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by

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Abstract. A space is od-compact (od-Lindelöf, respectively) provided any cover by open dense sets has a finite (countable, respectively) subcover. We first show with simple examples that these properties behave quite poorly under finite or countable unions. We then investigate the relations between Lindelöfness, od-Lindelöfness, and linear Lindelöfness (and similar relations with "compact"). We prove, in particular, that if a T_1 space is od-compact, then the subset of its non-isolated points is compact. If a T_1 space is od-Lindelöf, we only get that the subset of its non-isolated points is linearly Lindelöf, though Lindelöfness follows if the space is moreover locally openly Lindelöf (i.e., each point has an open Lindelöf neighborhood).

1. Introduction

In the middle of an argument involving Baire theorem, we noticed that we did not need the space under scrutiny to be really Lindelöf, but rather that any cover of it by open *dense* sets had a countable subcover. We then wondered whether this alternative definition of Lindelöfness, called here *od-Lindelöfness*, was interesting in itself, as well as the similarly defined notion of od-compactness. These notes are the results of our musings, which may be summarized as follows.

• od-compact spaces behave quite horribly when taking unions, even when just two subspaces are involved and there are even completely metrizable spaces that behave badly in this respect. A finite union of od-compact *closed* spaces is od-compact, though. On the other hand, a countable union of od-Lindelöf closed spaces does not need to be od-Lindelöf.

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- The image of an od-compact space under a continuous map is not always od-compact, and the same holds for od-Lindelöf spaces. However, the properties are preserved when the map is open. Moreover, the image of a T_1 od-compact space by a closed map is od-compact.

- Trivial examples of od-compact spaces are the discrete ones. But, in a way, they are the only non-compact ones. In fact, the subset of non-isolated points of a T_1 od-compact space is compact. For od-Lindelöfness, our results are not that strong. First, an od-Lindelöf T_1 space that does not contain a clopen uncountable discrete subset *and* which is locally openly Lindelöf is Lindelöf (see below for undefined terminology). If we drop the last assumption, then we could obtain only that the space is linearly Lindelöf. (In fact, the result for od-compact spaces follows from the equivalence of the linearly compact and compact notions.) Moreover, the examples we know of linearly Lindelöf spaces that are *not* Lindelöf happen to be non-od-Lindelöf as well.

We have not found older references to these od-notions, but since our examples and proofs are rather elementary, we would not be surprised if some of our results have already appeared somewhere. Perhaps the above points provide an explanation for this absence in the literature: od-compact and od-Lindelöf properties are not "robust" at all, and moreover (at least for the compact case), differ only slightly from the usual compact and Lindelöf notions. However, we would be interested in finding a non-trivial example of an od-Lindelöf non-Lindelöf space, or in showing that there is none.

This note is organized as follows. In section 2 we give the definitions and show some equivalences. In section 3 we investigate the behavior of the od-properties when taking unions. Then, in section 4, we prove the above-mentioned theorem relating od-Lindelöfness with Lindelöfness, while the relation with linear Lindelöfness is shown in section 5. The short section 6 contains the above-mentioned results about images of od-Lindelöf and od-compact spaces under open and closed maps. We conclude with a short appendix containing classical results featuring compactness, Lindelöfness, and complete accumulation points.

Most of this note does not contain or use technicalities beyond the basics of topology and elementary ordinal/cardinal manipulation and is fairly self contained. However, some of the examples we shall give are classical spaces of set-theoretic topology for which we will just give a reference, and we shall have a few words about more recent constructions of linearly Lindelöf non-Lindelöf spaces. We shall refer to the articles where these spaces were described for more details.

2. Definitions

"Space" always means "topological space." We use the Greek letters α, β, and γ for ordinals, and κ, λ, and τ for cardinals. We denote by $\overline{B}$ and $int(B)$ the closure and interior, respectively, of a subset of a space.

Definition 2.1. Let X be a space.

- $L(X)$ is the smallest cardinal κ such that any open cover of X has a subcover of cardinality $< \kappa$. X is compact if $L(X) \leq \omega$ and Lindelöf if $L(X) \leq \omega_1$, and more generally Lindelöf$_\kappa$ if $L(X) \leq \kappa$.
- $\ell L(X)$ is the smallest cardinal κ such that any open cover of X, which is a chain for the inclusion (in short, a chain-cover), has a subcover of cardinality $< \kappa$. X is linearly compact if $\ell L(X) \leq \omega$ and linearly Lindelöf if $\ell L(X) \leq \omega_1$, and more generally linearly Lindelöf$_\kappa$ if $\ell L(X) \leq \kappa$.
- $odL(X)$ is the smallest cardinal κ such that any cover of X by open dense sets (in short, an od-cover) has a subcover of cardinality $< \kappa$. X is od-compact if $odL(X) \leq \omega$ and od-Lindelöf if $odL(X) \leq \omega_1$, and more generally od-Lindelöf$_\kappa$ if $odL(X) \leq \kappa$.

Be aware that in a lot of texts, the similar Lindelöf degree of a space is defined a bit differently (for instance, $L(\mathbb{R}) = \omega_1$, while its Lindelöf degree is ω). We chose this definition because it seems to enable shorter statements when compact spaces are also involved. Of course, Lindelöf$_\omega$ and Lindelöf$_{\omega_1}$ are synonyms of compact and Lindelöf. Notice also that we do *not* assume any separation axiom for compactness and Lindelöfness, though it is not difficult to show that one can assume our spaces to be T_0 by taking Kolmogorov quotients. It was shown long ago that linearly compact spaces are compact; see the appendix.

Example 2.2. (1) Any Lindelöf$_\kappa$ space is od-Lindelöf$_\kappa$ and linearly Lindelöf$_\kappa$.

(2) Any space with the discrete topology is od-compact (in fact, $odL(X) = 2$).

Recall the following elementary lemma.

Lemma 2.3. (a) *For a topological space X and $\kappa \geq \omega$, $L(X) = \kappa$ if and only if $L(Y) = \kappa$ for each closed $Y \subset X$, if and only if, given a family of closed sets with empty intersection, there is a subfamily of cardinality $< \kappa$ with empty intersection.*

(b) *If X is a union of κ spaces X_α with $L(X_\alpha) \leq \lambda$ for a regular λ, then $L(X) \leq \kappa \cdot \lambda$.*

When od-properties are concerned, we obtain the following.

Lemma 2.4. *The following are equivalent:*

(a) *$odL(X) \leq \kappa$;*
(b) *any cover of X by open sets such that at least one is dense has a subcover of cardinality $< \kappa$;*
(c) *$odL(Y) \leq \kappa$ for each closed $Y \subset X$;*
(d) *$L(Y) \leq \kappa$ for each $Y \subset X$ closed and nowhere dense.*

In particular, a space is od-compact (od-Lindelöf, respectively) if and only if each of its closed nowhere dense subsets is compact (Lindelöf, respectively).

Proof. (a) and (b) are easily seen to be equivalent: Given an open cover $\{U_\alpha : \alpha \in \lambda\}$ with U_0 dense, then the sets $U_0 \cup U_\alpha$ form an od-cover.

(a) $\Rightarrow$ (c) If C is closed in X and U is an open dense set in C, then there is a V open in X with $C \cap V = U$, and $V \cup (X - C)$ is dense.

(c) $\Rightarrow$ (a) Immediate.

(a) $\Rightarrow$ (d) If $L(Y) > \kappa$ for some nowhere dense closed Y, then, given a cover of Y witnessing this fact, we find an od-cover of X taking the union of each member with $X - Y$.

(d) $\Rightarrow$ (b) Given an open cover $\{U_\alpha : \alpha \in \lambda\}$ of X such that U_0 is dense, set $B_\alpha = X - U_\alpha$. Then B_0 is nowhere dense, and $\cap_{\alpha\in\lambda} B_\alpha = \cap_{\alpha\in\lambda}(B_0 \cap B_\alpha) = \varnothing$. Since $L(B_0) \leq \kappa$, there is a subfamily of the B_α of cardinality $< \kappa$ with empty intersection by Lemma 2.3(a). The corresponding family of U_α covers X. $\square$

3. UNIONS OF OD-LINDELÖF$_\kappa$ SPACES

The od-covering properties behave in a quite horrible manner when taking unions.

Example 3.1. For each cardinal $\kappa \geq \omega$, there is a T_1 space X with $odL(X) = \kappa^+$, which satisfies $X = X_0 \sqcup X_1$, where X_0 is compact and X_1 closed and discrete (so $odL(X_0) = \omega$, $odL(X_1) = 2$).

If $\kappa = \omega$, set $\gamma = \omega{\cdot}\omega$; otherwise, set $\gamma = \kappa$. X is given by $(\gamma+1)\times\{0,1\}$ with the following topology. Let the topology on $X_0 = (\gamma+1)\times\{0\}$ be the usual order topology of $\gamma + 1$; X_0 is thus compact. The neighborhoods of $(\alpha, 1)$ are all the subsets of X that can be written as $(U-F)\times\{0\}\sqcup F\times\{1\}$, where U is open in $\gamma + 1$, and $F \subset U$ is a finite set containing α. (It is not difficult to check that the intersection of two sets of this kind is also of this kind.) Then $X_1 = (\gamma + 1) \times \{1\}$ is discrete in X. One shows easily that X is T_1 (but not Hausdorff). Set U to be the open set given by X_0 union $\{(\alpha, 1) : \alpha \text{ is successor}\}$. (Recall that $\{\alpha\}$ is open in $\gamma + 1$ if and only if α is successor.) U is dense in X. For each limit $\alpha \in \gamma + 1$, set

$U_\alpha = U \cup (\alpha \times \{1\})$; U_α is then open and dense. The od-cover by the U_α's does not have any subcover of cardinality $< \kappa$.

The same type of idea can be used to obtain the following.

Example 3.2. For each cardinal $\kappa \geq \omega$, there is a completely metrizable space X with $odL(X) \geq \kappa^+$, which satisfies $X = X_0 \sqcup X_1$, where X_0 and X_1 are discrete and X_0 is closed (so $odL(X_0) = odL(X_1) = 2$).

Take X to be a disjoint union of clopen copies J_α ($\alpha \in \kappa$) of $\{0\} \cup \{\frac{1}{m} : m \in \omega\}$, each with its usual topology. A complete metric on X is given by the usual distance for two points in the same J_α, while two points in two different J_α's are at distance 2. Then X_0 is the union of the 0 points, while X_1 is its complement. The od-cover given by the U_α, defined as $X_1 \cup J_\alpha$, has no proper subcover.

Still another example in the same vein, this time for (non)-od-Lindelöf spaces, showing that we cannot even trust a subspace of a (non-metrizable) 2-manifold.

Example 3.3. There is a subspace S of a 2-manifold with $odL(S) = L(S) = (2^\omega)^+$, such that $S = A \sqcup B$, with A closed discrete, and $B \simeq \mathbb{R}^2$ (so $odL(A) = 2$, $odL(B) = \omega_1$).

This example is the subset of the (separable version of the) Prüfer surface, with A being given by taking one point in each boundary component and B is the interior (i.e., the surface minus the boundary components). See, for instance, the appendix in [12] for a description. The idea is essentially a "manifold equivalent" to the tangent disk topology on the half plane which is described in [13, Example 82]. Both contain a closed nowhere dense discrete subset of cardinality 2^ω and are thus non-od-Lindelöf$_{2^\omega}$ by Lemma 2.4.

Examples 3.1–3.3 all make use of a closed discrete subset whose complement is dense. It is easy to see that one cannot hope to find two closed sets whose union behaves that badly.

Lemma 3.4. *Let κ be an infinite cardinal. If $X = X_0 \cup \cdots \cup X_n$ is a finite union of closed od-Lindelöf$_\kappa$ subsets for $i = 1, \ldots, n$, then $odL(X) \leq \kappa$.*

Proof. We prove it for two subsets; the general case follows by induction. Thus, let $X = X_0 \cup X_1$ and let $B \subset X$ be closed and nowhere dense. We shall show that $L(B \cap X_i) \leq \kappa$ for $i = 0, 1$, which implies $L(B) \leq \kappa$ and the result by Lemma 2.4. We may thus assume first that $B \subset X_0$, the other case being entirely symmetric.

Denote by int_0 the interior for the induced topology in X_0. If $int_0(B)$ is empty, then B is nowhere dense in X_0 and $L(B) \leq \kappa$ by Lemma 2.4. If not, let $U \subset X$ be open with $U \cap X_0 = int_0(B)$ (as in Figure 1). Notice that $L(B - int_0(B)) \leq \kappa$, since $B - int_0(B)$ is closed and nowhere dense in X_0. If $(U \cap X_0) - X_1 \neq \varnothing$, then $(U - X_1) \subset U \cap X_0 \subset B$, so B is not nowhere dense in X. Thus, $(U \cap X_0) \subset X_1$, so

$$U = (U \cap X_0) \cup (U \cap X_1) \subset X_1,$$

$U \cap X_1$ is open in X, and contains $int_0(B)$. It follows that $int_0(B)$ is nowhere dense in X_1 (otherwise, for some W open in U and thus in X, $W \subset int_0(B) \subset B$), so $L(\overline{int_0(B)}) \leq \kappa$, where the closure is taken in X_1 (or in X since X_1 is closed). Since $B = \overline{int_0(B)} \cup \big(B - int_0(B)\big)$, $L(B) \leq \kappa$. □

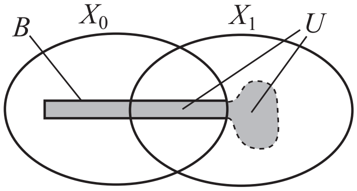

Figure 1. Proof of Lemma 3.4.

Also, there is no Hausdorff space that has the properties of Example 3.1.

Lemma 3.5. (a) *If $X = X_0 \cup X_1$ with X_0 closed, $L(X_0) \leq \kappa$, and $odL(X_1) \leq \kappa$, then $odL(X) \leq \kappa$.*

(b) *If $X = X_0 \cup X_1$ is Hausdorff, with X_0 compact and $odL(X_1) \leq \kappa$, then $odL(X) \leq \kappa$.*

Proof. (a) Take $B \subset X$ to be nowhere dense and closed. By Lemma 2.3, $L(B \cap X_0) \leq \kappa$, and since $X_1 - X_0$ is open, $B \cap (X_1 - X_0)$ is nowhere dense in X_1, so $L(B \cap (X_1 - X_0)) \leq \kappa$. Thus, $L(B) \leq \kappa$.

(b) Since X is Hausdorff, X_0 is closed, and we apply (a). □

The situation with countable unions is bad even for σ-discrete (i.e., a countable union of closed discrete subspaces) spaces.

Example 3.6. There are locally compact σ-discrete non-od-Lindelöf spaces.

Such a space is thus a countable union of closed od-compact subspaces but is non-od-Lindelöf. Any Hausdorff Aronszajn special ω_1-tree T with the order topology is such an example, since it is a countable union of antichains which are closed discrete subspaces and thus od-compact. (See, for instance, [11] for definitions, especially Theorem 4.11.) Moreover, such a tree is locally compact and Hausdorff, and thus Tychonoff. However, if one denotes the members of T at height α by T_α and the set of limit ordinals by Λ, the od-cover given by $U_\alpha = T - \cup_{\beta\in\Lambda,\,\beta>\alpha} T_\alpha$ has no countable subcover.

4. od-Lindelöfness in Locally (Openly) Lindelöf Spaces

From now on, "cardinal" means "infinite cardinal." There are various definitions of local Lindelöfness in the literature. We opted for the following terminology for clarity.

Definition 4.1. Let τ be a regular cardinal. A space X is *locally (openly) Lindelöf*$_\tau$ provided each of its points possesses a Lindelöf$_\tau$ neighborhood (which is open).

Recall that the notions agree for regular spaces (and regular cardinals $\tau \geq \omega_1$).

Lemma 4.2. *Let $\tau \geq \omega_1$ be regular and let X be a regular space. Then X is locally Lindelöf$_\tau$ if and only if it is locally openly Lindelöf$_\tau$, if and only if it has a basis of closed Lindelöf neighborhoods.*

A proof can be found, for instance, by combining [6, Theorem 2.3] and [5, Proposition 1.1] (the result is stated for $\tau = \omega_1$, but the proof works in general). When the space is not regular, the result does not hold anymore, as seen in the example below.

Example 4.3. The everywhere doubled line (see [4]) is a locally Euclidean T_1 space which is locally openly Lindelöf but does not have a basis of closed Lindelöf neighborhoods.

The half disk topology ([13, Example 78]) is a Hausdorff example of such a space.

(Neither example is od-Lindelöf, though.)

The goal of this section is the proof of the following theorem.

Theorem 4.4. *Let τ be a regular cardinal and let X be a T_1 locally openly Lindelöf$_\tau$ space with $odL(X) \leq \tau$. Then either $L(X) \leq \tau$ or there is a clopen discrete subset of cardinality $\geq \tau$ in X.*

(Note that when $\kappa = \omega$, Theorem 5.1(c) below is much stronger.) The core of the proof is essentially contained in the next lemma.

Lemma 4.5. *Let τ and λ be regular cardinals, and let X be a T_1 space with $odL(X) \leq \tau$. Let $Y \subset X$ be closed and let Z_α be open for $\alpha \in \lambda$, such that $Y \subset \cup_{\alpha\in\lambda} Z_\alpha$ and $Z_\alpha \subset Z_\beta$ whenever $\alpha < \beta < \lambda$ and $\overline{Z_\alpha} \not\supset Y$ for each α. Then either $\lambda < \tau$ or $\lambda \geq \tau$ and there is a discrete subset $D \subset Y$, clopen in X, of cardinality $\geq \lambda$.*

Proof. We shall define $x_\alpha \in Y$ and $f : \lambda \to \lambda$ as follows. Set $f(0) = 0$. Given $f(\alpha)$, choose $x_\alpha \in Y - \overline{Z_{f(\alpha)}}$, and set $f(\alpha+1)$ to be the smallest β such that $Z_\beta \ni x_\alpha$. When α is a limit, set $f(\alpha) = \sup_{\beta<\alpha} f(\beta)$. Since λ is regular, $f(\alpha)$ and x_α are defined for each $\alpha < \lambda$. Set $U_\alpha = Z_{f(\alpha)}$; the U_α's have the same properties as the Z_α's, and

$$x_\alpha \in U_{\alpha+1} - \overline{U_\alpha}. \tag{4.1}$$

Let E be the set of α such that $(\overline{U_\alpha} - U_\alpha) \cap Y \neq \varnothing$. If E is cofinal in λ, then letting

$$V_\alpha = X - \cup_{\alpha<\beta<\lambda}(\overline{U_\beta} - U_\beta) \cap Y,$$

we get a cover of X by open dense subsets without any subcover of cardinality $< \lambda$, which implies $\lambda < \tau$. The V_α's are indeed open, since any point y in the closure of $\cup_{\alpha<\beta<\lambda}(\overline{U_\beta} - U_\beta) \cap Y$ must be in Y and thus in U_γ for some γ which we can take minimal, γ is therefore a successor and equal to some $\xi + 1$, with $\xi > \alpha$. Then $y \in \overline{U_\xi}$; otherwise, $U_\gamma - \overline{U_\xi}$ is a neighborhood of y that intersects no $(\overline{U_\beta} - U_\beta)$, and $y \notin U_\xi$ by minimality of γ. So $y \in (\overline{U_\xi} - U_\xi) \cap Y$.

We may thus assume that $E \subset \alpha < \lambda$ for some α, and in fact that $E = \varnothing$. Set $B = \{x_\alpha : \alpha \in \lambda\}$. By (4.1), $\{x_\alpha\}$ is open in $\overline{B}$ for the induced topology. Since X is T_1 and the U_α's cover Y which is closed, $\overline{B} - B$ is contained in the union of the $(\overline{U_\alpha} - U_\alpha) \cap Y$ for limit α, which is empty; thus, B is closed, as well as any of its subsets. If the interior of B is contained in some U_α, then $B - U_\alpha$ is closed and nowhere dense, so by Lemma 2.4, $L(B - U_\alpha) \leq \tau$. But the U_β for $\beta < \lambda$ cover it and there is no subcover of cardinality $< \lambda$ by (4.1), and thus $\lambda < \tau$. So let us assume now that $int(B)$ is not contained in any U_α. Then the α for which $\{x_\alpha\}$ is open in X are cofinal in λ. Call D the union of all these open $\{x_\alpha\}$, then D is clopen and $|D| = \lambda$. □

Another auxiliary result that we shall use is the following lemma.

Lemma 4.6. *If there is a subset $U \subset X$ which is open, Lindelöf$_\tau$ and such that $\overline{U}$ is not Lindelöf$_\tau$, then $odL(X) > \tau$.*

Proof. Otherwise, the nowhere dense closed subset $\overline{U} - U$ would be Lindelöf$_\tau$ by Lemma 2.4, and $\overline{U} = (\overline{U} - U) \cup U$ as well. □

We now start the proof of Theorem 4.4 in earnest.

Proof of Theorem 4.4. Suppose that $L(X) > \tau$. By Lemma 4.6, we can assume that

$$\overline{U} \text{ is Lindelöf}_\tau \text{ whenever } U \text{ is open and Lindelöf}_\tau. \tag{4.2}$$

We will build open subsets X_α for ordinals α. Let $X_0 \subset X$ be any open Lindelöf$_\tau$ subset and build X_α as follows. If α is a limit, take

$X_\alpha = \cup_{\beta<\alpha} X_\beta$. If $\alpha = \beta + 1$ and $\overline{X_\beta} - X_\beta \neq \varnothing$, take a Lindelöf$_\tau$ open neighborhood U_x of each $x \in \overline{X_\beta} - X_\beta$. If $\overline{X_\beta}$ is Lindelöf$_\tau$ (and thus, $\overline{X_\beta} - X_\beta$ as well), extract a subcover U_{x_i} ($i \in \tau_0 < \tau$) and set $X_\alpha = X_\beta \cup (\cup_{i\in\tau_0} U_{x_i})$ (X_α is then Lindelöf$_\tau$ by Lemma 2.3). If such a subcover does not exist, set $X_\alpha = X_\beta \cup (\cup_{x\in\overline{X_\beta}-X_\beta} U_x)$. If $\overline{X_\beta} = X_\beta \neq X$, choose an open Lindelöf$_\tau$ set U disjoint from X_β and set $X_\alpha = X_\beta \cup U$.

By construction, we have $\overline{X_\beta} \subset X_\alpha$ whenever $\beta < \alpha$. For some α, $X = X_\alpha$. Take α to be minimal with this property. Let β be the supremum of $\{\gamma < \alpha : \overline{X_\gamma} \text{ is Lindelöf}_\tau\}$. Then $\overline{X_\beta}$ is not Lindelöf$_\tau$; otherwise, by construction and (4.2), $\overline{X_{\beta+1}}$ would be Lindelöf$_\tau$ too. Likewise, X_β is not Lindelöf$_\tau$. If β is a successor, $\overline{X_{\beta-1}}$ would be Lindelöf$_\tau$, so X_β would be as well, and similarly, if β is a limit with $cf(\beta) < \tau$, $X_\beta = \cup_{\gamma<\beta}\overline{X_\gamma}$ would be a union of less than τ Lindelöf$_\tau$ spaces, and therefore Lindelöf$_\tau$ by Lemma 2.3. Thus, $cf(\beta) \geq \tau$. We now have two cases. (The case $\beta = \alpha$ is contained in the first one, with $V = \varnothing$.)

Case 1: There is an open $V \supset (\overline{X_\beta} - X_\beta)$ such that the set $\{\gamma < \beta : (X_\beta - (V \cup X_\gamma)) \neq \varnothing\}$ is cofinal in β.

Then X satisfies the assumptions of Lemma 4.5 with $Y = \overline{X_\beta} - V$ and $\lambda = cf(\beta) \geq \tau$, and X contains a clopen discrete subset of cardinality $\geq \tau$.

Case 2: For any open set $V \supset (\overline{X_\beta} - X_\beta)$, there is a $\gamma < \beta$ such that $(X_\beta - X_\gamma) \subset V$.

Suppose that $\overline{X_\beta} - X_\beta$ is Lindelöf$_\tau$ and let $\langle U_i : i \in I\rangle$ be an open cover of $\overline{X_\beta}$. Extract a subcover of $\overline{X_\beta} - X_\beta$ of cardinality $< \tau$ and choose $\gamma < \beta$ such that X_γ is Lindelöf$_\tau$ and $(X_\beta - X_\gamma)$ is included in the union of this subcover. Adding a subcover of $\overline{X_\gamma}$ of cardinality $< \tau$ and putting everything together yields a subcover of $\overline{X_\beta}$ of the same cardinality, so $\overline{X_\beta}$ is Lindelöf$_\tau$ and $\overline{X_{\beta+1}}$ as well, contradicting the definition of β. Therefore, $\overline{X_\beta} - X_\beta$ is not Lindelöf$_\tau$.

Thus, let U_i ($i \in I$) be a cover of $\overline{X_\beta} - X_\beta$ without subcover of cardinality $< \tau$. Set $W_i = U_i \cup X_\beta \cup (X - \overline{X_\beta})$, which yields a cover of X by open dense sets with the same property, a contradiction since X is od-Lindelöf$_\tau$. □

In view of the impressive list given in [7], it might be interesting to notice the following corollary.

Corollary 4.7. *A connected manifold is metrizable if and only if it is od-Lindelöf.*

Proof. A manifold is locally compact (and thus locally openly Lindelöf) and its singletons are not open, so it cannot possess an open discrete subset. Hence, it is Lindelöf if and only if od-Lindelöf. We conclude by recalling that Lindelöfness and metrizability are equivalent for connected manifolds. □

The next lemma yields more consequences of Theorem 4.4.

Lemma 4.8. *Let τ be a regular cardinal, let X be an od-Lindelöf$_\tau$ space, and let D be the subspace of its isolated points. Then $X-D$ does not contain a clopen discrete subset of cardinality $\geq \tau$.*

Proof. Notice that D is open and discrete, so $X-D$, being closed, is od-Lindelöf$_\tau$ by Lemma 2.4. Suppose that $X-D$ contains a clopen (in $X-D$) discrete subset D_0 of cardinality $\geq \tau$. Then $D\cup\{x\}$ is a neighborhood of x for each $x\in D_0$ and setting $V_x=\{x\}\cup(X-D_0)$ yields an od-cover without subcover of cardinality $<\tau$. □

The next corollary follows immediately.

Corollary 4.9. *Let τ be a regular cardinal and let X be a locally openly Lindelöf$_\tau$ space with $odL(X)\leq\tau$. Let $D\subset X$ be the subset of isolated points. Then $X-D$ is Lindelöf$_\tau$.*

We shall later relax the local openly Lindelöfness assumption, so let us introduce a notation.

Definition 4.10. Let $\tau>\omega$ be a regular cardinal and let X be a topological space.

$$\begin{array}{rcl}\mathsf{L}_\tau(X) & = & \{x\in X : \exists \text{ an open Lindelöf}_\tau\ U\ni x\}\\ \mathsf{NL}_\tau(X) & = & X-\mathsf{L}_\tau(X).\end{array}$$

We denote by $\mathsf{C}(X)$ the subset containing the points possessing a compact neighborhood and set $\mathsf{NC}(X)=X-\mathsf{C}(X)$.

It is immediate from the definition that $\mathsf{L}_\tau(X)$ and $\mathsf{C}(X)$ are open. There are simple spaces with $\mathsf{NC}(X)$ ($\mathsf{NL}_\tau(X)$, respectively) consisting of just one point: the cone $[0,1]\times Y/\ (0,y)\sim(0,z)$ over any locally compact (locally openly Lindelöf$_\tau$, respectively) Y which is not compact (Lindelöf$_\tau$, respectively).

Theorem 4.11. *Let $\kappa\geq\omega_1$ be a regular cardinal and let X be a T_1 space such that $odL(X)\leq\kappa$ and $L(\mathsf{NL}_\kappa(X))\leq\kappa$. Then either $L(X)\leq\kappa$ or X contains a clopen discrete subset of cardinality $\geq\kappa$.*

Proof. Notice that if $L(X)\leq\kappa$, then $\mathsf{NL}_\kappa(X)=\varnothing$. We have two cases.

Case 1: There is some open $U\supset\mathsf{NL}_\kappa(X)$ such that $L(X-U)>\kappa$.

We repeat the proof of Theorem 4.4 in $X - U$ (which is od-Lindelöf$_\kappa$) and apply Lemma 4.5(1) for $Y = (\overline{X_\beta} - V) \cap (X - U)$, yielding the same result.

Case 2: For all open $U \supset \mathsf{NL}_\kappa(X)$, $L(X - U) \leq \kappa$.

In this case, $L(X)$ will be $\leq \kappa$. Indeed, given a cover of X by V_i, $i \in I$, let V_{i_k} for $k \in J$ be a subcover of $\mathsf{NL}_\kappa(X)$ of cardinality $< \kappa$. Then $X - \cup_{k \in J} V_{i_k}$ being Lindelöf$_\kappa$, is covered by $< \kappa$ many more V_i. □

5. od- and Linear-Lindelöfness

Here, we show the relations between od- and linear-Lindelöfness. First, an easy theorem.

Theorem 5.1. *Let κ be a regular cardinal.*

(a) *The subspace of non-isolated points of a T_1 od-Lindelöf$_\kappa$ space is linearly Lindelöf$_\kappa$.*
(b) *If the subspace of non-isolated points of a space is Lindelöf$_\kappa$, then the space is od-Lindelöf$_\kappa$.*
(c) *A T_1 space is od-compact if and only if the subspace of its non-isolated points is compact.*

Proof. (a) Let D be the subset of all isolated points of X and set $Z = X - D$. Then, by Lemma 2.4, $odL(Z) \leq \kappa$, and Z does not have a clopen discrete subset of cardinality $\geq \kappa$ by Lemma 4.8. Let $\{U_\alpha : \alpha \in \lambda\}$ be a chain-cover of Z. We may assume λ to be regular. If some U_α is dense in Z, then each U_β for $\beta \geq \alpha$ is such, so there is a subcover of Z of cardinality $< \kappa$. We may now assume that none of the U_α is dense in Z. But then X satisfies the hypotheses of Lemma 4.5 for $Y = Z$, which yields $\lambda < \kappa$.

(b) By Lemma 3.5(a).

(c) By Corollary 7.3 below, a linearly compact space is compact; the result follows thus from (a) and (b). □

When $\kappa > \omega$, one can get a finer result (though not as good as in the compact case).

Theorem 5.2. *Let X be a T_1 space with $odL(X) \leq \kappa$ for a regular $\kappa \geq \omega_1$ and let $D \subset X$ be the subset of isolated points. Then $X = D \sqcup X_0 \sqcup X_1$, where $X_0 \cup D$ is open, $L(\overline{X_0}) \leq \kappa$, X_1 is closed, $\ell L(X_1) \leq \kappa$, and any open set U with $U \cap X_1 \neq \varnothing$ satisfies $L(U) > \kappa$.*

Proof. Set $Z = X - D$, again $odL(Z) \leq \kappa$, and Z does not have a clopen discrete subset of cardinality $\geq \kappa$. Set $X_0 = \mathsf{L}_\kappa(Z)$ and $X_1 = \mathsf{NL}_\kappa(Z)$. By Lemma 2.4, $odL(\overline{X_0}) \leq \kappa$ and $odL(X_1) \leq \kappa$. Notice that $\mathsf{NL}_\kappa(\overline{X_0}) =$

$\overline{X_0} \cap X_1$ is closed and nowhere dense, so $L(\mathsf{NL}_\kappa(\overline{X_0})) \leq \kappa$, and by Theorem 4.11, $L(\overline{X_0}) \leq \kappa$.

We now repeat the proof of Theorem 5.1: Let U_α $(\alpha \in \lambda)$ be a chain-cover of X_1. As above, we may assume that none of the U_α is dense in X_1. But then Z satisfies the hypotheses of Lemma 4.5 for $Y = \overline{X_1} \supset \cup_{\alpha\in\lambda} U_\alpha$, which yields again $\lambda < \kappa$. □

Example 5.3. There are linearly Lindelöf non-od-Lindelöf spaces.

These spaces are examples of linearly Lindelöf non-Lindelöf (abbreviated ℓLnL below) spaces found in the literature, which happen to be non-od-Lindelöf. Probably the first example of an ℓLnL space was given by A. Miščenko in [10]. It is a Tychonoff space, defined as the subset of $R = \Pi_{i\in\omega}(\omega_i + 1)$ by the union $\cup_{k\in\omega} R_k$ with $R_k = \big(\Pi_{i=0,\dots,k-1}(\omega_i + 1)\big) \times \big(\Pi_{i\in\omega,\, i\geq k}\omega_i\big)$. (As usual, we denote by ω_i the i-th cardinal above $\omega = \omega_0$ and by ω_ω the sup of these ω_i.) The proof given in [10] can be easily adapted to show that the od-cover given by the $\Gamma_{\alpha,i}$, defined for $i \in \omega$ and $\alpha \in \omega_\omega$ as the subset of points whose i-th coordinate is not a limit ordinal $\geq \alpha$, does not admit a subcover of cardinality $< \aleph_\omega$, so this space is not od-Lindelöf.

A. V. Arhangel'skii and R. Z. Buzyakova [3, Example 4.1] gave a description of another Tychonoff ℓLnL space X, which is a subspace of $\mathcal{D}^A$, where $\mathcal{D}$ is the discrete space $\{0,1\}$ and A is discrete with cardinality $\aleph_\omega$. X is the subspace consisting of the points that have less than $\aleph_\omega$ coordinates equal to 1. They show that X is pseudocompact since it contains a dense countably compact subspace and non-compact since it is not closed in $\mathcal{D}^A$. It happens that X is non-od-Lindelöf. Indeed, fix an uncountable $A_0 \subset A$ such that $|A - A_0| = \aleph_\omega$ and let B be the subset of X consisting of points whose coordinates in A_0 are all 0. Then B is closed and nowhere dense (since it does not contain a basic open set, where only a finite number of coordinates are fixed). But B is homeomorphic to X, and thus non Lindelöf, so by Lemma 2.4 X is non-od-Lindelöf. The modified version in [2] has the same property.

Kenneth Kunen [8], [9] found locally compact ℓLnL spaces. Recall that a locally compact space is Tychonoff and thus regular, so by Lemma 4.2, X is locally openly Lindelöf; thus, $\mathsf{NL}_{\omega_1}(X) = \varnothing$. A linearly Lindelöf space does not contain an uncountable clopen discrete subset, so by Theorem 4.4, X is not od-Lindelöf.

These results and examples raise the following questions.

Question 5.4. Is there a T_1 space which does not contain a clopen uncountable discrete subset that is od-Lindelöf and non-Lindelöf?

Question 5.5. What conditions should be added to linear Lindelöfness to ensure that a space is od-Lindelöf?

6. Images of od-Lindelöf Spaces

Notice that the continuous image of an od-compact space may be violently non-od-compact.

Example 6.1. Denote by κ_d the cardinal κ with the discrete topology, while κ is endowed with the usual order topology. Then $odL(\kappa_d) = 2$, while $odL(\kappa) = cf(\kappa)$, and the identity map $\kappa_d \to \kappa$ is continuous.

However, we have preservation if the map is open and also if the map is closed and X is T_1 and od-compact. The proof of the latter fact uses Theorem 5.1(c). We found neither an easier proof (which we believe should exist) nor a general result for od-Lindelöf$_\kappa$ spaces with uncountable κ.

Lemma 6.2. *Let X and Y be spaces and let $f : X \to Y$ be continuous.*

(a) *If f is open, then $odL(f(X)) = odL(X)$.*
(b) *If f is closed and X is T_1 and od-compact, then $f(X)$ is od-compact.*

Proof. In both cases we may assume that $f(X) = Y$.

(a) First, $f(X)$ is open in Y, so a relatively open subset of $f(X)$ is indeed open. Let $\{U_j : j \in J\}$ be an od-cover of $f(X)$. If $f^{-1}(U_j)$ misses some open nonempty $W \subset X$, then $U_j \cap f(W) = \varnothing$, which is impossible. Thus, $\{f^{-1}(U_j) : j \in J\}$ is an od-cover, and we conclude by extracting a subcover and mapping it through f.

(b) Let D be the set of isolated points of X, then $X - D$ is closed and compact by Theorem 5.1(c), so $f(X - D)$ is closed and compact as well. We now show that the points in $Y - f(X - D)$ are isolated; by Lemma 3.5(a) this yields that $f(X)$ is od-compact. Let $x \in D$ be such that $f(x) \notin f(X - D)$. Define the open subset

$$Z_x = \{z \in D : f(z) = f(x)\},$$

then $\{f(x)\} = Y - f(X - Z_x)$ is open, which shows that $f(x)$ is isolated. □

7. APPENDIX: CLASSICAL RESULTS ON LINEARLY LINDELÖF AND COMPACT SPACES

Here we recall some classical basic results due to P. Alexandroff and P. Urysohn [1]. Consider the following properties for a space X and a regular infinite cardinal κ:

CAP(κ): If B is a subset of regular cardinality $\geq \kappa$, it has a point of complete accumulation.

CAP$^+$(κ): If B is a subset of cardinality $\geq \kappa$, it has a point of complete accumulation.

Then we have the following lemma.

Lemma 7.1. *X satisfies CAP(ω) if and only if it satisfies CAP$^+$(ω).*

Proof. CAP$^+$(ω) implies trivially CAP(ω); we thus show the other implication. Let $\kappa \geq \omega$ be minimal such that there is some $B \subset X$ with $|B| = \kappa$ without complete accumulation point; κ must be singular and $> \omega$ by CAP(ω). Thus, for all infinite $\lambda < \kappa'$, there is an accumulation point x_λ of B such that any open set containing x_λ intersects B in at least λ points. Let $\tau = cf(\kappa) < \kappa$ and let $f : \tau \to \kappa$ be a cofinal map. Since $C = \{x_{f(\alpha)} : \alpha \in \tau\}$ has a cardinality $< \kappa$ but $\geq \omega$, it possesses a complete accumulation point x. (In this part of the proof we really need ω.) Thus, any open $U \ni x$ contains x_λ for a subset of λ cofinal in κ. Hence, it intersects B in more than λ points for each $\lambda < \kappa$, and therefore in κ points. $\square$

Theorem 7.2. *Let X be a space and κ be regular.*

(a) *X satisfies CAP$^+$(ω) if and only if $L(X) = \omega$ (i.e., X is compact).*
(b) *If X satisfies CAP$^+$(κ), then $L(X) \leq \kappa$.*
(c) *X satisfies CAP(κ) if and only if $\ell L(X) \leq \kappa$.*

Proof. (a) Assume X to be compact and let $B \subset X$ be infinite. If there is no complete accumulation point for B, then for each $x \in X$ there is an open set $U_x \ni x$ with $|U_x \cap B| < |B|$. Taking a finite subcover, this yields that $|B|$ is a finite sum of smaller cardinals, which is impossible. The converse is included in (b).

(b) Let κ be regular. Suppose that $L(X) > \kappa$ and let κ' be minimal such that there exists an open cover $\langle U_\alpha : \alpha \in \kappa' \rangle$ of X without a subcover of cardinality $< \kappa$. Set $V_\alpha = \cup_{\beta<\alpha} U_\alpha$. If for some $\alpha < \kappa'$ we have $V_\alpha = X - E$, with $|E| < \kappa'$, then letting $\beta(x)$ be the smallest β such that $x \in U_\beta$, we get that

$$\langle U_\beta : \beta < \alpha \text{ or } \beta = \beta(x) \text{ for some } x \in E \rangle$$

is a cover of X by less than κ' open sets; thus, by minimality of κ' there is a cover of cardinality $< \kappa$, a contradiction. Thus, for each α, there is

$x_\alpha \notin V_\alpha$. Hence, $x_\alpha \notin U_\beta$ for each $\beta < \alpha$ and $B = \{x_\alpha : \alpha \in \kappa'\}$ has no complete accumulation point (because each $x \in X$ belongs to some U_β which contains $< \kappa'$ points of B).

(c) Assume that $\ell L(X) \leq \kappa$ and let $B = \{x_\alpha : \alpha < \kappa'\}$ for some regular $\kappa' \geq \kappa$. Set $B_\beta = \{x_\alpha : \beta \leq \alpha < \kappa'\}$ and $U_\beta = X - \overline{B_\beta}$. Then $\langle U_\beta : \beta \in \kappa' \rangle$ is a chain for the inclusion. If it covers X, we may extract a subcover of cardinal $< \kappa$, and since κ' is regular, there is some $\beta < \kappa'$ (the sup of the indices in the subcover) with $U_\beta = X$. Thus, B_α is empty for each $\alpha > \beta$, a contradiction. Therefore, there is some $x \in X$ such that $x \notin U_\beta$ (that is, $x \in \overline{B_\beta}$) for all β. Given an open set $U \ni x$, for each β there is an $\alpha \geq \beta$ with $x_\alpha \in U$. The regularity of κ' implies then that $|U \cap B| = \kappa'$, so x is a complete accumulation point.

Conversely, given an open cover $\langle U_j : j \in J \rangle$ of X which is a chain and does not possess a subcover of cardinality $< \kappa$, let λ be minimal such that there is a cofinal map $f : \lambda \to J$. Then λ is regular and, writing V_α for $U_{f(\alpha)}$, $\langle V_\alpha : \alpha \in \lambda \rangle$ is a cover of X, which does not possess a subcover of cardinality $< \kappa$. For each $\alpha \in \lambda$, let $x_\alpha \notin V_\alpha$, then $B = \{x_\alpha : \alpha \in \lambda\}$ has no complete accumulation point because each V_α contains less than λ points of B and they cover X. This contradicts CAP(κ). ☐

Notice that the last part of the proof does not work if one takes a cover that is not a chain. Moreover, the converse implication of (b) does not hold: ω_ω induced with the order topology is Lindelöf but it does not possess a point of complete accumulation.

The corollary we used in Theorem 5.1 follows immediately.

Corollary 7.3. *A space is compact if and only if it is linearly compact.*

REFERENCES

[1] P. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch., Amsterdam **14**(1929), no. 1, 1–96.

[2] A. V. Arhangel'skii and R. Z. Buzyakova, *Convergence in compacta and linear Lindelöfness*, Comment. Math. Univ. Carolin. **39** (1998), no. 1, 159–166.

[3] ———, *On linearly Lindelöf and strongly discretely Lindelöf spaces*, Proc. Amer. Math. Soc. **127** (1999), no. 8, 2449–2458.

[4] Mathieu Baillif and Alexandre Gabard, *Manifolds: Hausdorffness versus homogeneity*, Proc. Amer. Math. Soc. **136** (2008), no. 3, 1105–1111.

[5] Z. Balogh, *Locally nice spaces and axiom R*, Topology Appl. **125** (2002), no. 2, 335–341.

[6] U. N. B. Dissanayake and K. P. R. Sastry, *Locally Lindelöf spaces*, Indian J. Pure Appl. Math. **18** (1987), no. 10, 876–881.

[7] Gauld, David, *Metrisability of Manifolds.* Preprint arXiv:0910.0885 [math.GN].

[8] Kenneth Kunen, *Locally compact linearly Lindelöf spaces*, Comment. Math. Univ. Carolin. **43** (2002), no. 1, 155–158.

[9] ———, *Small locally compact linearly Lindelöf spaces*, Topology Proc. **29** (2005), no. 1, 193–198.

[10] A. Miščhenko, *Finally compact spaces*, Soviet Math. Dokl. **145** (1962), 1199–1202.

[11] Peter J. Nyikos, *Various topologies on trees*, in Proceedings of the Tennessee Topology Conference (Nashville, TN, 1996). Ed. P. R. Misra and M. Rajagopalan. River Edge, NJ: World Sci. Publ., 1997. 167–198.

[12] Michael Spivak, *A Comprehensive Introduction to Differential Geometry. Vol. I.* 2nd ed. Wilmington, Del.: Publish or Perish, Inc., 1979.

[13] Lynn Arthur Steen and J. Arthur Seebach, Jr., *Counterexamples in Topology.* 2nd ed. New York-Heidelberg: Springer-Verlag, 1978.

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