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CONCERNING THE MUTUALLY APOSYNDETIC DECOMPOSITION OF PRODUCTS OF HOMOGENEOUS CONTINUA

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Abstract. We prove that the product of a homogeneous mutually aposyndetic continuum and any continuum is mutually aposyndetic. We give some conditions for which a product of two homogeneous decomposable continua is mutually aposyndetic. Certain products involving continuous curves of pseudo-arcs are shown to be mutually aposyndetic. In particular, if a product of two solenoids is mutually aposyndetic, related products involving solenoids of pseudo-arcs are shown to be mutually aposyndetic. We also determine the mutually aposyndetic decomposition of a pseudo-arc and certain other homogeneous continua.

1. Introduction

A continuum X is *aposyndetic at x with respect to y* if there exists a subcontinuum of X containing x in its interior, but not containing y. A stronger condition is that X be *mutually aposyndetic with respect to x and y*, which means that there are two disjoint subcontinua of X, with x contained in the interior of one of the subcontinua and y contained in the interior of the other subcontinuum. Since the product of two nondegenerate continua is always aposyndetic, it is interesting to study the mutual aposyndesis of products.

C. L. Hagopian proved that the product of two aposyndetic continua is always *mutually aposyndetic*, which means it is mutually aposyndetic with respect to any two of its distinct points [1, Theorem 1]. He proved that the

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product of three non-degenerate continua is always mutually aposyndetic [1, Theorem 2]. He also proved that the product of two pseudo-arcs is *semi-indecomposable*, which means it is not mutually aposyndetic with respect to any pair of its points [1, Theorem 10]. (Hagopian used the term "strictly nonmutually aposyndetic" rather than "semi-indecomposable.")

Alejandro Illanes proved that the product of a p-adic solenoid and a q-adic solenoid is mutually aposyndetic if p and q are relatively prime [2]. Janusz R. Prajs proved that the product of any two solenoids is either mutually aposyndetic or semi-indecomposable, and he characterized which ones are mutually aposyndetic and which are semi-indecomposable [5, Corrollary 4.2].

Prajs also determined the mutually aposyndetic decomposition of the product of a pseudo-arc and a continuous curve of pseudo-arcs with quotient a circle or a Menger curve [7, Example 9.5].

In this paper, we will continue the study of the mutual aposyndesis of products of homogeneous continua.

2. Preliminaries

A *continuum* is a compact, connected metric space, and a *curve* is a one-dimensional continuum. A continuum X is *homogeneous* if, for every $x, y \in X$, there exists a homeomorphism $h : (X, x) \to (X, y)$.

A continuum is *decomposable* if it is the union of two of its proper subcontinua; otherwise, it is indecomposable. It is known that every proper subcontinuum of an indecomposable continuum has empty interior. This follows from [4, Proposition 6.3].

A continuum is *hereditarily indecomposable* if every subcontinuum of the continuum is indecomposable. It is easy to prove that if A and B are subcontinua of a hereditarily indecomposable continuum and $A \cap B \neq \emptyset$, then $A \subseteq B$ or $B \subseteq A$.

We use the symbol $N(A, \epsilon)$ to represent the set of points less than ϵ distance from the set A.

A *map* is a continuous function. A map is *monotone* if the inverse image of each point is connected. It is known that the inverse image of a continuum under a monotone surjective map between continua is a continuum. This follows from [4, Exercise 8.46].

For a product $X \times Y$, we will use $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ to represent the projections. The domain will be determined from the context.

A *decomposition* of a space is a partition of the space, where each partition element is considered to be a point in a space endowed with the quotient topology. If the quotient map is closed, the decomposition is *upper-semicontinuous.* If it is both open and closed, the decomposition is *continuous.*

A subcontinuum T is *terminal* in X if, for every subcontinuum C such that $C \cap T \neq \emptyset$, then $C \subset T$ or $T \subset C$. A *decomposition of X is terminal* if each decomposition element is a terminal subcontinuum. In this case, each subcontinuum of X is either contained in a decomposition element or is a union of decomposition elements.

A subcontinuum S of a continuum X is *semi-terminal* in X if whenever K_1 and K_2 are subcontinua of X each intersecting both S and its complement, $K_1 \cap K_2 \neq \emptyset$. We will use the fact that any proper semi-terminal subcontinuum of a mutually aposyndetic homogeneous continuum has empty interior [7, Theorem 5.4].

The continuum $\hat{X}$ is a *continuous curve of pseudo-arcs with quotient X* if it has a continuous terminal decomposition into pseudo-arcs with quotient space X. A *circle of pseudo-arcs* is a continuous curve of pseudo-arcs with quotient a circle, and a *solenoid of pseudo-arcs* is a continuous curve of pseudo-arcs with quotient a solenoid. It is known that any two continuous curves of pseudo-arcs with the same quotient are homeomorphic, and that for each homogeneous curve X, there is a continuous curve of pseudo-arcs with quotient X [3].

The aposyndetic decomposition of a homogeneous continuum X is the decomposition $\{L(x) : x \in X\}$, where $L(x)$ is the set of points y in X such that X is not aposyndetic at y with respect to x. It is known that the aposyndetic decomposition of a homogeneous decomposable continuum X is continuous and has elements that are terminal, mutually homeomorphic, homogeneous, indecomposable continua, and the quotient space is a homogeneous aposyndetic continuum [8, Theorem 1]. If X is decomposable but not aposyndetic, the quotient space is a curve [9, p. 3285].

The *mutually aposyndetic decomposition* of a homogeneous continuum X is the decomposition $\{M(x) : x \in X\}$, where $M(x)$ is the set of points y in X such that X is not mutually aposyndetic with respect to x and y. It is known that the mutually aposyndetic decomposition of a homogeneous continuum is a continuous decomposition whose elements are closed, homogeneous, and mutually homeomorphic, and the quotient space is a homogeneous mutually aposyndetic continuum [6, Theorem 3.1].

It follows easily from the definitions that a mutually aposyndetic continuum is aposyndetic. Since an indecomposable continuum contains no proper subcontinuum with interior, it follows easily that an indecomposable continuum is semi-indecomposable.

A subcontinuum A of a continuum X is *ample* in X if, for every open set U such that $A \subseteq U$, there is a subcontinuum L of X such that $A \subseteq \mathrm{int}(L) \subseteq U$. A subcontinuum with non-empty interior in a homogeneous continuum is always ample. This follows from [6, Proposition 1.1]. A homogeneous continuum is mutually aposyndetic with respect to x and y if and only if there exist disjoint ample continua A_1 and A_2 such that $x \in A_1$ and $y \in A_2$ [6, p. 4].

We will make use of the following three lemmas below.

Lemma 2.1 (Prajs [6, Remark 4.7])**.** *If X is a homogeneous continuum that is not mutually aposyndetic, then the closure of each arc component of X is contained in a single element of the mutually aposyndetic decomposition of X.*

Lemma 2.2 (Prajs [7, Proposition 4.7])**.** *Let A_1 and A_2 be subcontinua of the product $P \times P$ of two pseudo-arcs. If $\pi_1(A_1) = P = \pi_2(A_2)$, then $A_1 \cap A_2 \neq \emptyset$.*

The following easy lemma was assumed but not explicitly proven by Prajs in [7, Example 9.5]. We provide the proof for the convenience of the reader.

Lemma 2.3. *Let $\mathcal{D}$ be an upper semi-continuous decomposition of the homogeneous continuum X whose elements are subcontinua of X. If the quotient space for $\mathcal{D}$ is mutually aposyndetic, then each element of the mutually aposyndetic decomposition of X is contained in a single element of $\mathcal{D}$.*

Proof. Let $x, y \in X$ such that x and y belong to different elements of $\mathcal{D}$. Let $q : X \to Y$ be the quotient map for $\mathcal{D}$. Then $q(x) \neq q(y)$. Since Y is mutually aposyndetic, there exist disjoint subcontinua K_1 and K_2 of Y such that $q(x) \in \mathrm{int}(K_1)$ and $q(y) \in \mathrm{int}(K_2)$. Since the elements of $\mathcal{D}$ are continua, q is monotone. Then $q^{-1}(K_1)$ and $q^{-1}(K_2)$ are disjoint subcontinua of X. Since $x \in q^{-1}(\mathrm{int}(K_1)) \subseteq q^{-1}(K_1)$ and $y \in q^{-1}(\mathrm{int}(K_2)) \subseteq q^{-1}(K_2)$, and $q^{-1}(\mathrm{int}(K_1))$ and $q^{-1}(\mathrm{int}(K_2))$ are open, X is mutually aposyndetic with respect to x and y. Therefore, each element of the mutually aposyndetic decomposition of X must be contained in a single element of $\mathcal{D}$. □

3. Main Results

Our first theorem concerns the case where one of the factor spaces in a product is a mutually aposyndetic continuum.

Theorem 3.1. *The product of a non-degenerate homogeneous mutually aposyndetic continuum and any continuum is mutually aposyndetic.*

Proof. Let X be a non-degenerate homogeneous mutually aposyndetic continuum, and let Y be a continuum. Since $X \times Y$ is clearly mutually aposyndetic if Y is degenerate, we may assume Y is non-degenerate. Let (x_1, y_1) and (x_2, y_2) be distinct points in $X \times Y$.

Case 1: $x_1 \neq x_2$. Since X is mutually aposyndetic, there exists disjoint subcontinua K_1 and K_2 of X such that $x_1 \in \text{int}(K_1)$ and $x_2 \in \text{int}(K_2)$. Then, since projection onto the first coordinate is a monotone map, $C_1 = \pi_1^{-1}(K_1)$ and $C_2 = \pi_1^{-1}(K_2)$ are disjoint subcontinua of $X \times Y$. Also, since $\pi_1^{-1}(\text{int}(K_1)) \subseteq C_1$ and $\pi_1^{-1}(\text{int}(K_2)) \subseteq C_2$, $(x_1, y_1) \in \text{int}(C_1)$ and $(x_2, y_2) \in \text{int}(C_2)$.

Case 2: $x_1 = x_2$. Let $x = x_1 = x_2$. Since X is aposyndetic, there exists a proper subcontinuum C of X such that $x \in \text{int}(C)$. Since X is a homogeneous mutually aposyndetic continuum, every proper semi-terminal subcontinuum of X has empty interior [7, Theorem 5.4]. Then C is not semi-terminal. Hence, there exist disjoint subcontinua K_1 and K_2 of X that each intersects both C and its complement. Then $A_1 = C \cup K_1$ and $A_2 = C \cup K_2$ are subcontinua of X, and $x \in \text{int}(A_1)$ and $x \in \text{int}(A_2)$.

Let $p_1 \in K_1 - C$. Then, since K_1 and K_2 are disjoint, $p_1 \in A_1 - A_2$. A similar argument shows there exists $p_2 \in A_2 - A_1$. Clearly, $p_1 \neq p_2$.

Since (x_1, y_1) and (x_2, y_2) are distinct and $x_1 = x_2$ and $y_1 \neq y_2$, there exist open neighborhoods U_1 and U_2 in Y of y_1 and y_2, respectively, such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Then $(x, y_1) \in \text{int}(A_1 \times \bar{U}_1)$, and $A_1 \times \bar{U}_1$ is closed, and hence compact. Likewise, $(x, y_2) \in \text{int}(A_2 \times \bar{U}_2)$ and $A_2 \times \bar{U}_2$ is compact. Also, $A_1 \times \bar{U}_1 = \cup\{A_1 \times \{y\} : y \in \bar{U}_1\}$ and $A_2 \times \bar{U}_2 = \cup\{A_2 \times \{y\} : y \in \bar{U}_2\}$ are each unions of continua. Since $p_1 \in A_1$ and $p_2 \in A_2$, $C_1 = (A_1 \times \bar{U}_1) \cup (\{p_1\} \times Y)$ and $C_2 = (A_2 \times \bar{U}_2) \cup (\{p_2\} \times Y)$ are continua, and $(x_1, y_1) = (x, y_1) \in \text{int}(C_1)$ and $(x_2, y_2) = (x, y_2) \in \text{int}(C_2)$.

Since $\bar{U}_1 \cap \bar{U}_2 = \emptyset$, $(A_1 \times \bar{U}_1) \cap (A_2 \times \bar{U}_2) = \emptyset$. Since $p_1 \neq p_2$, $(\{p_1\} \times Y) \cap (\{p_2\} \times Y) = \emptyset$. Since $p_1 \notin A_2$, $(\{p_1\} \times Y) \cap (A_2 \times \bar{U}_2) = \emptyset$. Since $p_2 \notin A_1$, $(\{p_2\} \times Y) \cap (A_1 \times \bar{U}_1) = \emptyset$. Then C_1 and C_2 are disjoint.

Since in both cases there exist disjoint subcontinua C_1 and C_2 of $X \times Y$ such that $(x_1, y_1) \in \text{int}(C_1)$ and $(x_2, y_2) \in \text{int}(C_2)$, it follows that $X \times Y$ is mutually aposyndetic with respect to (x_1, y_1) and (x_2, y_2). Since these two points are arbitrary distinct points in $X \times Y$, it follows that $X \times Y$ is mutually aposyndetic. □

The following example, suggested by the referee of this paper, shows that the requirement in Theorem 3.1 that the mutually aposyndetic continuum be homogeneous is necessary.

Example 3.2. Let P be a pseudo-arc, and consider the product of the arc $[0, 1]$ and P. The arc is mutually aposyndetic and nearly homogeneous, but we show that $[0, 1] \times P$ is not mutually aposyndetic.

Let p_1 and p_2 be distinct points in P. Let K_1 and K_2 be subcontinua of $[0,1] \times P$ such that $(0,p_1) \in \operatorname{int}(K_1)$ and $(0,p_2) \in \operatorname{int}(K_2)$. Suppose K_1 and K_2 are disjoint.

Since K_1 and K_2 each has non-empty interior, $\pi_1(K_1)$ and $\pi_1(K_2)$ are non-degenerate subcontinua of $[0,1]$. Then both $\pi_1(K_1)$ and $\pi_1(K_2)$ are subarcs of $[0,1]$ containing 0. It follows that either $\pi_1(K_1) \subseteq \pi_1(K_2)$ or $\pi_1(K_2) \subseteq \pi_1(K_1)$. Without loss of generality, assume $\pi_1(K_1) \subseteq \pi_1(K_2)$.

Also, since K_1 and K_2 each has non-empty interior, $\pi_2(K_1)$ and $\pi_2(K_2)$ are each subcontinua of P with non-empty interior. Since P is indecomposable, $\pi_2(K_1) = \pi_2(K_2) = P$.

Since K_1 and K_2 are disjoint, there exists an $\epsilon > 0$ such that the ϵ-neighborhood of K_1 does not intersect K_2. Since P is chainable, there exists a surjective ϵ-map $g : P \to [0,1]$. If $\mathrm{id} : [0,1] \to [0,1]$ is the identity map, then $f = \mathrm{id} \times g : [0,1] \times P \to [0,1] \times [0,1]$ is an ϵ-map.

Let $\rho_1 : [0,1]\times[0,1] \to [0,1]$ and $\rho_2 : [0,1]\times[0,1] \to [0,1]$ be projections onto the first and second coordinates, respectively. Then $\rho_1(f(K_1)) = \pi_1(K_1) \subseteq \pi_2(K_2)$ and $\rho_1(f(K_2)) = \pi_1(K_2)$. Then both $f(K_1)$ and $f(K_2)$ are subcontinua of $\pi_1(K_2) \times [0,1]$, a product of two arcs. Also note that $\rho_2(f(K_1)) = g(\pi_2(K_1)) = g(P) = [0,1]$. Since $\rho_1(f(K_2)) = \pi_1(K_2)$ and $\rho_2(f(K_1)) = [0,1]$, $f(K_1)$ and $f(K_2)$ must intersect. Since f is an ϵ-map, there is a point of K_1 and a point of K_2 such that the distance between the points is less than ϵ. This is impossible, since the ϵ-neighborhood of K_1 does not intersect K_2. Hence, K_1 and K_2 cannot be disjoint, so $[0,1] \times P$ is not mutually aposyndetic with respect to $(0,p_1)$ and $(0,p_2)$.

Now we consider products involving spaces with dense arc components.

Theorem 3.3. *Let X and Y be non-degenerate homogeneous continua.*

(1) *If X has dense arc components, and $q : Y \to Q$ is the quotient map for an upper-semicontinuous decomposition of Y into subcontinua, and $X \times Q$ is mutually aposyndetic, then $X \times Y$ is mutually aposyndetic.*

(2) *If $q_1 : X \to Q_1$ and $q_2 : Y \to Q_2$ are quotient maps for upper-semicontinuous decompositions consisting of subcontinua, and both Q_1 and Q_2 are homogeneous and have dense arc components, and $Q_1 \times Q_2$ is mutually aposyndetic, then $X \times Y$ is mutually aposyndetic.*

Proof. We prove statement (1): Let $\mathcal{D} = \{\{x\} \times q^{-1}(z) : (x,z) \in X \times Q\}$. Since the mutually aposyndetic space $X \times Q$ is the quotient space for $\mathcal{D}$, and the elements of $\mathcal{D}$ are continua, it follows from Lemma 2.3 that each element of the mutually aposyndetic decomposition of $X \times Y$ is contained in a single element of $\mathcal{D}$.

Let $(x, y) \in X \times Y$, and let M be the element of the mutually aposyndetic decomposition of $X \times Y$ containing (x, y). Since X has dense arc components, the closure of the arc component of $X \times Y$ containing (x, y) contains $X \times \{y\}$. If $X \times Y$ is not mutually aposyndetic, it follows from Lemma 2.1 that $X \times \{y\} \subseteq M$. However, since the element of $\mathcal{D}$ containing (x, y) is $\{x\} \times q^{-1}(q(y))$, we must have $M \subseteq \{x\} \times q^{-1}(q(y))$. This is impossible, since $X \times \{y\}$ is not a subset of $\{x\} \times q^{-1}(q(y))$. Hence, $X \times Y$ is mutually aposyndetic.

Now we prove statement (2): Let $\mathcal{D}_1 = \{\{x\} \times q_2^{-1}(z) : (x, z) \in X \times Q_2\}$ and $\mathcal{D}_2 = \{q_1^{-1}(z) \times \{y\} : (z, y) \in Q_1 \times Y\}$. Since Q_2 and X and Q_1 and Y each satisfies the hypotheses of (1), $X \times Q_2$ and $Q_1 \times Y$ are each mutually aposyndetic. Then, by Lemma 2.3, since $X \times Q_2$ is the quotient space for $\mathcal{D}_1$ and $Q_1 \times Y$ is the quotient space for $\mathcal{D}_2$, each element of the mutually aposyndetic decomposition of $X \times Y$ must be contained in a single element of $\mathcal{D}_1$ and also in a single element of $\mathcal{D}_2$. Then, if M is the element of the mutually aposyndetic decomposition of $X \times Y$ containing a point (x, y), $M \subseteq \left[\{x\} \times q_2^{-1}(q_2(y))\right] \cap \left[q_1^{-1}(q_1(x)) \times \{y\}\right] = \{(x, y)\}$. Hence, each element of the mutually aposyndetic decomposition is degenerate. Therefore, $X \times Y$ is mutually aposyndetic. □

Corollary 3.4. *Let X be a non-degenerate homogeneous continuum with dense arc components, Y be a homogeneous curve, and $\hat{Y}$ be the continuous curve of pseudo-arcs with quotient Y. If either X or Y is mutually aposyndetic, or if both X and Y are aposyndetic, then $X \times \hat{Y}$ is mutually aposyndetic. If, in addition, Y has dense arc components, X is one-dimensional, and $\hat{X}$ is the continuous curve of pseudo-arcs with quotient X, then $\hat{X} \times \hat{Y}$ is mutually aposyndetic.*

Proof. The corollary follows easily from Theorem 3.3, Theorem 3.1, and the fact that the product of two aposyndetic continua is mutually aposyndetic [1, Theorem 1]. □

Since every known homogeneous aposyndetic curve is mutually aposyndetic and has dense arc components, and solenoids have dense arc components, the above corollary has many applications. For example, the product of a solenoid and a circle of pseudo-arcs is mutually aposyndetic, and the product of a circle of pseudo-arcs and a continuous curve of pseudo-arcs with quotient a Menger curve is mutually aposyndetic.

Corollary 3.5. *Let S_1 and S_2 be solenoids, and let $\hat{S}_1$ and $\hat{S}_2$ be solenoids of pseudo-arcs with quotients S_1 and S_2, respectively. If $S_1 \times S_2$ is mutually aposyndetic, then $S_1 \times \hat{S}_2$, $\hat{S}_1 \times S_2$, and $\hat{S}_1 \times \hat{S}_2$ are mutually aposyndetic.*

Proof. The corollary follows easily from Theorem 3.3 and the fact that solenoids have dense arc components. □

Now we prove a theorem about the product of two homogeneous decomposable continua.

Theorem 3.6. *Let X and Y be homogeneous decomposable continua. Then $X \times Y$ is mutually aposyndetic if at least one of the following holds:*

(1) *Either X or Y is mutually aposyndetic.*
(2) *X and Y are both aposyndetic.*
(3) *Either X or Y is an aposyndetic continuum with dense arc components.*
(4) *Each of the quotient spaces for the aposyndetic decompositions of X and Y is either mutually aposyndetic or has dense arc components.*

Proof. If X and Y satisfy (1), the conclusion follows from Theorem 3.1.

If X and Y satisfy (2), the conclusion follows from [1, Theorem 1].

Suppose X and Y satisfy (3). Without loss of generality, assume X is an aposyndetic continuum with dense arc components. If Y is also aposyndetic, (2) is satisfied, so we may assume Y is not aposyndetic. The aposyndetic decomposition of Y is an upper-semicontinuous decomposition of Y into subcontinua, and the quotient space is a homogeneous aposyndetic curve Q. Since both X and Q are aposyndetic, $X \times Q$ is mutually aposyndetic. It follows from Theorem 3.3 that $X \times Y$ is mutually aposyndetic.

Suppose (4) is satisfied. Since we have already shown that $X \times Y$ is mutually aposyndetic if (1), (2), or (3) holds, we may assume, without loss of generality, that neither X nor Y is aposyndetic. Let $q_1 : X \to Q_1$ and $q_2 : Y \to Q_2$ be the quotient maps for the aposyndetic decompositions of X and Y, respectively. Since each of Q_1 and Q_2 is either mutually aposyndetic or aposyndetic and has dense arc components, X and Q_2, and Q_1 and Y, each satisfies either (1) or (3). Then $X \times Q_2$ and $Q_1 \times Y$ are mutually aposyndetic.

Let $\mathcal{D}_1 = \{\{x\} \times q_2^{-1}(z) : (x, z) \in X \times Q_2\}$ and $\mathcal{D}_2 = \{q_1^{-1}(z) \times \{y\} : (z, y) \in Q_1 \times Y\}$. Since the elements of both $\mathcal{D}_1$ and $\mathcal{D}_2$ are continua, and the quotient spaces for $\mathcal{D}_1$ and $\mathcal{D}_2$ are the mutually aposyndetic continua $X \times Q_2$ and $Q_1 \times Y$, each element of the mutually aposyndetic decomposition of $X \times Y$ must be contained in a single element of $\mathcal{D}_1$, and also in a single element of $\mathcal{D}_2$, by Lemma 2.3. Then, if M is the element of the mutually aposyndetic decomposition of $X \times Y$ containing a point (x, y), $M \subseteq \left[\{x\} \times q_2^{-1}(q_2(y))\right] \cap \left[q_1^{-1}(q_1(x)) \times \{y\}\right] = \{(x, y)\}$. It follows that $X \times Y$ is mutually aposyndetic. □

Prajs asked if every homogeneous aposyndetic curve is mutually aposyndetic [6, Question 2]. If the answer is yes, then it would follow from Theorem 3.6 that the product of homogeneous decomposable curves is always mutually aposyndetic. Also, if it is true that every homogeneous aposyndetic curve has dense arc components, it would follow that the product of homogeneous decomposable curves is always mutually aposyndetic.

The quotient space of the aposyndetic decomposition of a homogeneous decomposable, nonaposyndetic continuum is a curve. Then, if it is true that either every homogeneous aposyndetic curve is mutually aposyndetic or every homogeneous aposyndetic curve has dense arc components, it would follow that the product of homogeneous decomposable nonaposyndetic continua is mutually aposyndetic.

Prajs determined the mutually aposyndetic decomposition of the product of a pseudo-arc and a continuous curve of pseudo-arcs with quotient a circle or a Menger curve [7, Example 9.5]. In explaining this example, Prajs essentially proved the lemma below, although the hypothesis he used is less general than the one in the lemma. We provide the proof for the convenience of the reader.

Lemma 3.7 (Prajs)**.** *Suppose X is a homogeneous continuum with a continuous terminal decomposition into pseudo-arcs with quotient map $q : X \to Y$, and P is a pseudo-arc. Let $\mathcal{D} = \{q^{-1}(y) \times \{p\} : (y, p) \in Y \times P\}$. Then $X \times P$ is not mutually aposyndetic with respect to any pair of points belonging to the same element of $\mathcal{D}$.*

Proof. Let (x_1, p) and (x_2, p) be distinct points belonging to the same element of $\mathcal{D}$. Then $x_1 \neq x_2$ and $q(x_1) = q(x_2)$. Let $y = q(x_1)$. Let K_1 and K_2 be subcontinua of $X \times P$ such that $(x_1, p) \in \mathrm{int}(K_1)$ and $(x_2, p) \in \mathrm{int}(K_2)$. Let C_1 be the component of $\pi_1^{-1}(q^{-1}(y)) \cap K_1$ containing (x_1, p), and let C_2 be the component of $\pi_1^{-1}(q^{-1}(y)) \cap K_2$ containing (x_2, p).

Since K_1 has nonempty interior, and π_1 and q are open, $q(\pi_1(K_1))$ is non-degenerate. Then C_1 is a proper subcontinuum of K_1. Then, for each positive integer n, there exists a subcontinuum L_n of K_1 such that C_1 is a proper subset of L_n and $L_n \subseteq N(C_1, \frac{1}{n})$. Then $\pi_1(L_n)$ is not a subset of $q^{-1}(y)$, since L_n is connected and C_1 is a component of $\pi_1^{-1}(q^{-1}(y)) \cap K_1$. Then $\pi_1(L_n)$ must intersect more than one element of the terminal decomposition of X into pseudo-arcs. Then $\pi_1(L_n)$ must contain each element of that terminal decomposition that it intersects. In particular, $q^{-1}(y) \subseteq \pi_1(L_n)$. Since this is true for each n, $\pi_1(C_1) = q^{-1}(y)$. A similar argument shows $\pi_1(C_2) = q^{-1}(y)$.

Since $p \in \pi_2(C_1) \cap \pi_2(C_2)$ and $\pi_2(C_1)$ and $\pi_2(C_2)$ are subcontinua of the hereditarily indecomposable continuum P, either $\pi_2(C_1) \subseteq \pi_2(C_2)$ or $\pi_2(C_2) \subseteq \pi_2(C_1)$. Without loss of generality, assume $\pi_2(C_1) \subseteq \pi_2(C_2)$.

Case 1: $\pi_2(C_2) = \{p\}$. Then $\pi_2(C_1) = \{p\}$, so $C_1 = q^{-1}(y) \times \{p\} = C_2$. In this case, clearly $C_1 \cap C_2 \neq \emptyset$.

Case 2: $\pi_2(C_2)$ is a pseudo-arc. Then both C_1 and C_2 are subcontinua in the product of pseudo-arcs $q^{-1}(y) \times \pi_2(C_2)$. Since $\pi_1(C_1) = q^{-1}(y)$, by Lemma 2.2, $C_1 \cap C_2 \neq \emptyset$.

Since, in both cases, $C_1 \cap C_2 \neq \emptyset$, we have $K_1 \cap K_2 \neq \emptyset$. Hence, $X \times P$ is not mutually aposyndetic with respect to (x_1, p) and (x_2, p). □

The following is a generalization of Prajs's Example 9.5 [7].

Corollary 3.8. *Let X be a homogeneous decomposable, but not aposyndetic, continuum, such that the quotient space for the aposyndetic decomposition is mutually aposyndetic. Let $q : X \to Q$ be the quotient map for the aposyndetic decomposition of X, P be a pseudo-arc, and $\mathcal{D} = \{q^{-1}(y) \times \{p\} : (y,p) \in Q \times P\}$. Then each element of the mutually aposyndetic decomposition of $X \times P$ is contained in an element of $\mathcal{D}$. Furthermore, if the elements of the aposyndetic decomposition of X are pseudo-arcs, then the mutually aposyndetic decomposition of $X \times P$ is $\mathcal{D}$.*

Proof. Since Q is mutually aposyndetic, by Theorem 3.1, $Q \times P$ is mutually aposyndetic. Since $Q \times P$ is the quotient space for $\mathcal{D}$, and the elements of $\mathcal{D}$ are continua, by Lemma 2.3, each element of the mutually aposyndetic decomposition of $X \times P$ is contained in an element of $\mathcal{D}$. If the elements of the aposyndetic decomposition of X are pseudo-arcs, by Lemma 3.7, $X \times P$ is not mutually aposyndetic with respect to any pair of points belonging to the same element of $\mathcal{D}$. Then, in this case, $\mathcal{D}$ is the mutually aposyndetic decomposition of $X \times P$. □

Lemma 3.9. *Let X be a homogeneous continuum with a continuous decomposition into closed sets such that the quotient space is semi-indecomposable, and X is not mutually aposyndetic with respect to any pair of points belonging to the same decomposition element. Then X is semi-indecomposable.*

Proof. Let $\mathcal{D}$ be the continuous decomposition of X mentioned in the theorem, and let $q : X \to Y$ be the quotient map for the decomposition. Let K_1 and K_2 be subcontinua of X with nonempty interiors. Since $\mathcal{D}$ is a continuous decomposition, q is open. Then $q(K_1)$ and $q(K_2)$ are subcontinua of Y with nonempty interiors. Since Y is semi-indecomposable, $q(K_1) \cap q(K_2) \neq \emptyset$. Then there exists $x_1 \in K_1$ and $x_2 \in K_2$ such that x_1 and x_2 belong to the same element of $\mathcal{D}$. Then X is not mutually aposyndetic with respect to x_1 and x_2. Since K_1 and K_2 have nonempty interiors, they are ample subcontinua of X. Hence, $K_1 \cap K_2 \neq \emptyset$. Since

K_1 and K_2 were arbitrary subcontinua of X with nonempty interiors, it follows that X is semi-indecomposable. $\square$

We noted in the introduction that the product of two solenoids is either mutually aposyndetic or semi-indecomposable. We now generalize this statement.

Lemma 3.10. *The product of a homogeneous continuum with dense arc components and a homogeneous semi-indecomposable continuum is either mutually aposyndetic or semi-indecomposable.*

Proof. Let X be a homogeneous continuum with dense arc components and Y be a homogeneous semi-indecomposable continuum. If $X \times Y$ is not mutually aposyndetic, then, by Lemma 2.1, the closure of each arc component of $X \times Y$ must be contained in a single element of the mutually aposyndetic decomposition of $X \times Y$. Since X has dense arc components, $X \times \{y\}$ is contained in the closure of the arc component of a point $(x, y) \in X \times Y$. Let $\mathcal{D} = \{X \times \{y\} : y \in Y\}$. If $X \times Y$ is not mutually aposyndetic, then $X \times Y$ is not mutually aposyndetic at any pair of points in the same element of $\mathcal{D}$. Since the quotient map for $\mathcal{D}$ is π_2, which is open, this decomposition is continuous. Since the quotient space is Y, which is semi-indecomposable, if $X \times Y$ is not mutually aposyndetic, then it is semi-indecomposable by Lemma 3.9. $\square$

Corollary 3.11. *Let X be a homogeneous curve with dense arc components, $\hat{X}$ be the continuous curve of pseudo-arcs with quotient X, and P be a pseudo-arc. Then $X \times P$ is either mutually aposyndetic or semi-indecomposable. If $X \times P$ is semi-indecomposable, then $\hat{X} \times P$ is semi-indecomposable. If $X \times P$ is mutually aposyndetic, and $q : \hat{X} \to X$ is the quotient map for the decomposition of $\hat{X}$ into maximal pseudo-arcs, then $\{q^{-1}(x) \times \{p\} : (x, p) \in X \times P\}$ is the mutually aposyndetic decomposition of $\hat{X} \times P$.*

Proof. Since P is semi-indecomposable, and X has dense arc components, by Lemma 3.10, $X \times P$ is either mutually aposyndetic or semi-indecomposable.

Let $\mathcal{D} = \{q^{-1}(x) \times \{p\} : (x, p) \in X \times P\}$. Since the decomposition of $\hat{X}$ into maximal pseudo-arcs is a continuous, terminal decomposition, by Lemma 3.7, $\hat{X} \times P$ is not mutually aposyndetic with respect to any pair of points belonging to the same element of $\mathcal{D}$. Note that the quotient space for $\mathcal{D}$ is $X \times P$.

If $X \times P$ is semi-indecomposable, then so is $\hat{X} \times P$, by Lemma 3.9.

If $X \times P$ is mutually aposyndetic, then each element of the mutually aposyndetic decomposition of $\hat{X} \times P$ is contained in an element of $\mathcal{D}$ by

Lemma 2.3. Then, in this case, $\mathcal{D}$ is the mutually aposyndetic decomposition of $\hat{X} \times P$. $\square$

Note that the corollary above holds in particular if X is a solenoid.

4. Questions

There are several questions prompted by the results in this paper.

Question 4.1. If S_1 and S_2 are solenoids such that $S_1 \times S_2$ is semi-indecomposable, and $\hat{S}_1$ and $\hat{S}_2$ are solenoids of pseudo-arcs with quotients S_1 and S_2, respectively, can we conclude anything about the mutually aposyndetic decompositions of $S_1 \times \hat{S}_2$ and $\hat{S}_1 \times \hat{S}_2$?

Question 4.2. Is the product of two homogeneous decomposable continua always mutually aposyndetic? What about the product of two homogeneous decomposable curves?

Question 4.3. In Corollary 3.8 and Corollary 3.11, can we replace the pseudo-arc P with an arbitrary homogeneous hereditarily indecomposable continuum?

Question 4.4. From Lemma 3.10 it follows that the product of a solenoid and a pseudo-arc is either mutually aposyndetic or semi-indecomposable. Is such product always semi-indecomposable?

Question 4.5. Must the product of two homogeneous hereditarily indecomposable continua be semi-indecomposable? What about the product of a pseudo-arc and a homogeneous hereditarily indecomposable continuum?

Another relevant question is the following.

Question 4.6. If a homogeneous continuum has dimension at least 3, must it be mutually aposyndetic?

If the answer to the last question is yes, this would reduce the study of the mutually aposyndetic decompositions of products of homogeneous continua to the study of the decompositions of products of two homogeneous curves.

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