in particular. As will be shown in Section 5, the fact that continuous convergence is locally solid on L' implies that every complete, Hausdorff locally convex, locally solid Riesz space is isomorphic, both as a convergence vector space and as a vector lattice, to its second dual.

5. Completeness and Completion

We now turn to the issue of completion and completeness. In particular, we consider the completion $\widehat{L}$ of a Hausdorff locally solid convergence vector lattice L. By a completion of a convergence vector space L we mean a complete convergence vector space containing L as a dense subspace with the property that any continuous linear mapping $T : L \to K$, with K a complete, Hausdorff convergence vector space, admits a continuous extension $\widehat{T} : \widehat{L} \to K$. It should be noted that not every Hausdorff convergence vector space admits such a completion [4]. In particular, a completion $\widehat{L}$ of L exists if and only if every Cauchy filter in L contains a bounded set. We will restrict ourselves to the case when L admits a completion.

We now show that $\widehat{L}$ is a vector lattice. In this regard, we recall [6, Theorem 11.4] that a *cone* in a (real) vector space L is a set L^+ with the following properties:

$$\tag{5.1} \begin{array}{l} \forall \quad f, g \in L^+, \ \alpha \in \mathbb{R}, \ \alpha \geq 0 \ : \\ \qquad 1) \quad f + g \in L^+ \\ \qquad 2) \quad \alpha f \in L^+ \\ \qquad 3) \quad -f \in L^+ \Rightarrow f = 0. \end{array}$$

Every cone in L induces a partial order on L through the relation

$$\tag{5.2} f \leq g \Leftrightarrow g - f \in L^+.$$

With respect to the order (5.2) L is an ordered vector space. L is a vector lattice if and only if $\sup\{f, 0\}$ exists in L for every $f \in L$, see [6, Theorem 11.5 (v)].

Theorem 5.1. *Let L be a Hausdorff locally solid convergence vector lattice admitting a completion $\widehat{L}$, and denote by $\widehat{L}^+$ the adherence $a_{\widehat{L}}(L^+)$ of L^+ in $\widehat{L}$. Then $\widehat{L}^+$ is a cone in $\widehat{L}$, and $\widehat{L}$ is a vector lattice with respect to the order induced by $\widehat{L}^+$. Furthermore L is a vector lattice subspace of $\widehat{L}$.*

Proof. The first two properties in (5.1) follow from the continuity of addition and scalar multiplication on $\widehat{L}$, and the corresponding properties of the positive cone L^+ in L. To verify the third condition, consider $f \in \widehat{L}^+$ such that $-f \in \widehat{L}^+$. Then there exists Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ in L, with bases $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{G}}$, respectively, consisting of sets in L^+, such that $\mathcal{F}$ converges to f and $\mathcal{G}$ converges to $-f$ in $\widehat{L}$, see [10, Proposition 2.3]. Then $\mathcal{F} + \mathcal{G}$ converges to 0 in $\widehat{L}$ so that there is a filter $\mathcal{H} \subseteq \mathcal{F} + \mathcal{G}$ with a basis consisting of solid sets which converges to 0 in L. Then for each $H \in \mathcal{H}$ there exist $F \in \mathcal{B}_{\mathcal{F}}$ and $G \in \mathcal{B}_{\mathcal{G}}$ such that $F + G \subseteq H$. Since $F, G \subseteq L^+$ it follows that $0 \leq h \leq h + g$ for all $h \in F$ and $g \in G$.

Since H may be taken to be a solid set, it thus follows that $F \subseteq H$. Therefore $\mathcal{H} \subseteq \mathcal{F}$ so that $\mathcal{F}$ converges to 0 in L, and hence in $\widehat{L}$. Since $\widehat{L}$ is Hausdorff, it follows that $f = 0$. Therefore $\widehat{L}^+$ is a cone in $\widehat{L}$.
Now we show that $\widehat{L}$ is a vector lattice with respect to the order (5.2) induced by $\widehat{L}^+$. The mapping $L \ni f \mapsto f^+ \in L$ is uniformly continuous by Proposition 2.5 (v). Therefore there exists a unique uniformly continuous mapping $p : \widehat{L} \to \widehat{L}$ which extends $L \ni f \mapsto f^+ \in L$. We claim that $p(f) = \sup\{f, 0\}$ for all $f \in \widehat{L}$. In this regard, we note that $p(f) \in \widehat{L}^+$ by [3, Lemma 1.3.5], thus $p(f) \geq 0$. Now pick a Cauchy filter $\mathcal{F}$ in L which converges to f in $\widehat{L}$. Then $p(\mathcal{F}) - \mathcal{F} = \mathcal{F}^+ - \mathcal{F}$ in L, and $p(\mathcal{F}) - \mathcal{F}$ converges to $p(f) - f$ in $\widehat{L}$. Let $\mathcal{G}$ be the filter in L with basis $[\{\{g^+ - g \ : \ g \ inF\} \ : \ F \in \mathcal{F}\}]$. Then $p(\mathcal{F}) - \mathcal{F} \subseteq \mathcal{G}$ so that $\mathcal{G}$ converges to $p(f) - f$ in $\widehat{L}$. But $\mathcal{G}$ has a basis consisting of sets in L^+. Therefore $p(f) - f \in \widehat{L}^+$ so that $f \leq p(f)$. Suppose $f \leq g$ and $0 \leq g$ in $\widehat{L}$ for some $g \in \widehat{L}$. Consider Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ in L, with bases $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{G}}$, respectively, consisting of sets in L^+, such that $\mathcal{F}$ converges to $g - f$ and $\mathcal{G}$ converges to g in $\widehat{L}$. Then the filter $\mathcal{H} = \mathcal{G} - p(\mathcal{G} - \mathcal{F})$ converges to $g - p(f)$ in $\widehat{L}$. But $\mathcal{H}$ is contained in the filter

$$\mathcal{K} = [\{\{u - u^+ + v^+ \ : \ u \in G, \ v \in F\} \ : \ G \in \mathcal{B}_{\mathcal{G}}, \ F \in \mathcal{B}_{\mathcal{F}}\}].$$

Since $\widehat{L}$ is Hausdorff and p is uniformly continuous, $\mathcal{K}$ converges to $g - p(f)$ in $\widehat{L}$. But each $G \in \mathcal{B}_{\mathcal{G}}, \ F \in \mathcal{B}_{\mathcal{F}}$ is contained in L^+, hence $u - u^+ + v^+ \geq 0$ in L for $u \in G, \ v \in F$. Therefore $\mathcal{K}$ has a basis consisting of sets in L^+, thus $g - p(f) \in \widehat{L}^+$ so that $p(f) \leq g$. Therefore $p(f) = \sup\{f, 0\}$ for all $f \in \widehat{L}$ so that $\widehat{L}$ is a vector lattice.
To see that L is a sublattice of $\widehat{L}$, it is sufficient to observe that, for $f, g \in L$

$$f \leq g \text{ in } \widehat{L} \Leftrightarrow g - f \in \widehat{L}^+ \Leftrightarrow g - f \in L^+ \Leftrightarrow f \leq g \text{ in } L,$$

and $\sup\{f, 0\}$ in L is $f^+ = p(f)$, which is $\sup\{f, 0\}$ in $\widehat{L}$. □

As mentioned, the convergence vector space completion of a locally solid convergence vector lattice need not be locally solid, as the following example shows.

Example 5.2. Consider the Archimedean vector lattice $\mathcal{C}(\mathbb{R})$, equipped with the order convergence structure. $\mathcal{C}(\mathbb{R})$ is not complete as a convergence vector space, see for instance [2, Example 21]. If $\mathcal{C}(\mathbb{R})^\sharp$ denotes the Dedekind order completion of $\mathcal{C}(\mathbb{R})$, then the completion $\widehat{\mathcal{C}(\mathbb{R})}$ of $\mathcal{C}(\mathbb{R})$ consists of the set $\mathcal{C}(\mathbb{R})^\sharp$ equipped with the vector space convergence structure defined as

$$\mathcal{F} \in \lambda_o^\sharp(0) \Leftrightarrow \left(\begin{array}{ll} \exists & (l_n),\ (u_n) \subset \mathcal{C}(\mathbb{R})\ : \\ & 1)\quad l_n \leq l_{n+1} \leq u_{n+1} \leq u_n,\ n \in \mathbb{N} \\ & 2)\quad \sup\{l_n\ :\ n \in \mathbb{N}\} = 0 = \inf\{u_n\ :\ n \in \mathbb{N}\} \\ & 3)\quad \{f \in \mathcal{C}(\mathbb{R})\ :\ l_n \leq f \leq u_n\} \in \mathcal{F},\ n \in \mathbb{N}, \end{array}\right.$$

see [11, Theorem 29]. Note that the sets $\{f \in \mathcal{C}(\mathbb{R})\ :\ l_n \leq f \leq u_n\}$ are not solid in $\widehat{\mathcal{C}(\mathbb{R})}$. Therefore the convergence structure $\lambda_o^\sharp$ is not locally solid.

Remark 5.3. We note that while the completion $\widehat{L}$ of a Hausdorff, locally solid convergence vector lattice is not locally solid, $\widehat{L}$ is a convergence vector lattice in the sense that the mapping $\widehat{L}\times\widehat{L} \ni (f,g) \mapsto \sup\{f,g\} \in \widehat{L}$ is uniformly continuous. Indeed, it is clear from the proof of Theorem 5.1 that the mapping $\widehat{L} \ni f \mapsto f^+ \in \widehat{L}$ is uniformly continuous. According to Remark 2.6, $\widehat{L}$ is a convergence vector lattice.
Example 5.2 therefore shows that conditions (ii) to (vi) in Proposition 2.5 do not imply that a vector space convergence structure on a vector lattice is locally solid.

6. Applications

In this section we present two nontrivial applications of the concepts introduced and results obtained in Sections 2 and 3. In particular, we present Closed Graph Theorems for a class of locally solid convergence vector lattices, as well as a duality result for locally convex, locally solid Riesz spaces.

6.1. **A Closed Graph Theorem.** In this section we present Closed Graph Theorems for a class of locally solid convergence vector lattices. We recall that a convergence vector space F is *ultracomplete* [3, Definition 6.1.1] whenever F is strongly first countable [3, Definition 1.6.5] and for all $\mathcal{F} \in \lambda(0)$ there is a countable subset $\{W_n\ :\ n \in \mathbb{N}\} \subseteq \mathcal{F}$ such that $[\{\sum_{n=k}^{\infty} W_n\ :\ k \in \mathbb{N}\}] \in \lambda(0)$.

The general Closed Graph Theorem in convergence vector spaces [3, Theorem 6.2.6] is now formulated as follows.

Theorem 6.1. *Let E be an inductive limit of Fréchet spaces and let F be a convergence vector space admitting a finer vector space convergence structure which is ultracomplete. Any closed linear mapping $T : E \to F$ is continuous.*

The basic result of this section is the following.

Theorem 6.2. *Let K and L be Archimedean locally solid convergence vector lattices and $T : K \to L$ a closed linear operator. Assume that there is a countable set $B \subseteq L$ such that the ideal generated by B in L is the whole space L. If K is the inductive limit of Fréchet spaces and L is relatively uniformly complete, then T is continuous.*

Proof. By Proposition 2.4 λ_r is finer than than the convergence structure on L. Since L is the union of countably many of its principle ideals, it follows from [12, Corollary 3.9] that λ_r is ultracomplete. The result now follows from Theorem 6.1. □

The following particular cases of Theorem 6.2 are of interest.

Corollary 6.3. *Let K and L be Archimedean locally solid convergence vector lattices and $T : K \to L$ a regular, closed linear operator. Assume that K has strong order unit e. If K is the inductive limit of Fréchet spaces and L is relatively uniformly complete, then T is continuous.*

Proof. Since T is regular, we may express T as $T = S - P$, where S and P are positive linear operators. Since P and S are positive operators $S(K)$ is contained in the ideal generated in L by $S(e)$, while $P(K)$ is contained in the ideal generated by $P(e)$ in L. Hence the Dominated Decomposition Property [6, Corollary 15.6 (i)] implies that $T(K)$ is contained in the ideal I generated by $\sup\{S(e), P(e)\}$ in K. Since L is relatively uniformly complete, the ideal I is also relatively uniformly complete [6, Exercise 59.5]. The result now follows from Theorem 6.2. □

Corollary 6.4. [12, Corollary 3.11] *Let K and L be relatively uniformly complete Archimedean vector lattices, and $T : K \to L$ a linear mapping with the property that*

$$\forall \quad f \in K : \\ \left(\begin{array}{ll} 1) & (f_n) \text{ converges relatively uniformly to } f \\ 2) & (Tf_n) \text{ converges relatively uniformly to } g \end{array} \right) \Rightarrow Tf = g \ .$$

If there is a countable set $B \subset L$ such that L is the ideal generated by B, then T is order bounded.

6.2. **Duality for locally solid Riesz spaces.** In this section we apply the results obtained in Section 3 to locally convex, locally solid Riesz spaces. For a convergence vector space E we denote by E'_c its topological dual, equipped with the continuous convergence structure, and by E''_c the dual of E'_c, again equipped with the continuous convergence structure.

The mapping

$$j_E : E \to E''_c,$$

where for $x \in E$ we have

$$j_E(x) : E'_c \ni \varphi \mapsto \varphi(x) \in \mathbb{R},$$

is well defined, linear and continuous [3, Lemma 4.2.1]. The mapping j_E is injective if and only if E'_c separates the points of E. If E is a Hausdorff, locally convex topological vector space, more can be said, see [3, Theorem 4.3.19 & Corollary 4.3.21].

Theorem 6.5. *Let E be a Hausdorff, locally convex topological vector space. Then E''_c is a complete, Hausdorff locally convex topological vector space, and j_E is an isomorphism onto a dense subspace of E''_c.*
In particular, if E is complete, then $j_E(E) = E''_c$.

The following is an application of Theorem 6.5, Proposition 4.1 and Theorem 4.3.

Theorem 6.6. *Let L be a complete, Hausdorff locally convex locally solid Riesz space. Then L''_c is a complete locally convex, locally solid Riesz space and the mapping $j_L : L \to L''_c$ is a convergence space isomorphism and a Riesz isomorphism into $(L'_c)^\sim \supseteq L''_c$.*

Proof. That L''_c is a locally solid convergence vector lattice follows from Proposition 4.1 and Theorem 4.3. Theorem 6.5 implies that L''_c is complete and locally convex, and that j_L is a convergence vector space isomorphism. Since L'_c is an ideal in $L^\sim$ which separates the points of L, it follows from [13, Lemma 109.1] that j_L is a Riesz isomorphism. $\square$

Theorem 6.6 may be compared with the following: If L is a norm reflexive Banach lattice, then the natural mapping $j_L : L \to L''$ is also a Riesz isomorphism into $(L'_c)^\sim \supseteq L''_c$. More generally, if L is a locally solid locally convex space which is reflexive with respect to the strong topology on L' and L'', then the mapping $j_L : L \to L''$ is a Riesz isomorphism into $(L')^\sim \supseteq L''$. These results both follow from [13, Lemma 109.1]. Note that, for L a locally solid locally convex space, it may also happen that $j_L : L \to L''$ is a lattice isomorphism without it being also a topological isomorphism, see for instance [1, Theorem 22.4] for a characterization of such spaces.

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