TOPOLOGY PROCEEDINGS
Volume 42, 2013
Pages 57–72
http://topology.auburn.edu/tp/

JSJ Decompositions of Coxeter Groups over FA Subgroups

by

John G. Ratcliffe and Steven T. Tschantz

Electronically published on August 3, 2012

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 42 (2013)
Pages 57-72

E-Published on August 3, 2012

JSJ DECOMPOSITIONS OF COXETER GROUPS OVER FA SUBGROUPS

JOHN G. RATCLIFFE AND STEVEN T. TSCHANTZ

Abstract. A group G has property FA if G fixes a point of every tree on which G acts isometrically without inversions. We prove that every Coxeter system of finite rank has a visible JSJ decomposition over subgroups with property FA. As an application, we reduce Bernhard Mühlherr's twist conjecture to Coxeter systems that are indecomposable with respect to amalgamated products over visible subgroups with property FA.

1. Introduction

JSJ decompositions first appeared in 3-manifold theory as secondary decompositions of 3-manifolds over tori. A primary decomposition of 3-manifolds is a connected sum decomposition over 2-spheres. There is a close relationship between the topology of a 3-manifold and the algebraic properties of its fundamental group, so it was natural for JSJ decompositions to migrate to group theory.

A JSJ decomposition of a group G, over a class of subgroups $\mathcal{A}$, is a graph of groups decomposition of G, with edge groups in $\mathcal{A}$, that has certain universal properties. JSJ decompositions of groups over various classes of subgroups have been studied by a number of authors. For an introduction to JSJ decompositions of groups, see Vincent Guirardel and Gilbert Levitt [5]. Michael Mihalik [8] recently described nice JSJ decompositions of Coxeter groups over virtually abelian subgroups. In this paper, we describe nice JSJ decompositions of Coxeter groups over subgroups with property FA. Our JSJ decompositions of Coxeter groups

2010 *Mathematics Subject Classification.* 20F55.
Key words and phrases. Coxeter group, JSJ decomposition, property FA.

are more primary than Mihalik's and are analogous to the primary connect sum decompositions of 3-manifolds, whereas Mihalik's JSJ decompositions of Coxeter groups are analogous to the JSJ decompositions of 3-manifolds.

Let (W, S) be a Coxeter system of finite rank. By [1, Lemma A], the graph of any graph of groups decomposition of W is a tree, since the abelianization of W is finite. Therefore, a graph of groups decomposition Ψ of W is *reduced* if and only if no edge group of Ψ is equal to a vertex group of Ψ. A *visible subgroup* of (W, S) is a subgroup of W generated by a subset of S. A *visible graph of groups decomposition* of (W, S) is a graph of groups decomposition Ψ of W such that all the vertex and edge groups of Ψ are visible subgroups of (W, S). A *vertex system* of a visible graph of groups decomposition Ψ of (W, S) is a subsystem (V, R) of (W, S) such that V is a vertex group of Ψ, and an *edge system* of Ψ is a subsystem (E, T) of (W, S) such that E is an edge group of Ψ.

A group G has *property* FA if G fixes a point of every tree on which G acts without inversions. As references for groups with property FA, see [1] and §I.6 of Jean-Pierre Serre's arboreal book [14]. A subset C of S is said to be *complete* if the product of any two elements of C has finite order. Note that the empty set is complete. If C is a complete subset of S, we call $\langle C\rangle$ a complete visible subgroup of W. A visible subgroup $\langle C\rangle$ of (W, S) has property FA if and only if C is complete by [14, §I.6: Theorem 15 and Exercise 3]. Mihalik and Steven T. Tschantz [9] proved that if a subgroup H of W has property FA, then H is contained in a conjugate of a complete visible subgroup of (W, S). Let $\mathcal{F}A$ be the set of all subgroups of W that are contained in some subgroup of W with property FA. Then $\mathcal{F}A$ is closed with respect to subgroups and conjugation.

In this paper, we prove that (W, S) has a visible reduced JSJ decomposition Ψ over the class of subgroups $\mathcal{F}A$. All the edge groups of Ψ are complete visible subgroups, and so have property FA. The vertex groups of Ψ are the maximal visible subgroups of (W, S) that are indecomposable as an amalgamated product over a complete visible subgroup.

We prove that the sets of conjugacy classes of the vertex groups and the edge groups of a visible reduced JSJ decomposition Ψ of (W, S) over $\mathcal{F}A$ do not depend on the choice of the set of Coxeter generators S of W, and that the sets of isomorphism classes of the vertex groups and the edge groups of our JSJ decompositions are isomorphism invariants of W.

We prove that all the vertex groups of a visible reduced JSJ decomposition of (W, S) over $\mathcal{F}A$ are complete if and only if (W, S) is a chordal Coxeter system [13]. As an application to the isomorphism problem for Coxeter groups, we reduce Bernhard Mühlherr's twist conjecture [11] to

Coxeter systems that are indecomposable as an amalgamated product over a visible complete subgroup.

2. Existence of Visible JSJ Decompositions

We will use presentation diagrams to graphically represent Coxeter systems rather than Coxeter diagrams. The *presentation diagram* (P-*diagram*) of a Coxeter system (W,S) is the labeled undirected graph $\mathrm{P}(W,S)$ with vertices S and edges $\{(s,t) : s,t \in S \text{ and } 1 < m(s,t) < \infty\}$ such that an edge (s,t) is labeled by the *order* $m(s,t)$ of st in W. Note that a subset C of S is complete if and only if the underlying graph of $\mathrm{P}(\langle C\rangle, C)$ is complete.

Let (W,S) be a Coxeter system of finite rank. Suppose that $S_1, S_2 \subseteq S$, with $S = S_1 \cup S_2$ and $S_0 = S_1 \cap S_2$, are such that $m(a,b) = \infty$ for all $a \in S_1 - S_0$ and $b \in S_2 - S_0$. Then we can write W as a visible amalgamated product

$$W = \langle S_1\rangle *_{\langle S_0\rangle} \langle S_2\rangle.$$

We say that S_0 *separates* S if $S_1 - S_0 \neq \emptyset$ and $S_2 - S_0 \neq \emptyset$. The amalgamated product decomposition of W will be nontrivial if and only if S_0 separates S. If S_0 separates S, we call the triple (S_1, S_0, S_2) a *separation* of S, and S_0 a *separator* of S. Note that S_0 separates S if and only if S_0 separates $\mathrm{P}(W,S)$, that is, there are a,b in $S - S_0$ such that every path in $\mathrm{P}(W,S)$ from a to b must pass through S_0. A subset S_0 of S is a *minimal separator* of S, if S_0 separates S, and no other subset of S_0 separates S.

Lemma 2.1. *Let (W,S) be a Coxeter system, and let (S_1,S_0,S_2) be a separation of S such that S_0 is complete. If $T \subset S_1$ separates S_1, then T separates S.*

Proof. On the contrary, suppose that T does not separate S. As T separates S_1, there exist x,y in $S_1 - T$ such that every path in $\mathrm{P}(\langle S_1\rangle, S_1)$ from x to y passes through T. As T does not separate S, there is a path in $\mathrm{P}(W,S)$ from x to y that avoids T. The path must exit $\mathrm{P}(\langle S_1\rangle, S_1)$ through S_0 at some first element a of S_0 before entering $S - S_1$ and must pass back through S_0 at some last element b of S_0. As S_0 is complete, we can short circuit the path by going directly from a to b. This gives a path from x to y in $\mathrm{P}(\langle S_1\rangle, S_1)$ that avoids T, which is a contradiction. Thus, T must separate S. □

Lemma 2.2. *Let (W,S) be a Coxeter system of finite rank. Then (W,S) has a visible reduced graph of groups decomposition Ψ such that for each vertex system (V,R) of Ψ, the set R is not separated by a complete subset and such that each edge group of Ψ is a complete visible subgroup of (W,S).*

Proof. The proof is by induction on $|S|$. Suppose that S is not separated by a complete subset. This includes the case $|S| = 1$. Let Ψ be the trivial graph of groups decomposition of W with one vertex and no edges. Then Ψ satisfies the requirements of the lemma. Suppose the lemma is true for all Coxeter systems of rank less than $|S|$ and S is separated by a complete subset S_0. As every subset of S_0 is complete, we may assume that S_0 is a minimal separator of S. Let (S_1, S_0, S_2) be a separation of S. Then $|S_i| < |S|$ for each $i = 1, 2$. By the induction hypothesis, $(\langle S_i \rangle, S_i)$ has a visible reduced graph of groups decomposition Ψ_i satisfying the requirements of the lemma for each $i = 1, 2$. As S_0 is complete, S_0 is not separated by any subset. Hence, S_0 is contained in a vertex group V_i of Ψ_i for each $i = 1, 2$. We define a visible graph of groups decomposition of (W, S) whose graph is obtained by joining the graph of Ψ_1 to the graph of Ψ_2 by an edge from the vertex of the graph of Ψ_1 corresponding to V_1 to the vertex of the graph of Ψ_2 corresponding to V_2. The vertex groups of Ψ are the vertex groups of Ψ_1 and Ψ_2 assigned to their previous vertices. The edge groups of Ψ are the edge groups of Ψ_1 and Ψ_2 assigned to their previous edges, together with the group $\langle S_0 \rangle$ assigned to the new edge.

We next show that Ψ is reduced. First assume $V_i = \langle S_i \rangle$ for some $i = 1, 2$. Then $\langle S_0 \rangle \neq V_i$, since $S_0 \neq S_i$. Now assume $V_i \neq \langle S_i \rangle$. Then there is an edge group E of Ψ_1 incident to V_1. Let $T \subset S_1$ be the set of visible generators of E. Then T separates S_1, and so T separates S by Lemma 2.1. Now $\langle S_0 \rangle \neq V_i$, since otherwise S_0 would contain T as a proper subset contradicting the fact that S_0 is a minimal separating subset of S. Hence, Ψ is reduced. Thus, Ψ has all the required properties. This completes the induction. □

Lemma 2.3. *Let (W, S) be a Coxeter system of finite rank; let Ψ be a visible reduced graph of groups decomposition of (W, S) such that, for each vertex system (V, R) of Ψ, the set R is not separated by a complete subset and such that each edge group of Ψ is a complete visible subgroup of (W, S); and let Φ be a graph of groups decomposition of W with edge groups in $\mathcal{F}A$. Then each vertex group of Ψ is contained in a conjugate of a vertex group of Φ.*

Proof. Let (V, R) be a vertex system of Ψ. By [9, Theorem 1], the Coxeter system (V, R) has a visible graph of groups decomposition Λ such that each vertex group of Λ is contained in a conjugate of a vertex group of Φ and each edge group of Λ is contained in a conjugate of an edge group of Φ. As R is finite, we may assume that the graph of Λ has only finitely many vertices and edges, and that Λ is reduced. Let (E, T) be an edge system of Λ. Then $E \in \mathcal{F}A$, since the edge groups of Φ are in $\mathcal{F}A$. Hence, there is a complete subset C of S such that E is contained in a conjugate of $\langle C\rangle$ by [9, Lemma 25]. This implies that T is conjugate to a subset of C by [10, Lemma 4.3]. Therefore, T is complete. Hence, R is separated by a complete subset, which is a contradiction. Therefore, the graph of Λ consists of a single point, and so V is contained in a conjugate of a vertex group of Φ. □

The next theorem, together with Lemma 2.2, implies that visible reduced JSJ decompositions over $\mathcal{F}A$ of a Coxeter system of a finite rank exist.

Theorem 2.4. *Let (W,S) be a Coxeter system of finite rank, and let Ψ be a visible reduced graph of groups decomposition of (W,S) such that for each vertex system (V,R) of Ψ, the set R is not separated by a complete subset, and such that each edge group of Ψ is a complete visible subgroup of (W,S). Then Ψ is a JSJ decomposition of W over the class $\mathcal{F}A$.*

Proof. According to Guirardel and Levitt [5], we need to show that Ψ is minimal, universally elliptic, and that Ψ dominates every minimal, universally elliptic graph of groups decomposition of W over $\mathcal{F}A$. For a discussion of minimal graph of groups decompositions, see [3, §2]. The graph of groups decomposition Ψ is minimal, since it is reduced, and universally elliptic, since the edge groups of Ψ have property FA. By Lemma 2.3, the graph of groups decomposition Ψ dominates every minimal graph of groups decomposition of W over $\mathcal{F}A$. Thus, Ψ is a JSJ decomposition of W over $\mathcal{F}A$. □

Let (W,S) be a Coxeter system of finite rank. Let $S_0\subset S$, and let $a,b\in S-S_0$. We say that S_0 is an *(a,b)-separator* of S if there is a separation (S_1,S_0,S_2) of S such that $a\in S_1-S_0$ and $b\in S_2-S_0$. Note that S_0 is an (a,b)-separator of S if and only if a and b lie in different connected components of $\mathrm{P}(\langle S-S_0\rangle, S-S_0)$; moreover, S_0 separates S if and only if there are elements $a,b\in S-S_0$ such that S_0 is an (a,b)-separator of S. We say that S_0 is a *minimal (a,b)-separator* of S if S_0 is an (a,b)-separator of S and no other subset of S_0 is an (a,b)-separator of S. We say that S_0 is a *relative minimal separator* of S if there exists elements $a,b\in S$ such that S_0 is a minimal (a,b)-separator of S. Note that every minimal separator of S is a relative minimal separator of S, but a relative minimal separator of S need not be a minimal separator of S.

The next theorem characterizes the vertex groups and the edge groups of our JSJ decompositions.

Theorem 2.5. *Let (W,S) be a Coxeter system of finite rank, and let Ψ be a visible reduced graph of groups decomposition of (W,S) such that for* *each vertex system* (V, R) *of* Ψ*, the set* R *is not separated by a complete subset, and such that each edge group of* Ψ *is a complete visible subgroup of* (W, S)*. Let* $\mathcal{V}$ *be the set of all maximal subsets of* S *that are not separated by a complete subset, and let* $\mathcal{E}$ *be the set of all complete relative minimal separators of* S*. Then all the subgroups generated by sets in* $\mathcal{V}$ *are the vertex groups of* Ψ*, and all the subgroups generated by sets in* $\mathcal{E}$ *are the edge groups of* Ψ*.*

Proof. If (E, T) is an edge system of Ψ, then T is a separating subset of S, since the graph of Ψ is a tree. Let (V, R) be a vertex system of Ψ. Clearly, every subset of S that contains R properly is separated by a complete subset C of S that is contained in some edge group of Ψ that is incident to V. Therefore, R is a maximal subset of S that is not separated by a complete subset, and so $R \in \mathcal{V}$.

Now suppose $R \in \mathcal{V}$. We claim that $\langle R \rangle$ is a vertex group of Ψ. Every element of R is in some vertex group of Ψ. Let $R' \subseteq R$ be a maximal subset of R that is contained in some vertex group of Ψ. If $R - R' \neq \emptyset$, say $x \in R - R'$, then R' and x are not both contained in a vertex group of Ψ. Take vertex groups V and V' of Ψ, with $x \in V$ and $R' \subseteq V'$, which are closest together in the graph of Ψ. Let E be an edge group of the path between V and V'. Then E is generated by a complete subset T of S by assumption. Let $C = R \cap T$. Then C is a complete subset of S. Now $x \notin C$; otherwise, x would also be in a vertex group closer to V' on the path between V and V'. Likewise, $R' \nsubseteq C$ or else R' would be contained in a vertex group closer to V on a path between V and V'. But then $\mathrm{P}(\langle R - C \rangle, R - C)$ would have at least two connected components, one containing x and one containing some element of $R' - C$. This contradicts the assumption that $R \in \mathcal{V}$. Instead all of R must be contained in a vertex group V of Ψ. By the maximality of R, we have that $\langle R \rangle = V$.

Let (E, T) be an edge system of Ψ. As the graph of Ψ is a tree, there are distinct vertex systems (V_1, R_1) and (V_2, R_2) such that $R_1 \cap R_2 = T$. As Ψ is reduced, there is an $a \in R_1 - T$ and a $b \in R_2 - T$. As the graph of Ψ is a tree, T is an (a, b)-separator of S. Let $t \in T$, and let T' be any subset of T not containing t. Then T' is complete. Hence, T' does not separate R_1 or R_2. Therefore, there is a path in $\mathrm{P}(\langle R_1 - T' \rangle, R_1 - T')$ from a to t and there is a path in $\mathrm{P}(\langle R_2 - T' \rangle, R_2 - T')$ from t to b. Thus, T' is not an (a, b)-separator of S. Hence, T is a minimal (a, b)-separator of S. Thus, $T \in \mathcal{E}$.

Finally, suppose $T \in \mathcal{E}$. Then T is a minimal (a, b)-separator of S for some $\{a, b\} \subseteq S - T$. Let (S_1, T, S_2) be a separation of S with $a \in S_1 - T$ and $b \in S_2 - T$. Each $R \in \mathcal{V}$ generates a vertex group of Ψ and is not separated by any subset of T, and so each $R \in \mathcal{V}$ is contained in either S_1 or S_2.

Pick vertex groups V_1 and V_2 as close together in Ψ as possible such that V_1 is generated by a subset of S_1 and V_2 is generated by a subset of S_2. Then V_1 and V_2 are adjacent since every vertex group in a path between V_1 and V_2 is generated by a subset of either S_1 or S_2. Now $V_1 \cap V_2$ is an edge group E of Ψ which is generated by a subset T' of T. The set T' is an (a, b)-separator of S since the graph of Ψ is a tree. Hence, $T' = T$, since T is a minimal (a, b)-separator of S. Thus, the sets in $\mathcal{E}$ generate the edge groups of Ψ. $\square$

Example 2.6. Consider a Coxeter system (W, S) such that the underlying graph of $\mathrm{P}(W, S)$ is as given in Figure 1. Then (W, S) has two visible reduced JSJ decompositions over $\mathcal{FA}$, namely,

$$W = \langle a, b\rangle *_{\langle b\rangle} \langle b, c, e\rangle *_{\langle b,e\rangle} \langle b, d, e\rangle,$$
$$W = \langle a, b\rangle *_{\langle b\rangle} \langle b, d, e\rangle *_{\langle b,e\rangle} \langle b, c, e\rangle.$$

By Theorem 2.5, both decompositions have the same vertex groups and the same edge groups. The only difference between the two decompositions is their graphs. In the first decomposition the edge group $\langle b\rangle$ is attached to the vertex group $\langle b, c, e\rangle$, whereas in the second decomposition, the edge group $\langle b\rangle$ is attached to the vertex group $\langle b, d, e\rangle$. The two decompositons are related by a slide move [5, Definition 7]. It is worth noting that $\{b, e\}$ is a relative minimal separator of S, but $\{b, e\}$ is not a minimal separator of S, since $\{b\}$ separates S.

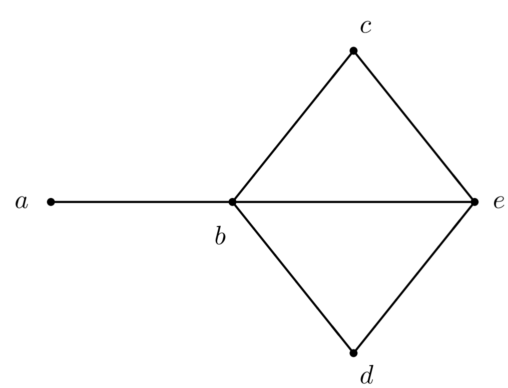

Figure 1. A graph with five vertices a, b, c, d, e

Remark 2.7. Let (W, S) be a Coxeter system of finite rank. The decomposition of the underlying graph of $\mathrm{P}(W, S)$ determined by a visible reduced JSJ decomposition of (W, S) over $\mathcal{FA}$ was first described in a graph theoretic context by Hanns-Georg Leimer [7]. In particular, an efficient algorithm for finding such a decomposition is given in Leimer's paper.

It is interesting that our visible reduced JSJ decompositions of (W, S) over $\mathcal{F}A$ depend only on the underlying graph of $\mathrm{P}(W, S)$ and not on the edge labels of $\mathrm{P}(W, S)$.

3. Uniqueness Properties

We now turn our attention to uniqueness properties of reduced JSJ decompositions of a Coxeter group W. Let $\mathcal{A}$ be a class of subgroups of W which is closed with respect to taking subgroups and conjugation.

Theorem 3.1. *Let W be a Coxeter group of finite rank, and let Ψ and Ψ' be reduced JSJ decompositions of W over $\mathcal{A}$. Then for each vertex group V of Ψ, there is a unique vertex group V' of Ψ' such that V is conjugate to V' in W. Therefore, the graphs of Ψ and Ψ' have the same number of vertices and the same number of edges.*

Proof. Let V be a vertex group of Ψ. Then there is a $w \in W$ and a vertex group V' of Ψ' such that $V \subseteq wV'w^{-1}$, since, by [5, Theorem 12], Ψ dominates Ψ'. Moreover, there is a $w' \in W$ and a vertex group V'' of Ψ such that $V' \subseteq w'V''w'^{-1}$, since, by [5, Theorem 12], Ψ' dominates Ψ. Hence, $V \subseteq ww'V''(ww')^{-1}$. By [9, Lemma 3], we have that $V = V''$ and $ww' \in V$. As $V \subseteq wV'w^{-1} \subseteq ww'V(ww')^{-1} = V$, we have that $V = wV'w^{-1}$; moreover, by [9, Lemma 3], V' is unique. Thus, the graphs of Ψ and Ψ' have the same number of vertices. As the graphs of Ψ and Ψ' are both trees, they also have the same number of edges. □

Lemma 3.2. *Let (W, S) be a Coxeter system of finite rank. Then S is separated by a complete subset if and only if W has a nontrivial amalgamated product decomposition over a subgroup in $\mathcal{F}A$.*

Proof. Suppose S_0 is a complete separator of S. Then there is a separation (S_1, S_0, S_2) of S and we have a nontrivial amalgamated product decomposition $W = \langle S_1 \rangle *_{\langle S_0 \rangle} \langle S_2 \rangle$ with $\langle S_0 \rangle \in \mathcal{F}A$.

Conversely, suppose W has a nontrivial amalgamated product decomposition $W = A *_C B$ with $C \in \mathcal{F}A$, and on the contrary, S has no complete separator. By Lemma 2.3, we have that W is contained in a conjugate of A or B, which is a contradiction. Therefore, S has a complete separator. □

The next theorem, together with Theorem 2.4, characterizes a visible reduced JSJ decomposition over $\mathcal{F}A$ of a Coxeter system (W, S) of finite rank.

Theorem 3.3. *Let Ψ be a visible reduced JSJ decomposition of over $\mathcal{F}A$ of a Coxeter system (W, S) of finite rank. Then for each vertex system (V, R) of Ψ, the set R is not separated by a complete subset, and each edge group of Ψ is a complete visible subgroup of (W, S).*

Proof. By Lemma 2.2, the system (W, S) has a visible reduced graph of groups decomposition Ψ' such that, for each vertex system (V', R') of Ψ', the set R' is not separated by a complete subset and such that each edge group of Ψ' is a complete visible subgroup of (W, S). By Theorem 2.4, the decomposition Ψ' is a JSJ decomposition of W over the class $\mathcal{F}A$. Let (V, R) be a vertex system of Ψ. By Theorem 3.1, there is a vertex system (V', R') of Ψ' such that V is conjugate to V'. By Lemma 3.2, the group V' is indecomposable as a nontrivial amalgamated product over a subgroup in $\mathcal{F}A$. Hence, V is indecomposable as a nontrivial amalgamated product over a subgroup in $\mathcal{F}A$. By Lemma 3.2, the set R is not separated by a complete subset.

Let (E, T) be an edge system of Ψ. Then $E \in \mathcal{F}A$. Hence, E is contained in an FA subgroup H of W. By [9, Lemma 25], there is a complete subset C of S and a $w \in W$ such that $H \subseteq w\langle C\rangle w^{-1}$. Hence, $E \subseteq w\langle C\rangle w^{-1}$. By [10, Lemma 4.3], the set T is conjugate to a subset of C. Hence, T is complete and E is a complete visible subgroup (W, S). □

Let (W, S) be a Coxeter system of finite rank. A subset S_0 of S is a *c-minimal separator* of S if S_0 separates S and no conjugate of another subset of S_0 separates S. Note that if S_0 is a c-minimal separator of S, then S_0 is a minimal separator of S. Also if S_0 and S_0' are conjugate separators of S, then S_0 is c-minimal if and only if S_0' is c-minimal.

Example 3.4. Consider the Coxeter system (W, S) whose P-diagram is given in Figure 2. Observe that $\{c, d\}$ is a minimal separator of S, but $\{c, d\}$ is not a c-minimal separator of S, since c is conjugate to b and $\{b\}$ separates S.

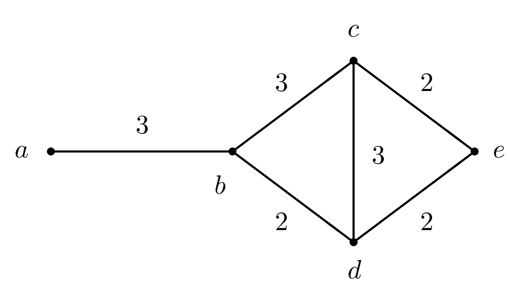

Figure 2. The P-diagram of a Coxeter system

Lemma 3.5. *Let (W,S) be a Coxeter system of finite rank, and let S' be another set of Coxeter generators of W. If S_0 is a c-minimal separator of S, then there exists a c-minimal separator of S_0' of S' such that $\langle S_0\rangle$ is conjugate to $\langle S_0'\rangle$ in W.*

Proof. Let (S_1, S_0, S_2) be a separation of S. Then we have a nontrivial visible amalgamated product decomposition $W = \langle S_1\rangle *_{\langle S_0\rangle} \langle S_2\rangle$. By [10, Theorem 6.1], there is a c-minimal separator S_0'' of S and a c-minimal separator S_0' of S' such that $\langle S_0''\rangle$ is conjugate to $\langle S_0'\rangle$ and $\langle S_0''\rangle$ is conjugate to a subgroup of $\langle S_0\rangle$. By [10, Lemma 4.3], we have that S_0'' is conjugate to a subset of S_0. As S_0 is a c-minimal separator of S, we deduce that S_0'' is conjugate to S_0. Hence, $\langle S_0\rangle$ is conjugate to $\langle S_0'\rangle$ in W. □

We now turn our attention to the uniqueness of the edge groups of a visible reduced JSJ decomposition over $\mathcal{F}A$ of a Coxeter system.

Theorem 3.6. *Let (W,S) be a Coxeter system of finite rank, and let Ψ be a visible reduced JSJ decomposition of (W,S) over $\mathcal{F}A$. Let S' be another set of Coxeter generators of W, and let Ψ' be a visible reduced JSJ decomposition of (W,S') over $\mathcal{F}A$. Then for each edge group E of Ψ, there is an edge group E' of Ψ' such that E is conjugate to E' in W. Moreover, for each edge system (E,T) of Ψ such that T is a c-minimal separator of S, there is an edge system (E',T') of Ψ' such that E is conjugate to E' and T' is a c-minimal separator of S'.*

Proof. According to Jacques Tits [15], Bass [1], and Guirardel and Levitt [6], there are five possible types of reduced JSJ decompositions of W over $\mathcal{F}A$: trivial, dihedral, linear abelian, genuine abelian, and irreducible. The abelian types do not apply to W, since the abelianization of W is finite. By [6, Proposition 3.10], the decompositions Ψ and Ψ' have the same type. If Ψ and Ψ' are both trivial, then they have no edge groups.

Suppose that Ψ and Ψ' are dihedral. By [14, §I.4, Theorem 6], we deduce that Ψ corresponds to a nontrivial visible amalgamated product decomposition $W = \langle A\rangle *_{\langle C\rangle} \langle B\rangle$ with C complete and $\langle C\rangle$ of index two in both $\langle A\rangle$ and $\langle B\rangle$, and Ψ' corresponds to a nontrivial visible amalgamated product decomposition $W = \langle A'\rangle *_{\langle C'\rangle} \langle B'\rangle$ with C' complete and $\langle C'\rangle$ of index two in both $\langle A'\rangle$ and $\langle B'\rangle$. Now $\langle C\rangle$ and $\langle C'\rangle$ are normal in W. By the main result of [12], we deduce that $A - C = \{a\}$ and $\langle A\rangle = \langle a\rangle \times \langle C\rangle$, and $B - C = \{b\}$ and $\langle B\rangle = \langle b\rangle \times \langle C\rangle$, and $A' - C' = \{a'\}$ and $\langle A'\rangle = \langle a'\rangle \times \langle C'\rangle$, and $B' - C' = \{b'\}$ and $\langle B'\rangle = \langle b'\rangle \times \langle C'\rangle$. Hence, A, B, A', and B' are all complete. Therefore, C is the unique separator of S and C' is the unique separator of S'. Hence, $\langle C\rangle$ is conjugate to $\langle C'\rangle$ by Lemma 3.5.

Now assume that Ψ and Ψ' are irreducible. Then Ψ and Ψ' are non-ascending [6], since the graphs of Ψ and Ψ' are trees, and so for each edge group E of Ψ, there is an edge group E' of Ψ' such that E is conjugate to E' in W by [6, Corollary 7.3].

Let (E, T) be an edge system of Ψ such that T is a c-minimal separator of S. Then there is a c-minimal separator T' of S' such that E is conjugate to $\langle T' \rangle$ by Lemma 3.5. As E has property FA, we have that $\langle T' \rangle$ has property FA. Therefore, T' is complete. Hence, T' generates an edge group E' of Ψ' by Theorem 2.5. $\square$

Remark 3.7. Let Ψ and Ψ' be as in Theorem 3.6. It is not necessary that a minimal edge group of Ψ is conjugate to a minimal edge group of Ψ'. We will give an example below.

Let (W, S) be a Coxeter system of finite rank. Suppose that $S_1, S_2 \subseteq S$, with $S = S_1 \cup S_2$ and $S_0 = S_1 \cap S_2$, are such that $m(a, b) = \infty$ for all $a \in S_1 - S_0$ and $b \in S_2 - S_0$. Let $\ell \in \langle S_0 \rangle$ such that $\ell S_0 \ell^{-1} = S_0$. The triple (S_1, ℓ, S_2) determines an *elementary twist* of (W, S) giving a new Coxeter generating set $S' = S_1 \cup \ell S_2 \ell^{-1}$ of W. Note that if $S_1 \subseteq S_2$, then $S' = \ell S \ell^{-1}$.

Example 3.8. Consider the Coxeter system (W, S) whose P-diagram is given in Figure 2. Let ℓ be the longest element of the visible subgroup $\langle b, c, d \rangle$. Then $(\{a, b, c, d\}, \ell, \{b, c, d, e\})$ is an elementary twist of (W, S). The P-diagram of the twisted system (W, S') is given in Figure 3. Let Ψ be the unique visible reduced JSJ decomposition of (W, S) over $\mathcal{F}A$, and let Ψ' be one of the two visible reduced JSJ decompositions of (W, S') over $\mathcal{F}A$. The minimal edge group $\langle c, d \rangle$ of Ψ is conjugate to the edge group $\langle b, c \rangle$ of Ψ'. The edge group $\langle b, c \rangle$ is not minimal, since $\langle b \rangle$ is an edge group of Ψ'.

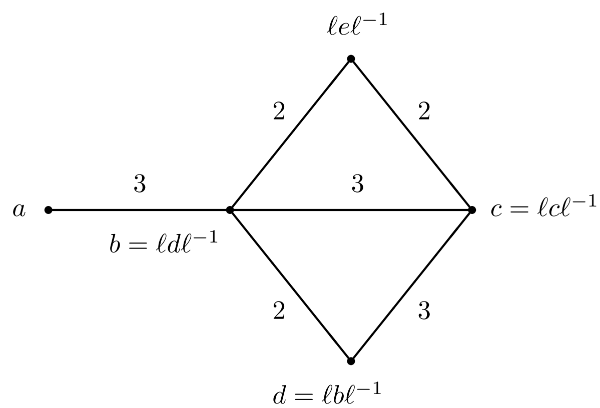

Figure 3. The P-diagram of a twisted Coxeter system

4. Chordal Coxeter Groups

A graph is said to be *chordal* if every cycle of length at least four has a chord. For example, the graph in Figure 1 is chordal. A Coxeter system (W, S) is said to be *chordal* if the underlying graph of the P-diagram of (W, S) is chordal.

Theorem 4.1. *Let (W, S) be a Coxeter system of finite rank, and let Ψ be a visible reduced JSJ decomposition of (W, S) over $\mathcal{F}A$. Then (W, S) is chordal if and only if each vertex group of Ψ is a complete visible subgroup of (W, S).*

Proof. Suppose (W, S) is chordal. We prove that each vertex system of Ψ is complete by induction on $|S|$. Suppose that (W, S) is complete. This includes the case $|S| = 1$. Then, by theorems 2.5 and 3.3, (W, S) is the only vertex system of Ψ, and (W, S) is complete.

Now suppose that (W, S) is incomplete and that each vertex system of a visible reduced JSJ decomposition over $\mathcal{F}A$ of a chordal Coxeter system, of rank less than $|S|$, is complete. Then S has a separating subset. Let S_0 be a minimal separator of S. By [2, Theorem 1], the set S_0 is complete. Now, by theorems 2.5 and 3.3, $\langle S_0 \rangle$ is an edge group of Ψ. As the graph of Ψ is a tree, Ψ is a nontrivial amalgamated product of two visible reduced graph of group decompositions Ψ_1 and Ψ_2 amalgamated along $\langle S_0 \rangle$. By Theorem 2.4 and Theorem 3.3, we deduce that Ψ_i is a visible reduced JSJ decomposition over $\mathcal{F}A$ of a proper subsystem (W_i, S_i) of (W, S) for $i = 1, 2$. Each subsystem of (W, S) is chordal. By the induction hypothesis, each vertex system of Ψ_1 and Ψ_2 is complete. Therefore, each vertex system of Ψ is complete. This completes the induction.

Conversely, suppose that each vertex system of Ψ is complete. We prove that (W, S) is chordal by induction on the number of vertices in the graph of Ψ. Suppose that the graph of Ψ has only one vertex. Then the graph of Ψ is a point, since the graph is a tree. Hence, (W, S) is complete. Therefore, (W, S) is chordal.

Now suppose that the graph of Ψ has more than one vertex and that all finite rank Coxeter systems, with a visible reduced JSJ decomposition over $\mathcal{F}A$ with fewer vertices than Ψ and all vertex systems complete, are chordal. Let (E, T) be an edge system of Ψ. As the graph of Ψ is a tree, Ψ is a nontrivial amalgamated product of two visible reduced graph of group decompositions Ψ_1 and Ψ_2 amalgamated along $\langle S_0 \rangle$. By Theorem 2.4 and Theorem 3.3, we deduce that Ψ_i is a visible reduced JSJ decomposition over $\mathcal{F}A$ of a proper subsystem (W_i, S_i) of (W, S) for $i = 1, 2$. By the induction hypothesis, (W_i, S_i) is chordal for $i = 1, 2$. Now $\mathrm{P}(W, S) = \mathrm{P}(W_1, S_1) \cup \mathrm{P}(W_2, S_2)$ and $\mathrm{P}(W_1, S_1) \cap \mathrm{P}(W_2, S_2) = \mathrm{P}(E, T)$

with $\mathrm{P}(E,T)$ complete. Therefore, (W,S) is chordal by [2, Theorem 2]. This completes the induction. $\square$

5. Application to the Isomorphism Problem

Let (W,S) be a Coxeter system of finite rank. Define

$$S^W = \{wsw^{-1} : s \in S \text{ and } w \in W\}.$$

Let S' be another set of Coxeter generators of W. The set of generators S is said to be *sharp-angled* with respect to S' if for each pair $s,t \in S$ such that $2 < m(s,t) < \infty$, there is a $w \in W$ such that $w\{s,t\}w^{-1} \subseteq S'$. The Coxeter systems (W,S) and (W,S') are said to be *twist equivalent* if there is a finite sequence of elementary twists that transforms S into S'. If (W,S) and (W,S') are twist equivalent, then $S' \subseteq S^W$ and S is sharp-angled with respect to S'.

The following conjecture is due to Mühlherr [11, Conjecture 2].

Conjecture 5.1 (Twist Conjecture)**.** *Let (W,S) be a Coxeter system of finite rank, and let S' be another set of Coxeter generators of W such that $S' \subseteq S^W$ and S is sharp-angled with respect to S'. Then (W,S) is twist equivalent to (W,S').*

Lemma 5.2. *If (W,S) is a complete Coxeter system of finite rank, then (W,S) satisfies the twist conjecture.*

Proof. Let S' be another set of Coxeter generators for W such that $S' \subseteq S^W$ and S is sharp-angled with respect to S'. We need to prove that (W,S) is twist equivalent to (W,S'). As S has no separating subsets, (W,S) can only be twisted by conjugating S. Now W has property FA, since S is complete. Hence, S' is complete by [9, Lemma 25]. Let $(W,S) = (W_1,S_1) \times \cdots \times (W_n,S_n)$ be the factorization of (W,S) into irreducible factors, and let $(W,S') = (W_1',S_1') \times \cdots \times (W_m',S_m')$ be the factorization of (W,S') into irreducible factors. By [4, Lemma 14], $m = n$ and by reindexing, we may assume that $W_i' = W_i$ for each $i = 1,\ldots,n$. As S is sharp-angled with respect to S', there is a $w_i \in W$ such that $w_iS_iw_i^{-1} \subseteq S'$ for each i by [13, Lemma 7.1]. As the jth component of w_i, for $j \neq i$, centralizes W_i, we may assume that $w_i \in W_i$. Then $w_iS_iw_i^{-1} = S_i'$ for each i. Let $w = w_1 \cdots w_n$. Then $wSw^{-1} = S'$. Therefore, (W,S) is twist equivalent to (W,S'). Thus, (W,S) satisfies the twist conjecture. $\square$

Theorem 5.3. *Let (W,S) be a Coxeter system of finite rank, and let Ψ be a visible reduced JSJ decomposition of (W,S) over $\mathcal{F}A$. If each vertex system (V,R) of Ψ satisfies the twist conjecture, then (W,S) satisfies the twist conjecture.*

Proof. Let S' be another set of Coxeter generators for W such that $S' \subseteq S^W$ and S is sharp-angled with respect to S'. We need to prove that (W,S) is twist equivalent to (W,S'). The proof is by induction on the number of vertices of the graph of Ψ. Suppose the graph of Ψ has only one vertex. Then the graph of Ψ is a single point, since the graph of Ψ is a tree. Hence, (W,S) satisfies the twist conjecture by hypothesis. Therefore, (W,S) is twist equivalent to (W,S').

Now assume that the graph of Ψ has more than one vertex, and the theorem is true for all Coxeter systems of finite rank whose JSJ decompositions over $\mathcal{F}A$ have fewer vertex systems than Ψ. Then S has a complete separating subset C. We now follow the argument in the proof of Theorem 8.4 of [13].

Let (A,C,B) be a separation of S. Then $W = \langle A\rangle *_{\langle C\rangle} \langle B\rangle$ is a nontrivial amalgamated product decomposition. By [10, Theorem 6.6], the Coxeter systems (W,S) and (W,S') are twist equivalent to Coxeter systems (W,R) and (W,R'), respectively, such that there exists a nontrivial visible reduced graph of groups decomposition Φ of (W,R) and a nontrivial visible graph of groups decomposition Φ' of (W,R') having the same graphs and the same vertex and edge groups and all edge groups equal and a subgroup of a conjugate of $\langle C\rangle$. Now $R' \subseteq R^W$ and R is sharp-angled with respect to R', since R' is twist equivalent to S'.

Let $\{(W_i,R_i)\}_{i=1}^k$ be the Coxeter systems of the vertex groups of Ψ, and let (W_0,R_0) be the Coxeter system of the edge group of Ψ. Then $k \geq 2$, and $R = \cup_{i=1}^k R_i$, and $\cap_{i=1}^k R_i = R_0$, and $R_i - R_0 \neq \emptyset$ for each $i > 0$, and $m(a,b) = \infty$ for each $a \in R_i - R_0$ and $b \in R_j - R_0$ with $i \neq j$. By [10, Lemma 4.3, Theorem 6.1, and Theorem 6.6], we have that R_0 is conjugate to a subset of C, and so R_0 is complete. By [10, Theorem 6.1 and Theorem 6.6], we have that R_0 is conjugate to a c-minimal separator of S. By Lemma 3.5, there is a c-minimal separator R_0'' of R such that $\langle R_0\rangle$ is conjugate to $\langle R_0''\rangle$. By [10, Lemma 4.3], we have that R_0 is conjugate to R_0''. As R_0 separates R, we conclude that R_0 is a c-minimal separator of R.

Let Φ_i be a visible reduced JSJ decomposition of (W_i,R_i) over $\mathcal{F}A$ for each $i = 1,\dots,k$. As R_0 is a complete minimal separator of R, we can amalgamate $\Phi_1,\dots,\Phi_k$ to give a visible reduced JSJ decomposition Φ of (W,R) over $\mathcal{F}A$ with the same vertex groups and the edge group $\langle R_0\rangle$ joining a vertex group of Ψ_i to a vertex group of Ψ_{i+1}, for each $i = 1,\dots,k-1$, by the same argument as in the proof of Lemma 2.2. Hence, the number of vertices in the graph of Φ_i is less than the number of vertices of the graph of Φ for each $i = 1,\dots,k$. By Theorem 3.1, the graphs of Φ and Ψ have the same number of vertices.

Let $\{(W_i', R_i')\}_{i=1}^k$ be the Coxeter systems of the vertex groups of Ψ' indexed so that $W_i' = W_i$ for each i, and let (W_0', R_0') be the Coxeter system of the edge group of Ψ'. Then $W_0' = W_0$, and $R' = \cup_{i=1}^k R_i'$, and $\cap_{i=1}^k R_i' = R_0'$, and $R_i' - R_0' \neq \emptyset$ for each $i > 0$, and $m(a', b') = \infty$ for each $a' \in R_i' - R_0'$ and $b' \in R_j' - R_0'$ with $i \neq j$.

By [13, Lemma 8.1], we have $R_i' \subseteq R_i^{W_i}$ and R_i is sharp-angled with respect to R_i' for each i. Hence, by the induction hypothesis, (W_i, R_i) is twist equivalent to (W_i, R_i') for each i. As R_0 is complete, there is an element w_0 of W_0 such that $w_0 R_0 w_0^{-1} = R_0'$ by the proof of Lemma 5.2. By conjugating W by w_0, we may assume that $R_0 = R_0'$. By the same argument as in the last paragraph of the proof of Theorem 8.4 of [13], we have that (W, R) is twist equivalent to (W, R'), and so (W, S) is twist equivalent to (W, S'). This completes the induction. □

Corollary 5.4 ([13, Theorem 8.4]). *All Chordal Coxeter systems of finite rank satisfy the twist conjecture.*

Proof. This follows from Theorem 4.1, Lemma 5.2, and Theorem 5.3. □

References

[1] Hyman Bass, *Some remarks on group actions on trees*, Comm. Algebra **4** (1976), no. 12, 1091–1126.

[2] G. A. Dirac, *On rigid circuit graphs*, Abh. Math. Sem. Univ. Hamburg **25** (1961), 71–76.

[3] Max Forester, *Deformation and rigidity of simplicial group actions on trees*, Geom. Topol. **6** (2002), 219–267.

[4] W. N. Franzsen and R. B. Howlett, *Automorphisms of nearly finite Coxeter groups*, Adv. Geom. **3** (2003), no. 3, 301–338.

[5] Vincent Guirardel and Gilbert Levitt, *A general construction of JSJ decompositions*, in Geometric Group Theory. Ed. Goulnara N. Arzhantseva, Laurent Bartholdi, José Burillo, and Enric Ventura. Trends in Mathematics. Basel: Birkhäuser, 2007. 65–73.

[6] ———, *Deformation spaces of trees*, Groups Geom. Dyn. **1** (2007), no. 2, 135–181.

[7] Hanns-Georg Leimer, *Optimal decomposition by clique separators*, Discrete Math. **113** (1993), no. 1-3, 99–123.

[8] Michael Mihalik, *JSJ decompositions of Coxeter groups over virtually abelian splittings*. arXiv:0804.3963v2 [math.GR].(index uses abbreviated title: JSJ decompositions of Coxeter groups.)

[9] Michael Mihalik and Steven Tschantz, *Visual decompositions of Coxeter groups*, Groups Geom. Dyn. **3** (2009), no. 1, 173–198.

[10] Michael L. Mihalik, John G. Ratcliffe, and Steven T. Tschantz, *Matching theorems for systems of a finitely generated Coxeter group*, Algebr. Geom. Topol. **7** (2007), 919–956.

[11] Bernhard Mühlherr, *The isomorphism problem for Coxeter groups*, in The Coxeter Legacy. Ed. Chandler Davis and Erich W. Ellers. Providence, RI: Amer. Math. Soc., 2006. 1–15.

[12] Luis Paris, *Commensurators of parabolic subgroups of Coxeter groups*, Proc. Amer. Math. Soc. **125** (1997), no. 3, 731–738.

[13] John G. Ratcliffe and Steven T. Tschantz, *Chordal Coxeter groups*, Geom. Dedicata **136** (2008), 57–77.

[14] Jean-Pierre Serre, *Trees*. Translated from the French by John Stillwell. Berlin-New York: Springer-Verlag, 1980.

[15] Jacques Tits, *Sur le groupe des automorphismes d'un arbre*, in Essays on Topology and Related Topics (Mèmoires dédiés à Georges de Rham). Ed. André Haefliger and Raghavan Narasimhan. New York: Springer, 1970. 188–211.

(Ratcliffe and Tschantz) Mathematics Department; Vanderbilt University; Nashville TN 37240

E-mail address: j.g.ratcliffe@vanderbilt.edu

E-mail address: steven.tschantz@vanderbilt.edu