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ON FIXED POINTS OF PERIODIC SELF-HOMEOMORPHISMS OF COMPACT TOPOLOGICAL SURFACES

E. BUJALANCE AND G. GROMADZKI

Abstract. The two main reasons for this survey are to outline a certain research method, based on hyperbolic geometry, concerning a study of finite actions on compact topological surfaces through their self-homeomorphisms, and to give a survey on fixed points of periodic self-homeomorphisms of such surfaces. The results presented have a common feature that can be obtained by the Nielsen-Riemann-Macbeath approach, which allows us to see such homeomorphisms as conformal maps of compact topological surfaces with some conformal structures imposed, and then study it combinatorially. We shall also give samples of proofs to illustrate the method in question.

1. Introduction

In this survey we shall get a general audience with an interest in topology acquainted with a certain approach based on hyperbolic geometry, and concerning studies of finite actions on compact topological surfaces. We shall also give some illustrative examples of results concerning fixed points of periodic self-homeomorphisms of compact topological surfaces and, more generally, finite actions through self-homeomorphisms which can be proved in a particularly simple way.

2010 *Mathematics Subject Classification.* Primary 57E, 30F; Secondary 14H.

Key words and phrases. Riemann surfaces, automorphisms of Riemann surfaces, fixed points, Fuchsian groups, Hurwitz bound, invariant subsets.

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The starting point for our considerations is a classical Nielsen theorem which asserts that a finite group of orientation preserving self-homeomorphisms of a closed topological surface can be viewed as a group of holomorphic automorphisms of a Riemann surface of the same genus which in turn, due to the Riemann uniformization theorem and well developed theory of Fuchsian and non-euclidean crystallographic groups, allows combinatorial studies of some purely topological problems, by means of their " hyperbolization". First, we shall give a short outline of the approach mentioned in the spirit of [4], and then we will present some exemplary results to illustrate it. We focus our attention both on classical and more recent results. Among the first the main one, though perhaps not very widely known, is the Schwarz theorem concerning the order of a finite group of self-homeomorphisms of a closed topological surface of genus $g \geq 2$ [29]. Among the second there are qualitative and quantitative results concerning fixed points of periodic homeomorphisms of compact topological surfaces.

The first one, in modern times, to focus on studying fixed points of periodic orientation preserving self-homeomorphisms of closed orientable surfaces was Moore [24]. His results were generalized to the case of bordered and non-orientable surfaces and self-homeomorphisms reversing orientation by Etayo [9]. An important role has also been played by an earlier paper [20] of Harvey, which turned out later to be seminal in the studies of cyclic actions on compact surfaces. Later in [23], Macbeath found a beautiful, purely algebraic formula for finding the number of fixed points of a single automorphism φ of a Riemann surface X of genus $g \geq 2$, in terms of the group $\mathrm{Aut}(X)$ of its automorphisms and the topological type of the action. It seems not to be perceived for a long time, that the previous results can be seen as more or less direct consequences of the Macbeath formula, and our exposition is essentially based on the approach suggested by it.

2. Description of the approach

Here we shall describe and present all the ingredients of our approach, which reduces the problem of studying finite groups of self-homeomorphisms of compact topological surfaces to certain problems concerning combinatorial group theory.

2.1. **Riemann surfaces and their automorphisms.** By a Riemann surface we mean a closed topological surface with a holomorphic structure, which can be seen as a maximal holomorphic atlas $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$, meaning that all the functions $\varphi_j \circ \varphi_i^{-1}$constructed as in Figure 1, known as the transition functions between maps, are analytic in the usual sense.

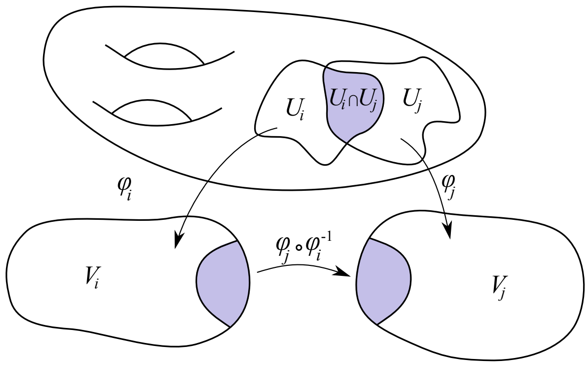

Figure 1. Transition function between maps (charts) (U_i, φ_i)

By an automorphism (analytic) of a Riemann surface we understand its self-homeomorphism f, whose local forms are holomorphic, which means that all the functions $\varphi_j \circ f \circ \varphi_i^{-1}$, constructed as in Figure 2, known as the local forms of f, are analytic in the usual sense.

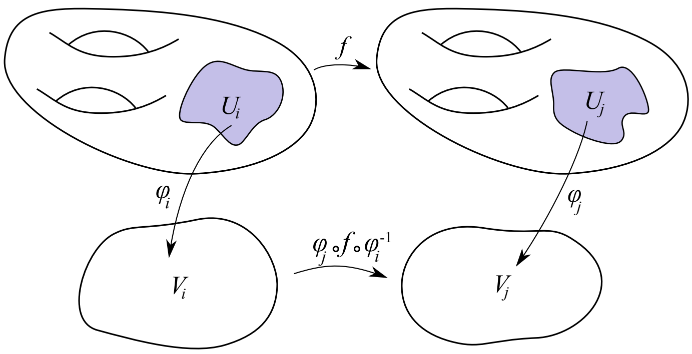

Figure 2. Local form of a continuous map

Similar concepts exist also for closed non-orientable surfaces and for surfaces with nonempty boundary. This time however, roughly speaking, the complex conjugation $z \mapsto \bar{z}$ in the character of transition maps and the folding map $a + bi \mapsto a + |b|i$ in local forms of maps between bordered surfaces are also involved (precise definitions can be found in [1]).

2.2. **Nielsen Theorem.** Let X be a closed surface of genus $g \geq 2$ and let G be a finite group of its self-homeomorphisms. Then, by the mentioned Nielsen theorem, see [25], there exists a structure of a Riemann surface on X, which by abuse of language will also be denoted by X, such that all the elements of G become holomorphic automorphisms.

2.3. **Riemann uniformization Theorem.** Now having a Riemann surface X, say of genus g, forget for a while about its holomorphic structure $\mathcal{A}$ and consider the universal covering $\pi : \widetilde{X} \to X$. We know that $\widetilde{X}$ is a simply connected surface and π is a local homeomorphism. Hence, by the maximality of $\mathcal{A}$, for every $x \in X$ there exists a map (U, φ) such that $x \in U$ and the inverse image of U is a disjoint sum of the sets U_i homeomorphic to U through the projection π restricted to it. Therefore (U_i, φ_i), where $\varphi_i = \varphi \circ \pi_{|U_i}$ constructed in this way for all the elements of X, becomes the holomorphic atlas on $\widetilde{X}$ for which π is a holomorphic map. So we have a simply connected Riemann surface $\widetilde{X}$ which, by the Riemann uniformization theorem, is either the sphere $\mathcal{S} = \mathbb{C}^*$, the complex plane $\mathcal{C} = \mathbb{C}$ or the upper half plane $\mathcal{H} = \mathbb{C}^+$ with their natural holomorphic structures for $g = 0, g = 1$ or $g \geq 2$ respectively. In this article we shall restrict ourselves to the last case.

Summing up, given a Riemann surface X, we have a holomorpic covering $\pi : \mathcal{H} \to X$ being a local homeomorphism. Now elementary covering theory asserts that the set of its deck transformations Γ is a group of holomorphic automorphisms of $\mathcal{H}$ and the orbit space $\mathcal{H}/\Gamma$ has the natural structure of a Riemann surface, inherited from $\mathcal{H}$, which is holomorphically isomorphic to X. Furthermore, if G is a group of automorphisms of a Riemann surface X represented in such a way and Δ is the set of all liftings of all the elements of G, then Δ is a discrete subgroup of $\mathrm{Aut}(\mathcal{H})$ for which $G \cong \Delta/\Gamma$.

2.4. **Fuchsian groups.** The space $\mathcal{H}$ can be seen as a model of the hyperbolic space while Γ and Δ turn out to be discrete subgroups of its group of isometries, which are known as *Fuchsian groups*. The point is that their algebraic structures are well known. Namely they have the presentations

$$\langle a_1, b_1, \ldots, a_g, b_g \,|\, [a_1, b_1] \ldots [a_g, b_g] \rangle \tag{1}$$

and

$$\langle a_1, b_1, \ldots, a_h, b_h, x_1, \ldots, x_r \,|\, x_1^{m_1}, \ldots, x_r^{m_r}, x_1 \ldots x_r [a_1, b_1] \ldots [a_h, b_h] \rangle \tag{2}$$

respectively.

A system of generators satisfying the above relations is called *canonical* and given such a system we shall call its elements *canonical generators.* Now a Fuchsian group has a fundamental region F, whose area $\mu(F)$ depends only on the algebraic presentation and for a group with presentation (2), it equals

$$2\pi\Big(2h-2+\sum_{i=1}^{r}\Big(1-\frac{1}{m_i}\Big)\Big). \tag{3}$$

Finally, an abstract group with the presentation (2) can be realized as a Fuchsian group if and only if (3) is positive and a subgroup Δ' of a Fuchsian group Δ of finite index is a Fuchsian group itself, and the following formula, known as the Hurwitz-Riemann formula, holds

$$[\Delta:\Delta'] = \frac{\mu(\Delta')}{\mu(\Delta)}. \tag{4}$$

2.5. **Generalization to non-orientable or bordered surfaces.** We can employ a similar approach to bordered and non-orientable surfaces. This time, however, the role of Fuchsian groups is being played by more general and more involved *non-euclidean crystallographic groups* (NEC-groups in short), which are discrete and cocompact groups of isometries of the hyperbolic plane $\mathcal{H}$, including orientation reversing ones; the interested reader can find this approach explained in [4].

2.6. **Summary.** Summing up, any finite action of a group G on a compact topological surface X of genus $g \geq 2$ is given by an epimorphism $\theta : \Delta \to G$, whose kernel is a group with the presentation (1) and Δ has the presentation (2). Such an epimorphism is usually called a *smooth epimorphism.*

3. Fixed points of periodic self-homeomorphisms of compact topological surfaces

3.1. **Macbeath formula and its proof.** The following theorem is crucial for our approach to studying fixed points of periodic homeomorphisms.

Theorem 3.1.1 (A.M. Macbeath 1973 ([23])). *Let $G = \Delta/\Gamma$ be the group of orientation preserving automorphisms of a Riemann surface $X = \mathcal{H}/\Gamma$ and let $x_1, \ldots, x_r$ be the set of canonical elliptic generators of Δ of orders $m_1, \ldots, m_r$ respectively. Let $\theta : \Delta \to G$ be the canonical smooth epimorphism. Then the number of points of X fixed by $g \in G$ is given by the formula*

$$|\mathrm{N}_G(\langle g\rangle)| \sum 1/m_i,$$

where N *stands for the normalizer and the sum is taken over those i for which g is conjugate to a power of $\theta(x_i)$. In particular the number of fixed points of g is finite.*

As we already remarked, this theorem is principal in the proofs of results presented in this survey and so we give a proof (actually different from the original proof of Macbeath, being more algebraic, but in the same spirit).

Proof. Let $\pi : \mathcal{H} \to X$ and $\theta : \Delta \to G$ be the canonical projections. Then the action of G on X is given by

$$gx = \pi(\delta h) \text{ if } g = \theta(\delta), x = \pi(h).$$

Therefore if $\varphi = \theta(\delta)$, then $x = \pi(h)$ is its fixed point if and only if $\delta h = \gamma h$ for some $\gamma \in \Gamma$, which means that h is a fixed point of $\gamma^{-1}\delta$. But then $\gamma^{-1}\delta$ is an elliptic element and therefore it is conjugate to a power of one of the canonical ones i.e., $\gamma^{-1}\delta = \rho x_i^{n_i}\rho^{-1}$ for some $\rho \in \Delta$ and $i \leq r$, which means that φ is conjugate to a power of $\theta(x_i)$ and for h_i being the unique fixed point of x_i, $h = \rho h_i$. Now, since conjugate elements have the same number of fixed points, we can assume that $\varphi = \theta(x_i^{n_i})$, i.e. $\rho = 1$ and so $\pi(h_i)$ is a fixed point of φ. But for $\pi(\rho h_i)$ being another fixed point of φ, we have $\gamma^{-1}\delta = \rho x_i^{n_i'}\rho^{-1}$, which means that $\theta(\rho)$ normalizes $\langle\varphi\rangle$ and therefore $\rho \in \theta^{-1}(\mathrm{N}_G(\langle\varphi\rangle)$.

Finally, $\pi(\rho h_i) = \pi(\rho' h_i)$ if and only if $\rho^{-1}\gamma\rho' h_i = h_i$ for some $\gamma \in \Gamma$, which means that $\rho^{-1}\rho'(\rho'^{-1}\gamma\rho') = x_i^{n_i}$. However, the last is equivalent to $\rho^{-1}\rho' \in \langle x_i\rangle\Gamma = \theta^{-1}(\langle\theta(x_i)\rangle)$. Hence for each i, for which φ is conjugate to a power of $\theta(x_i)$, we obtain

$$\left[\theta^{-1}(\mathrm{N}_G(\langle\varphi\rangle)) : \theta^{-1}(\langle\theta(x_i)\rangle)\right] = \left[\mathrm{N}_G(\langle\varphi\rangle) : \langle\theta(x_i)\rangle\right] = \left|\mathrm{N}_G(\langle\varphi\rangle)\right|/m_i$$

fixed points and hence the result. □

Remark. Formulas of similar type for the number of fixed points of self-homeomorphisms of closed non-orientable or bordered (orientable or not) compact surfaces are also known [11, 15]. However in the case of non-orientable surfaces, the assumption on orientation preserving character of φ (which became senseless) is replaced by explicit assumption that we deal with the fixed points of φ which are isolated in $\mathrm{Fix}(\varphi)$.

3.2. **The nature of the set of points fixed by self-homeomorphisms of compact surfaces.** A periodic, orientation preserving or orientation reversing but non-involutory, self-homeomorphism of a closed surface, has a finite set of fixed points. The set of points fixed by an orientation reversing involution of a closed orientable surface of genus g consists of k, where $0 \le k \le g+1$, connected components, each homeomorphic to a circle, (called *ovals* in the nineteenth century terminology of Hilbert, or *mirrors*, used by some authors in modern terminology) while for bordered surfaces also *arcs* (homeomorphic to the closed intervals with the ends on the boundary) can stand for the connected components of the set fixed points [3]. These cases however, will not be considered here, though they can be successfully studied using [11, 13]) where similar, like the above Macbeath formula, formulas hold.

3.3. **The total number of fixed points for finite actions.** We start this subsection with presenting a topological version of the Hurwitz bound, together with its proof, to illustrate the method that we use in the proofs of most results presented in this survey.

Theorem 3.3.1 (A. Hurwitz 1893 ([21]))**.** *A finite group G of self-homeomorphisms of a closed orientable surface X of genus $g \ge 2$ has at most $84(g-1)$ elements.*

Proof. Indeed, by the Nielsen theorem, G can be seen as a group of holomorphic automorphisms of a Riemann surface X. By the approach described above, $X = \mathcal{H}/\Gamma$ for some Fuchsian surface group Γ with the presentation (1) and $G = \Delta/\Gamma$ for a Fuchsian group Δ with the presentation (2). Now by the Hurwitz-Riemann formula, $|G| = 4\pi(g-1)/\mu(\Delta)$ and it is easy to check that the minimal positive area, as in (3), is equal $\pi/21$ and is attained for $h = 0, r = 3$ and $m_1 = 2, m_2 = 3, m_3 = 7$, hence the result. □

Let G be a finite group of self-homeomorphisms of a closed orientable topological surface X of genus $g \ge 2$. A point x on X is said to be a fixed point of G if it is a fixed point of some of its elements. Let $F_g(G)$ be the order of the set of fixed points of G and let

$$F(g) = \max F_g(G),$$

where G runs over all finite groups of self-homeomorphisms of X.

Theorem 3.3.2 (G. Gromadzki 2009 ([17]))**.** *The upper bound $F(g) \le 82(g-1)$ holds and it is attained if and only if there exists an order $84(g-1)$ group of self-homeomorphisms of a surface of genus g.*

3.4. **Order vs number of fixed points for periodic self-homeomorphisms of closed orientable surfaces.** As a good starting point and an example illustrating our approach, we supply the following classical result together with the sketch of its proof.

Theorem 3.4.1 (A. Wiman 1895 ([30])). *The order of a periodic self-homeomorphism φ of a closed orientable topological surface of genus $g \geq 2$ is not greater than $4g+2$. This bound is attained for arbitrary $g \geq 2$ and furthermore φ for which this is the case, must have just one fixed point.*

Proof. Let φ be a self-homeomorphism, say of order n, of a closed orientable topological surface X, say of genus $g \geq 2$. Then, by the Nielsen theorem, there is a structure of a Riemann surface on X such that φ is its conformal automorphism. Let $X = \mathcal{H}/\Gamma$ and $G = \langle\varphi\rangle = \Delta/\Gamma$, where Δ is a Fuchsian group, say with the presentation (2) and Γ is a Fuchsian surface group with the presentation (1). By the Hurwitz-Riemann formula (4), $n = |\varphi| = \mu(\Gamma)/\mu(\Delta) = 4\pi(g-1)/\mu(\Delta)$ and therefore the maximal n is obtained for Δ with minimal area for which there exists a smooth epimorphism $\theta : \Delta \to \mathrm{Z}_n = \langle\varphi\rangle$. It is not difficult, but somewhat cumbersome, to show that $\mu(\Delta) = 4\pi(g-1)/(4g+2)$ and the bound is attained for Δ with the presentation

$$\langle x_1, x_2, x_3 \,|\, x_1^2, x_2^{2g+1}, x_3^{4g+2}, x_1x_2x_3\rangle$$

for which $n = 4g+2$. Up to a generator φ of Z_n, the corresponding smooth epimorphism is given by

$$\theta(x_1) = \varphi^{2g+1}, \theta(x_2) = \varphi^2, \theta(x_3) = \varphi^{2g-1}$$

and so by the Macbeath theorem, φ has just one fixed point. Conversely, having $g \geq 2$, let $n = 4g+2$ and let Δ and θ be defined as above. Then, for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ is a Riemann surface having an automorphism of order n with a single fixed point. □

Later, in [28], it was realized that:

Theorem 3.4.2 (T. Szemberg 1991). *If a single self-homeomorphism has at least 2 fixed points, then the Wiman bound can be improved to $4g$ and again this bound is attained for arbitrary $g \geq 2$.*

The following two results are also in the same spirit. Namely:

Theorem 3.4.3 (Y. Izumi 1982 ([22])). *A self-homeomorphism of finite order n of a closed orientable topological surface of genus g has at most*

$$2\,\frac{g}{n-1} + 2$$

fixed points. In particular, a periodic self-homeomorphism of a closed orientable surface of genus $g \geq 2$ has at most $2g+2$ fixed points and this bound is attained only for self-homeomorphisms of order 2.

and:

Theorem 3.4.4 (H.M. Farkas, I. Kra 1980 ([10])). *A periodic self-homeomorphism of a closed orientable topological surface of genus $g \geq 2$, having $q \geq 3$ fixed points has order not exceeding*

$$M = 2\,\frac{g}{q-2} + 1.$$

Furthermore given $g \geq 2$ and $q \geq 3$ for which the above M is an integer there exists a closed orientable topological surface of genus g and its periodic self-homeomorphism having q fixed points and order M.

The two results and a conjecture below can be found in [6] and concern orientation preserving continuous involutions.

Theorem 3.4.5 (E. Bujalance, G. Gromadzki, E. Tyszkowska 2009). *Let $\tau_i, i = 1, 2$ be two orientation preserving involutions of a closed orientable topological surface X of genus $g \geq 2$ such that the orbit spaces X/τ_i have genera p_i and let $n < \infty$ be the order of their product. Then*

$$g \leq \frac{n}{n-1}\,(p_1 + p_2) + 1.$$

Theorem 3.4.6 (E. Bujalance, G. Gromadzki, E. Tyszkowska 2009). *Given k commuting involutions of a closed topological surface of genus $g \geq 2$, they have at most*

$$2g + 2^{k+1} - 2$$

fixed points in total and this bound is attained for arbitrary k and g for which

$$g \equiv 1 \bmod 2^{k-1} \text{ and } k \leq (g-1)/2^{k-2} + 3.$$

Remark and Conjecture. The above bound holds also for k, not necessarily commuting, continuous involutions generating finite groups of self-homeomorphisms. Even more, it is not attained in such a case and so can be significantly improved. Hence we obtain a problem of finding the sharp bound for a set of continuous involutions generating a finite group of self-homeomorphisms, without any assumptions on their commutativity.

3.5. **Order vs number of fixed points for periodic self-homeomorphisms of bordered compact surfaces.** Here we survey some results concerning self-homeomorphisms of bordered topological surfaces. By the *algebraic genus* of such a surface X we mean the genus of its canonical unbordered orientable double cover. It is equal to $\varepsilon g + k - 1$, where k is the number of the boundary components, $\varepsilon = 2$ or 1 according to X being orientable or not and g is the topological genus, that is the number of hands or cross-caps attached to the sphere, respectively.

Theorem 3.5.1 (C. Corrales, J.M. Gamboa, G. Gromadzki 1999 ([8])). *A periodic self-homeomorphism of a compact bordered topological surface of algebraic genus $p \geq 2$ (the genus of its canonical unbordered orientable double cover), having $q \geq 2$ fixed points has order not greater than*

$$N = \frac{p}{q-1} + 1. \tag{5}$$

Moreover, if the above bound (5) *is attained, then the corresponding surface X is orientable, the self-homeomorphism preserves the orientation of X, the number k of connected components of ∂X is a divisor of N and finally*

$$q \equiv k \bmod 2$$

if N is even.

The next result shows that these conditions are also sufficient for (5) to be attained:

Theorem 3.5.2 (C. Corrales, J.M. Gamboa, G. Gromadzki 1999 ([8])). *Let $p \geq 2$ and $q \geq 2$ be integers such that the above N is an integer. Let k be a positive divisor of N such that $q \equiv k \bmod 2$ if N is even. Then there exists an orientable compact bordered surface X of algebraic genus p, whose boundary has k connected components, admitting an orientation-preserving selfhomeomorphism φ of order N which fixes q points.*

In the next two results we investigate the case of bordered non-orientable surfaces, for which the above bound (5) is not attained:

Theorem 3.5.3 (C. Corrales, J.M. Gamboa, G. Gromadzki 1999 ([8])). *A periodic self-homeomorphism of a non-orientable, compact, bordered topological surface of algebraic genus $p \geq 2$ having $q \geq 2$ fixed points has order not greater than*

$$N = \frac{2p-1}{2q-1} + 1. \tag{6}$$

Theorem 3.5.4 (C. Corrales, J.M. Gamboa, G. Gromadzki 1999 ([8])). *Let $p \geq 2, q \geq 2$ and $k \geq 1$ be integers such that N is an integer. Then, there exists a non-orientable compact bordered topological surface X of algebraic genus p, with k boundary components, and a self-homeomorphism of X of order N, as in* (6)*, with q fixed points if and only if p is odd and $k = N/2$.*

3.6. **Finite dynamics on closed orientable surfaces.** Here we shall give a sample of results, proved recently in [18], describing singular orbits of finite cyclic groups actions, by which we understand orbits whose cardinality is smaller than the order of the acting group, on a closed orientable topological surface. Observe that this concept concerns actually fixed points, since each element of a singular orbit must be a fixed point of some element of the group acting.

To state the result for orientation reversing cyclic action on a closed orientable topological surface through its self-homeomorphisms, consider the following properties for a sequence of integers $m_1, \dots, m_r$ and N.

(0) All m_i divide N, all N/m_i are even and $\operatorname{lcm}(m_1, \dots, m_r) \neq N$.
(1) All m_i divide N, all N/m_i are even, $\operatorname{lcm}(m_1, \dots, m_r) = N/2$ and the number of i for which $\exp_2(N/m_i) = 1$ is odd.
(2) All m_i divide N, all N/m_i are even, $\operatorname{lcm}(m_1, \dots, m_r) \neq N$ and $N/2$ is odd or the number of i for which $\exp_2(N/m_i) = 1$ is even but they do not satisfy (1).
(3) The numbers $N, m_1, \dots, m_r \geq 2$ satisfy (0) but they do not satisfy (1) or (2).

Then we have the following result:

Theorem 3.6.1 (G. Gromadzki, W. Marzantowicz 2011 ([18]))**.** *Given $g \geq 2$ there exists an orientation reversing self-homeomorphism of order $2 < N < \infty$ of a closed orientable surface of some genus g with the periods (the sizes of all singular orbits)*

$$N_1, \overset{n_1}{\dots}, N_1, \dots, N_r, \overset{n_r}{\dots}, N_r$$

if and only if all the N_i are even, divide N and

$$\operatorname{lcm}(N/N_1, \dots, N/N_r) \neq N.$$

Furthermore, the minimal genus of such a surface is

- $\frac{1}{2}(\eta N - N_1) + 1$, *where $\eta = 1$ if N_1 is a multiple of 4 or N is not a multiple of 4 and $\eta = 2$ otherwise for $r = 1$,*
- $\frac{1}{2}N + 1$, *for $r = 2$ and $N_1 = N_2 = N/2$*
- $\frac{1}{2}N\big(\alpha + r - 2\big) - \frac{1}{2}\big(N_1 + \dots + N_r\big) + 1,$

where $\alpha = 1, 2$ or 3 according to whether $N, m_1, \dots, m_r$ satisfy respectively $(1), (2)$ or (3).

Similar results for orientation preserving self-homeomorphisms of orientable surfaces and for self-homeomorphisms of non-orientable surfaces were proved in [18].

3.7. **Invariant subsets on closed surfaces.** Let G be a finite group of self-homeomorphisms of a closed orientable topological surface. A G-invariant subset of X is said to be *irreducible* if it has no G-invariant proper subsets and a G-invariant subset of X is said to be *essential* if its cardinality is smaller than the order of G. Observe that on a closed topological surface with a finite group of homeomorphisms G there is only a finite number of singular orbits.

In the middle of the fifties, Oikawa [26] proved:

Theorem 3.7.1 (K. Oikawa 1956). *Let X be a closed orientable topological surface of genus $g \geq 2$ with a finite group of self-homeomorphisms G and with a G-invariant subset of cardinality k. Then $|G| \leq 12(g-1)+6k$ and this bound is attained if and only if G can be generated by two elements of order 2 and 3.*

Later Arakawa [2] showed that:

Theorem 3.7.2 (T. Arakawa 2000). *Let X be a closed orientable topological surface of genus $g \geq 2$ with a finite group of self-homeomorphisms G and with three G-invariant subsets of cardinalities k, l and m. Then $|G| \leq 2(g-1)+k+l+m$.*

Arakawa proved also that:

Theorem 3.7.3 (T. Arakawa 2000). *If X is a closed orientable topological surface of genus $g \geq 2$ with a finite group of self-homeomorphisms G and with two G-invariant subsets of cardinalities k and l, then $|G| \leq 8(g-1)+k+4l$.*

However, in [5] we have proved that the above Arakawa bound is never attained, as the next result shows:

Theorem 3.7.4 (E. Bujalance, G. Gromadzki 2011). *Let X be a closed orientable topological surface of genus $g \geq 2$ with a finite group of self-homeomorphisms G and with two G-invariant subsets of the cardinalities k and l. Then either $|G| \leq 2(g-1)+k+l$ or G has order precisely*

$$\frac{m}{m-1}\big(2(g-1)+k+l\big)$$

for some $m \geq 2$. Furthermore, given $m \geq 2$ there are infinitely many values of g for which there exists a group of self-homeomorphisms of the above order of a closed orientable topological surface of genus $g \geq 2$, admitting two proper irreducible invariant subsets of cardinalities k and l. In such a case, the orbit space X/G must be the sphere with exactly three cone points of orders $|G|/k, |G|/l$ and $|G|/m$.

Then, using another well known theorem of Oikawa from [26], concerning filling holes, extending self-homeomorphisms and the canonical Riemann double cover, we get similar results for bordered topological surfaces. Namely:

Theorem 3.7.5 (E. Bujalance, G. Gromadzki 2011)**.** *Let X be a closed orientable topological surface of genus $g \geq 2$ with a finite group of self-homeomorphisms G and with G-invariant irreducible subsets $B_1, \ldots, B_n$ of cardinalities $q_1 \leq \ldots \leq q_n$ and assume that $s \geq 4$ of them are essential. Then each q_i divides $|G|$ and*

$$|G| \leq \frac{2}{s-2}\big(g-1\big) + \frac{q_1 + \ldots + q_s}{s-2}.$$

Conversely, for each s, these bounds are attained for infinitely many values of g.

3.8. **The number of self-homeomorphisms of prime order vs their number of fixed points.** A self-homeomorphism φ of a prime order p of a closed orientable topological surface X of genus $g \geq 2$ for which the orbit space X/φ is a topological surface of genus n is said to be (p,n)*-gonal self-homeomorphism* and the group generated by such a self-homeomorphism to be (p,n)*-gonal group*. In such a case we can view X as a Riemann surface $\mathcal{H}/\Gamma$ and $\langle\varphi\rangle$ as its group of automorphisms Δ/Γ, where Δ has the presentation

$$\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_k \mid x_1^p, \ldots, x_k^p, x_1 \ldots x_k [a_1, b_1] \ldots [a_g, b_g]\rangle.$$

Here k is the number of points fixed by φ and, due to the Macbeath theorem,

$$k = 2\,\frac{g-1-p(n-1)}{p-1}$$

by the Hurwitz-Riemann formula. In particular we see that given g and (p,n)-gonal self-homeomorphism on a closed topological surface of genus g, the number p determines n and viceversa, and the classical Castelnuovo-Severi inequality [7, 27] asserts that:

Theorem 3.8.1 (Castelnuovo 1937, Severi 1921)**.** *If $g > 2pn + (p-1)^2$, then there is at most one (p,n)-gonal subgroup in arbitrary finite group of self-homeomorphisms of a closed topological orientable surface of genus $g \geq 2$.*

Then:

Theorem 3.8.2 (G. Gonzalez-Diez 1995, G. Gromadzki, A. Weaver, A. Wootton 2011). *If $g \leq 2pn+(p-1)^2$, then any two groups generated by two (p,n)-gonal self-homeomorphisms, which are contained in some finite group G of self-homeomorphisms of a closed topological orientable surface of genus $g \geq 2$ are conjugate in G.*

This was proved in [12] (see also [14] for alternative proof) for $n = 0$ and in [19] for $n = 1$.

The last results in this section and in the whole survey deal with the size of the above conjugacy classes. Namely, we have that for $n = 0$:

Theorem 3.8.3 (G. Gromadzki 2009 ([16])). *Given a prime p, a closed topological orientable surface X of genus $g \geq 2$, admits at most:*

$$6(g-1)(p-1)/(g+p-1)(p-6), 16(g-1)/(g+4) \text{ and } 28(g-1)/(g+2),$$

$(p,0)$-gonal groups, for a prime p, where $p \geq 7, p = 5$ and $p = 3$ respectively, provided that they are all contained in some finite group of self-homeomorphisms.

For arbitrary n we have similar results, under some extra assumption:

Theorem 3.8.4 (G. Gromadzki, A. Weaver, A. Wootton 2010 ([19])). *The maximum number of (p,n)-gonal groups, $p > 2n+1$, $n \geq 1$, contained in some finite group of self-homeomorphisms of a closed orientable topological surface of genus $g \geq 2$ is*

$$\begin{array}{ll} \dfrac{28(g-1)}{(g+2-3n)} & \text{if} \quad p = 3, \\ \dfrac{16(g-1)}{(g+4-5n)} & \text{if} \quad p = 5, \\ \dfrac{6(g-1)(p-1)}{(g+p-1-np)(p-6)} & \text{if} \quad p \geq 7. \end{array}$$

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