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ASSOUAD–NAGATA DIMENSION AND FINITE-TO-ONE LIPSCHITZ MAPS

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Abstract. W. Hurewicz characterized the covering dimension of a separable metric space in terms of a finite-to-one closed map from a zero-dimensional space onto the space. More recently, N. Brodskiy, J. Dydak, J. Higes, and A. Mitra proved a metric space (X, d) has Assouad–Nagata dimension 0 if it admits an ultrametric ρ on X so that the identity map $(X, \rho) \to (X, d)$ is bi-Lipschitz. Motivated by those results, we obtain a Hurewicz type characterization of the Assouad–Nagata dimension in this paper. More precisely, we show that a separable proper metric space X has Assouad–Nagata dimension $\leq n$ if and only if it is the image of an at most $(n+1)$-to-1 Lipschitz map from an ultrametric space such that the map satisfies some cobounded condition.

1. Introduction

One of the well-known Hurewicz characterizations of covering dimension [7] states the following:

Theorem 1.1. *Let X be a separable metric space and let n be a nonnegative integer. Then $\dim X \leq n$ if and only if there exist a zero-dimensional space Y and a closed surjective map $f : Y \to X$ such that $|f^{-1}(x)| \leq n+1$ for each $x \in X$.*

This result was generalized to a class of (nonseparable) metric spaces by Kiiti Morita [9, Theorem 4]. The "if" part is a special case of the dimension-raising theorem [9, Theorem 5] (see also [10] and [11] for more

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general cases). More recently, N. Brodskiy, J. Dydak, M. Levin, and A. Mitra [4] obtained a Hurewicz-type theorem concerning dimension-lowering maps for the Assouad–Nagata dimension. In this paper, we prove an analog of Hurewicz's theorem concerning finite-to-one maps (Theorem 1.1) for the Assouad–Nagata dimension which was introduced by Patrice Assouad [1] under the name of Nagata dimension and extensively studied by Urs Lang and Thilo Schlichenmaier [8].

On the other hand, ultrametric spaces play an important role in characterizing zero-dimensionality in appropriate categories. A separable metric space (X, d) has dimension 0 if and only if it admits an ultrametric ρ so that the identity map $(X, \rho) \to (X, d)$ is a homeomorphism (see [6] and [12]). A metric space (X, d) has Assouad–Nagata dimension ≤ 0 if and only if it admits an ultrametric ρ so that the identity map $(X, \rho) \to (X, d)$ is bi-Lipschitz [3, Theorem 3.3]. A metric space (X, d) has uniform dimension 0 if and only if it admits an ultrametric ρ so that the identity map $(X, \rho) \to (X, d)$ is bi-uniform [3, Theorem 4.3].

In this paper, we characterize the Assouad–Nagata dimension in terms of finite-to-one maps from ultrametric spaces. Here is the main theorem.

Main Theorem. *Let X be a proper separable metric space and let n be a nonnegative integer. Then the following conditions are equivalent.*

(1) *X has Assouad–Nagata dimension at most n.*
(2) *There exist an ultrametric space Z and a Lipschitz map $f : Z \to X$ onto X such that $|f^{-1}(x)| \leq n+1$, and they have the following property:*
 (B) *There exists a constant $c > 0$ such that for each $r > 0$ and for every subset B of X with diameter at most r, there exists a subset A of Z with diameter at most cr and $f(A) = B$.*

In the proof for the "only if" part of Theorem 1.1, Morita used the condition $\dim X \leq n$ to construct a decreasing sequence of closed covers, based on which he defined a map from a subset of Baire's zero-dimensional space to the metric space. In this paper, in order to construct a decreasing sequence of covers, we use the characterization of the Assouad–Nagata dimension which is a modification of the characterization of asymptotic dimension by G. Bell and A. Dranishnikov [2].

2. Preliminaries

Let (X, d) be a metric space and $r > 0$. For every subset A of X, let $\operatorname{diam} A$ denote the diameter of A; A is said to be *r-bounded* if $\operatorname{diam} A \leq r$. A family $\mathcal{U}$ of subsets of X is said to be *r-disjoint* if $d(x, x') > r$ for any x and x' that belong to different elements of $\mathcal{U}$. The *r-multiplicity* of $\mathcal{U}$ is defined as the largest number n so that no ball of radius r meets more than n elements of $\mathcal{U}$. Recall that the *mesh* of $\mathcal{U}$, denoted $\mathrm{mesh}(\mathcal{U})$, is defined as $\sup\{\operatorname{diam} U : U \in \mathcal{U}\}$, and the *Lebesgue number* of $\mathcal{U}$, denoted $L(\mathcal{U})$, is defined as the supremum of positive numbers r so that for every $A \subseteq X$ of $\operatorname{diam} A \leq r$, there exists $U \in \mathcal{U}$ with $A \subseteq U$.

A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is *Lipschitz* if there exists a constant $c > 0$ such that $d_Y(f(x), f(x')) \leq c d_X(x, x')$ holds for all $x, x' \in X$, and f is a λ-*Lipschitz* if the inequality holds for the constant $c = \lambda$. The *Lipschitz constant* of a Lipschitz map f, denoted $\mathrm{Lip}(f)$, is defined as $\inf\{\lambda : f \text{ is a } \lambda\text{-Lipschitz map}\}$.

For every nonnegative integer n, a metric space is said to have *Assouad–Nagata dimension at most* n, denoted $\dim_{AN} X \leq n$, provided there exists a constant $c > 0$ such that for every $r > 0$, there exists a cover $\mathcal{U} = \cup_{i=1}^{n+1} \mathcal{U}_i$ of X so that each $\mathcal{U}_i$ is r-disjoint and $\mathrm{mesh}(\mathcal{U}) \leq cr$.

For every countable simplicial complex K, let $|K|$ be its geometric realization. Embed $|K|$ into ℓ^2 by sending each vertex of K to an element of an orthogonal basis of ℓ^2, and let $|K|$ be equipped with the metric induced from that on ℓ^2.

The Assouad–Nagata dimension is characterized in many ways (see [8]). For our purpose, we modify the characterizations of asymptotic dimension by Bell and Dranishnikov [2, Theorem 1] to obtain the characterizations of the Assouad–Nagata dimension for separable metric spaces.

Proposition 2.1. *Let X be a separable metric space. Then the following are equivalent.*

(1) $\dim_{AN} X \leq n$.
(2) *There exists a constant $c_1 > 0$ such that for every $r > 0$, there exists a countable open cover $\mathcal{U}_r$ of X with r-multiplicity at most $n+1$ and $\mathrm{mesh}(\mathcal{U}_r) \leq c_1 r$.*
(3) *There exists a constant $c_2 > 0$ such that for every $r > 0$, there exists a countable open cover $\mathcal{V}_r$ of X with multiplicity at most $n+1$, $\mathrm{mesh}(\mathcal{V}_r) \leq c_2 r$, and $L(\mathcal{V}_r) \geq r$.*
(4) *There exists a constant $c_3 > 0$ such that for every $r > 0$, there exist a uniform countable simplicial complex K of dimension n and an r-Lipschitz map $\varphi : X \to |K|$ so that the family $\{\varphi^{-1}(\sigma) : \sigma \in K\}$ is c_3/r-bounded.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) are proved by the same argument as in the corresponding implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2) of [2, Theorem 1].

(1) is equivalent to the following condition:

(2a) There exists a constant $c_4 > 0$ such that for every $r > 0$, there exists a cover $\mathcal{W}_r$ of X with r-multiplicity at most $n+1$ and $\mathrm{mesh}(\mathcal{W}_r) \leq c_4 r$.

It remains to show (2a) $\Rightarrow$ (2). Suppose that (2a) holds; let $r > 0$. Let $c_4 > 0$ be as in (2a). Then there exists a cover $\mathcal{U}_{3r}$ of X with $3r$-multiplicity at most $n+1$ and $\mathrm{mesh}(\mathcal{U}_{3r}) \le 3c_4 r$. The cover $\mathcal{W}'_r = \{B_r(U) : U \in \mathcal{U}_{3r}\}$ has r-multiplicity at most $n+1$, and $\mathrm{mesh}(\mathcal{W}'_r) \le (3c_4+2)r$. Since X is separable, there exists a countable subcover $\mathcal{W}_r$ of $\mathcal{W}'_r$. This $\mathcal{W}_r$ has the desired property in (2). $\square$

We recall the notion of code space. For more details, the reader is referred to [5].

Let $\mathbb{N} = \{0, 1, \ldots\}$. Then the *code space* on $\mathbb{N}$ is denoted by $\Sigma = \prod_{i=1}^{\infty} \mathbb{N}$. For each $k \in \mathbb{N}$, let $\Sigma_k = \prod_{i=0}^{k} \mathbb{N}$ and let $\Sigma_* = \cup_{k=0}^{\infty} \Sigma_k$. If $\sigma = (a_0, a_1, \ldots) \in \Sigma$, then for each $k \in \mathbb{N}$, write $\sigma \restriction k$ for the element $(a_0, a_1, \ldots, a_k) \in \Sigma_k$.

For each real number s with $0 < s < 1$, a metric on Σ is defined as follows. If $\sigma = (a_0, a_1, \ldots)$ and $\tau = (b_0, b_1, \ldots)$, let $d_s(\sigma, \tau) = s^k$ if $a_i = b_i$ for $i \le k$ and $a_{k+1} \ne b_{k+1}$, and let $d_s(\sigma, \tau) = 0$ if $\sigma = \tau$. This defines an ultrametric d_s on Σ. Recall that a metric d is an ultrametric if it satisfies the ultra-triangle inequality: $d(\sigma, \tau) \le \max\{d(\sigma, \theta), d(\theta, \tau)\}$ for $\sigma, \tau, \theta \in \Sigma$.

The set Σ is represented as an infinite tree such that the nodes at the kth level are the elements of Σ_k. If $k \ge 1$, for each element $(a_0, \ldots, a_k) \in \Sigma_k$, let $(a_0, \ldots, a_{k-1}) \in \Sigma_k$ be its parent. Then the elements of Σ are in one-to-one correspondence with the infinite paths starting at the root.

3. Proof of Main Theorem

First, suppose that $\dim_{AN} X \le n$. By Proposition 2.1(3), there exists a constant $c > 0$ such that for each $r > 0$, there exists a countable open cover $\mathcal{V}_r$ of X with multiplicity at most $n+1$, $\mathrm{mesh}(\mathcal{V}_r) \le cr$, and $L(\mathcal{V}_r) \ge r$. Without loss of generality, we can assume $c > 1$. Let $\mathcal{U}_0 = \{X\}$, and for each $k \in \mathbb{N}$, put $\mathcal{U}_k = \mathcal{V}_{1/c^k}$. Then $\mathcal{U}_k$ has multiplicity at most $k+1$, $\mathrm{mesh}(\mathcal{U}_k) \le 1/c^{k-1}$, and $L(\mathcal{U}_k) \ge 1/c^k$. This implies that for each $U \in \mathcal{U}_k$, there exists $V \in \mathcal{U}_{k-1}$ such that $\overline{U} \subseteq V$. We construct a tree as follows. Let the nodes at the kth level be the elements of $\mathcal{U}_k$. If $k \ge 1$, for each node $U \in \mathcal{U}_k$, choose an element $V \in \mathcal{U}_{k-1}$ so that $\overline{U} \subseteq V$, and let V be the parent of U. The nodes of the tree are in one-to-one correspondence with the finite paths in the tree starting at the root. For each finite path α starting at the root, write U_α for the element $U \in \mathcal{U}_k$ which corresponds to α. Since each $\mathcal{U}_k$ is at most countable, every finite path α starting at the root can be represented by $\alpha = (a_0, a_1, \ldots, a_k) \in \Sigma_k$, and every infinite path σ starting at the root can be represented by $\sigma = (a_0, a_1, \ldots) \in \Sigma$. Let Z be the set of $\sigma = (a_0, a_1, \ldots) \in \Sigma$ so that $a_0 a_1 \cdots$ is an infinite path starting at the root. Then we can write $\mathcal{U}_k$ as $\{U_{\sigma \restriction k} : \sigma \in Z\}$. Let

Z be equipped with the ultrametric $d_{1/c}$. Define a map $f : Z \to X$ as follows. There is a decreasing sequence $\overline{U_{\sigma\restriction 0}} \supseteq U_{\sigma\restriction 0} \supseteq \overline{U_{\sigma\restriction 1}} \supseteq U_{\sigma\restriction 1} \supseteq \cdots \supseteq \overline{U_{\sigma\restriction k}} \supseteq U_{\sigma\restriction k} \supseteq \cdots$, and $\operatorname{diam} U_{\sigma\restriction k} \to 0$ as $k \to \infty$. Since X is proper, each $\overline{U_{\sigma\restriction k}}$ is compact. So, $\cap_{k=0}^{\infty} U_{\sigma\restriction k} = \cap_{k=0}^{\infty} \overline{U_{\sigma\restriction k}}$ consists of a single point, which we denote by $f(\sigma)$.

The map $f : Z \to X$ is Lipschitz. Indeed, let $\sigma, \sigma' \in Z$ and let $d_{1/c}(\sigma, \sigma') = 1/c^k$. Then $\sigma \restriction k = \sigma' \restriction k$. This implies that $f(\sigma), f(\sigma') \in U$ for some $U \in \mathcal{U}_k$, and hence $d_{1/c}(f(\sigma), f(\sigma')) \leq \operatorname{diam} U \leq \operatorname{mesh} \mathcal{U}_k \leq 1/c^{k-1} = c d_{1/c}(\sigma, \sigma')$.

The map f satisfies $|f^{-1}(x)| \leq n+1$ for each $x \in X$. Indeed, let $\sigma_1, \ldots, \sigma_{n+2} \in Z$, where $\sigma_i \neq \sigma_j$ for $i \neq j$. Then there exists k such that $\sigma_i \restriction k \neq \sigma_j \restriction k$ for $i \neq j$. If $x = f(\sigma_1) = \cdots = f(\sigma_{n+2})$, then $x \in \cap_{i=1}^{n+2} U_{\sigma_i \restriction k} \neq \emptyset$, contracting to the fact that $\mathcal{U}_k$ has multiplicity at most $n+1$.

To show assertion (2), it remains to verify condition (B). Let $r > 0$, and let B be a subset of X such that $\operatorname{diam} B \leq r$. We wish to show that there exists a subset A of Z such that $\operatorname{diam} A \leq cr$, and $f(A) = B$. If $r \geq 1$, then the assertion is obvious since $\operatorname{diam} Z \leq c$. So we assume $r < 1$. Let k be the nonnegative integer such that $1/c^{k+1} \leq r < 1/c^k$. There exist $U_{\alpha_i} \in \mathcal{U}_i$ where $i = 0, \ldots, k$, such that $B \subseteq U_{\alpha_k} \subseteq \cdots \subseteq U_{\alpha_0}$. Let $A = \{\sigma \in Z : \sigma \restriction k = \alpha_0 \cdots \alpha_k, f(\sigma) \in B\}$. Then $f(A) = B$, and for any $\sigma, \sigma' \in A$, $d_{1/c}(\sigma, \sigma') \leq 1/c^k < cr$, which shows $\operatorname{diam} A \leq cr$.

Conversely, let $f : Z \to X$ be a Lipschitz map from an ultrametric space Z onto X such that $|f^{-1}(x)| \leq n+1$ and it has property (B). Let $c_1 > 0$ be a constant such that for each $r > 0$ and for every subset B of X with diameter $\leq r$, there exists a subset A of Z with diameter at most $c_1 r$ and $f(A) = B$. Since $\dim_{AN} Z = 0$, then, by Proposition 2.1(3), there exists a constant $c_2 > 0$ such that for every $r > 0$, there exist a countable open cover $\mathcal{U}_r$ of Z with multiplicity at most $n+1$, $\operatorname{mesh}(\mathcal{U}_r) \leq c_2 r$, and $L(\mathcal{U}_r) \geq r$. For each $r > 0$, let $\mathcal{V}_r = f(\mathcal{U}_{c_1 r}) = \{f(U) : U \in \mathcal{U}_{c_1 r}\}$.

The multiplicity of $\mathcal{V}_r$ is at most $n+1$ since the multiplicity of $\mathcal{U}_{c_1 r}$ is at most 1 and f is at most $(n+1)$-to-1.

If $U \in \mathcal{U}_{c_1 r}$ and $x, x' \in f(U)$, and if $x = f(z)$ and $x' = f(z')$, where $z, z' \in U$, then $d(x, x') \leq \operatorname{Lip}(f) d(z, z') \leq \operatorname{Lip}(f) c_1 r$. This shows that $\operatorname{mesh}(\mathcal{V}_r) \leq \operatorname{Lip}(f) c_1 r$.

It remains to show that $L(\mathcal{V}_r) \geq r$. Let B be a subset of X such that $\operatorname{diam} B \leq r$. By condition (B), there exists a subset A of Z such that $f(A) = B$ and $\operatorname{diam} A \leq c_1 r$. Then $A \subseteq U$ for some $U \in \mathcal{U}_{c_1 r}$, and hence $B = f(A) \subseteq f(U)$ and $f(U) \in \mathcal{V}_r$. This shows that $L(\mathcal{V}_r) \geq r$, and completes the proof of the theorem.

Remark 3.1. Since every ultrametric space has Assouad–Nagata dimension 0 and since every metric space with Assouad–Nagata dimension 0 admits an ultrametric in the Lipschitz category (see [3, Theorem 3.3]), condition (2) in the theorem is equivalent to the following condition:

(3) There exist a metric space Z and a Lipschitz map $f : Z \to X$ onto X such that $\dim_{AN} Z \leq 0$, $|f^{-1}(x)| \leq n+1$, and they have property (B).

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