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ON TOPOLOGICAL GROUPS WITH A FIRST-COUNTABLE REMAINDER

A. V. ARHANGEL'SKII AND J. VAN MILL

Abstract. We establish the following estimate on the character of an arbitrary non-locally compact topological group with a first-countable remainder: it does not exceed ω_1. We also show that this estimate is the best possible by constructing a non-metrizable non-locally compact topological group with a first-countable remainder. At the beginning of the article, a brief survey of the properties of remainders of topological groups is provided.

1. Introduction

By *space*, we understand a Tychonoff topological space. By *remainder* of a space X, we mean the subspace $bX\setminus X$ of a Hausdorff compactification bX of X. We follow the terminology and notation in [7].

Recall that a π-*network* (π-*base*) of a space X at a point $x \in X$ is a family η of non-empty subsets (open subsets, respectively) of X such that every open neighbourhood of x contains a member of η.

A space is said to be ω-*bounded* (*strongly* ω-*bounded*) if the closure of every countable (σ-compact, respectively) subset is compact.

A series of results on remainders of topological groups have been obtained in [1], [2], and [4]. They show that the remainders of topological groups are much more sensitive to the properties of topological groups than the remainders of topological spaces are in general. Of course, there is an important exception to this rule: the case of locally compact topological groups. Indeed, every locally compact non-compact topological

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group has a remainder consisting of exactly one point. Thus, we will be interested only in the case of non-locally compact topological groups.

It was proved in [2] that if a non-locally compact topological group G has a remainder with a G_δ-diagonal, then both G and this remainder are separable metrizable spaces.

In this connection, the next general question has been posed in [3]: When does a topological group have a first-countable remainder?

It is a well-known theorem of Birkhoff and Kakutani that every first-countable topological group is metrizable (see, for example, [6]). One may expect that the first-countability of a remainder would also force the metrizability of the group itself, provided, of course, that the remainder is dense in the compactification. Notice that the remainder is dense in the compactification if the group is not locally compact. The next problem has not been formulated in the article [3] by a technical error, even though [3] is almost exclusively concerned with the conditions under which the answer to the next question is "yes."

Problem 1.1. Suppose that G is a non-locally compact topological group with a first-countable remainder. Is G metrizable?

Among the main results in [3], we find the following two theorems.

Theorem 1.2. *Let G be a non-compact topological group such that G^ω has a first-countable remainder. Then G is metrizable.*

Theorem 1.3. *A topological group G is metrizable if there is a non-locally compact metrizable space M such that the product space $G \times M$ has a first-countable remainder.*

The following special case of Problem 1.1 has been explicitly formulated in [3].

Problem 1.4. Is there a non-metrizable countable topological group with a first-countable remainder?

A convenient criterion for metrizability of a non-locally compact topological group has been given in [5].

Theorem 1.5. *A non-locally compact topological group G is metrizable if and only if some remainder Y of G is a non-countably compact space of countable π-character.*

In certain special classes of spaces, Problem 1.1 has been answered in the positive way. The following theorems two such results are from [3].

Theorem 1.6. *If G is a topological group with a countable network and G has a remainder of countable π-character, then either G is metrizable or G is σ-compact.*

Theorem 1.7. *For any uncountable space X, the π-character of any remainder of $C_p(X)$ is uncountable.*

We denote by (MA+¬CH) that Martin's Axiom and the negation of Continuum Hypothesis are assumed (see [8]).

The following two results have also been obtained in [3].

Theorem 1.8 (MA+¬CH). *Suppose that G is a σ-compact topological group with a remainder of countable tightness. Then either G is locally compact or G is metrizable.*

Theorem 1.9. *If G is a non-metrizable topological group with a remainder of countable π-character, then every remainder of G is ω-bounded.*

Below we answer Problem 1.1 in the negative and establish a new necessary condition for a non-locally compact topological group to have a first-countable remainder.

2. A Theorem

Theorem 2.1. *Suppose that G is a non-locally compact topological group with a first-countable remainder Y. Then the character of the space G doesn't exceed ω_1.*

To prove Theorem 2.1, we need the following statement.

Proposition 2.2. *Suppose that Y is a space of countable tightness satisfying the following condition: (s) for any subset A of Y such that $|A| \le \omega_1$, the closure of A in Y is compact. Then Y is compact.*

Proof. Assume the contrary. Then we can represent Y as a non-closed subspace of some compact Hausdorff space X. Fix $x \in \overline{Y} \setminus Y$. By condition (s), the closure in X of any countable subset A of Y is contained in Y. Hence, Y is countably compact. Therefore, the next statement holds.

Fact 1. For any G_δ-subset P of X such that $x \in P$, we have $P \cap Y \neq \emptyset$.

Using this fact, we define for every $\alpha < \omega_1$ a point $y_\alpha \in Y$ and a closed G_δ-subset P_α of X containing x, as follows. Let y_0 be any element of Y, and put $P_0 = X$. Now assume that $\alpha \in \omega_1$, and that the points $y_\beta \in Y$ and the closed G_δ-subsets P_β of X have been already defined for every $\beta < \alpha$. Denote by F_α the closure of the set $\{y_\beta : \beta < \alpha\}$ in X. Then $F_\alpha \subset Y$ and $x \notin F_\alpha$. Since F_α is closed in X, it follows that there exists a closed G_δ-subset V_α of x in X such that $x \in V_\alpha$ and $V_\alpha \cap F_\alpha = \emptyset$. Put $P_\alpha = V_\alpha \cap \bigcap_{\beta<\alpha} P_\beta$. Clearly, $x \in P_\alpha$, and P_α is a closed G_δ-subset of X. By Fact 1, we have $P_\alpha \cap Y \neq \emptyset$. Now define y_α to be an arbitrary point of the set $P_\alpha \cap Y$. The transfinite sequences $\{y_\alpha : \alpha \in \omega_1\}$ and

$\{P_\alpha : \alpha \in \omega_1\}$ are constructed. Obviously, the following statements hold for any $\alpha < \omega_1$ (Fact 4 follows directly from facts 2 and 3.

FACT 2. $\overline{\{y_\beta : \beta < \alpha\}} \cap P_\alpha = \emptyset$.

FACT 3. $\overline{\{y_\beta : \alpha \leq \beta < \omega_1\}} \subset P_\alpha$.

FACT 4. $\overline{\{y_\beta : \beta < \alpha\}} \cap \overline{\{y_\beta : \alpha \leq \beta < \omega_1\}} = \emptyset$.

Fact 4 means that $\eta = \{y_\alpha : \alpha \in \omega_1\}$ is a free sequence in X. Clearly, η is contained in Y, and the cardinality of the set η is ω_1. By condition (s), some point z of Y is a point of complete accumulation for the set η. However, since the tightness of Y is countable, it follows from Fact 4 that no point of Y is a point of complete accumulation for η, a contradiction. □

Proposition 2.3. *Suppose that X is a nowhere locally compact space with a first-countable remainder Y. Then the π-character of the space X doesn't exceed ω_1 at some point of X.*

Proof. Let bX be a compactification of the space X such that $Y = bX \setminus X$. The subspace X is not open in bX since X is not locally compact. Therefore, Y is not closed in bX, that is, Y is not compact. Since the tightness of Y is countable, it follows from Proposition 2.2 that Y doesn't satisfy condition (s). Hence, there exists a subset A of Y such that $|A| \leq \omega_1$, and the closure of A in Y is not compact. Then there exists $x \in X \setminus Y$ such that $x \in \overline{A}$. Observe that Y is dense in bX since X is nowhere locally compact. Since Y is first-countable, it follows that bX is first-countable at each $y \in Y$. Therefore, we can fix a countable base ξ_y of bX at y for every $y \in Y$. Put $\gamma = \bigcup_{y \in A} \xi_y$ and $\mathcal{P} = \{W \cap X : W \in \gamma\}$. Since X is dense in bX and x is in the closure of A, the family $\mathcal{P}$ is a π-base of X at the point x. Obviously, $|\mathcal{P}| \leq \omega_1$. □

Proof of Theorem 2.1. It follows from Proposition 2.3 that there exists a π-base $\mathcal{P}$ of G at the neutral element e of G such that $|\mathcal{P}| \leq \omega_1$. Then, clearly, the family $\mu = \{UU^{-1} : U \in \mathcal{P}\}$ is a base of G at e such that $|\mu| \leq \omega_1$. □

Theorem 2.4. *If G is a non-locally compact topological group with a first-countable remainder, then $|G| \leq 2^{\omega_1}$.*

Proof. Let bG be a compactification of the space G such that the remainder $Y = bG \setminus G$ is first-countable. By Theorem 2.1, the character of the space G doesn't exceed ω_1. Since Y is first-countable and Y and G are both dense in bG, we conclude that $\chi(bG) \leq \omega_1$. Since bG is compact, it follows that $|bG| \leq 2^{\omega_1}$. Hence, $|G| \leq 2^{\omega_1}$. □

Since it is consistent with ZFC that $2^{\omega_1} = 2^{\omega}$, it follows that the next statement holds.

Corollary 2.5. *It is consistent with* ZFC *that if G is any non-locally compact topological group with a first-countable remainder, then $|G| \leq 2^{\omega}$.*

The proof of Proposition 2.3 shows that the next statement is true.

Theorem 2.6. *Suppose that G is a non-locally compact topological group with a remainder Y such that the tightness of Y is countable and the π-character of Y doesn't exceed ω_1. Then the character of the space G doesn't exceed ω_1.*

3. The Example

We now show that Theorem 2.1 is best possible; i.e., there exists a non-locally compact topological group G of character ω_1 which has a compactification bG such that $bG \setminus G$ is first-countable.

Let X be a space with a dense subset D. We think of

$$X(D) = (X \times \{0\}) \cup (D \times \{1\})$$

as a subspace of the Alexandroff duplicate of X. That is, the points of $D \times \{1\}$ are isolated, and a basic neighborhood of $(x, 0) \in X \times \{0, 1\}$ has the form $(U \times \{0\}) \cup (((U \cap D) \setminus F) \times \{1\})$, where U is an arbitrary open neighborhood of x in X and F is finite.

Observe that $X(D)$ is compact if X is compact.

Now let X be a space with a dense subset D and let Y be an arbitrary non-empty space. We want to replace every isolated point of the form $(d, 1)$ in $X(D)$ by a copy of Y. It is not a problem to do that of course. Indeed, put

$$X(D, Y) = (X \times \{0\}) \cup (D \times Y \times \{1\}),$$

and topologize it as follows. Every set of the form $\{d\} \times Y \times \{1\}$ is clopen in $X(D, Y)$, for any d in D, and the function $y \mapsto (d, y, 1)$, $y \in Y$, is a homeomorphism. That is, Y and $\{d\} \times Y \times \{1\}$ are the same spaces.

A basic neighborhood of a point $(x, 0)$ in $X(D, Y)$ has the form

$$(U \times \{0\}) \cup (((U \cap D) \setminus F) \times Y \times \{1\}),$$

where U is an arbitrary open neighborhood of x in X and F is finite.

Observe that if both X and Y are compact, then so is $X(D, Y)$.

The function $\pi \colon X(D, Y) \to X \times \{0\}$ that collapses each set of the form $\{d\} \times Y \times \{1\}$ to $(d, 0)$ is a retraction. It is so standard that it will always be denoted by π regardless of what spaces X, D, and Y we are dealing with.

Our basic building block is $K = 2^\omega(2^\omega)$, i.e., the Alexandroff duplicate of the Cantor set 2^ω. Using this building block repeatedly, we will construct an inverse sequence of compact spaces as follows.

Let $X_0 = 2^\omega$, and let X_1 be $X_0(X_0)$, i.e., $X_1 = K$. Let $\pi^1_0\colon X_1 \to X_0$ be the standard retraction π.

Now we replace each isolated point of X_1 by a copy of K, i.e., we put

$$X_2 = X_1(X_0 \times \{1\}, K);$$

again, let $\pi^2_1\colon X_2 \to X_1$ be the standard retraction π.

Observe that if $p \in X_0 \times \{0\}$, then $(\pi^2_1)^{-1}(\{p\})$ equals $\{p\}$.

The points $q \in X_1$, for which $\pi_2^{-1}(\{q\})$ is non-trivial, are exactly the isolated points of X_1. Moreover, if $q \in X_1$ is isolated, then $(\pi^2_1)^{-1}(\{q\})$ contains $\mathfrak{c}$ isolated points of X_2.

In this way we continue to define X_n for $n < \omega$, at each step replacing each isolated point of X_n by a copy of K.

Let $X_\omega = \varprojlim\{X_n, \pi^n_m\}$. For each $n < \omega$, let $\pi^\omega_n\colon X_\omega \to X_n$ denote the projection.

Observe that if $p \in X_\omega$, then there are two cases. The first case is that $\pi^\omega_n(p)$ is not isolated for some $n < \omega$. In that case we get

$$(\pi^\omega_n)^{-1}(\{\pi^\omega_n(p)\}) = \{p\}.$$

That is, the point $\pi^\omega_n(p)$ is " left alone" in the next steps.

The second case is that $\pi^\omega_n(p)$ is isolated for every $n < \omega$. It is clear that the set of all such points D is dense.

Now put $X_{\omega+1} = X_\omega(D)$, and let $\pi^{\omega+1}_\omega$ be the standard projection. We continue as before, replacing each isolated point by a copy of K, etc. Let $X_{\omega+\omega}$ be the inverse limit of spaces $X_{\omega+n}$.

Continuing in this way for all $\alpha < \omega_1$, we get an inverse sequence $\{X_\alpha, \pi^\alpha_\beta\}$ of compact spaces that are clearly all first-countable. Let $X = \varprojlim\{X_\alpha, \pi^\alpha_\beta\}$ with projections $\pi^{\omega_1}_\alpha : X \to X_\alpha$ for all $\alpha < \omega_1$. The following fact holds.

> If $p \in X$ and there exists a successor ordinal number $\alpha < \omega_1$ such that $\pi^{\omega_1}_\alpha(p)$ is not isolated, then
>
> $$(\pi^{\omega_1}_\alpha)^{-1}(\{\pi^{\omega_1}_\alpha(p)\}) = \{p\}.$$
>
> Hence, X is first-countable at p.

The points $p \in X$, for which $\pi^{\omega_1}_\alpha(p)$ is isolated for every successor ordinal number $\alpha < \omega_1$, form a dense subspace G in X. The space G is easily seen to be homeomorphic to the space $\mathfrak{c}^{\omega_1}$ with the G_δ-topology. The reason that we get the G_δ-topology is clear: because if $p \in G$, then, for every $\alpha < \omega_1$, we have that $\pi^{\omega_1}_{\alpha+1}(p)$ is isolated.

Hence, G is a topological group, and so we are done.

Observe that G is non-discrete. Since every G_δ-set in G is open, the space G is not metrizable. It is also clear that the closure of any countable subset in X is compact. It is also worth mentioning that the compactification X of G is not a dyadic compactum since it contains a dense first-countable subspace $Y = X \setminus G$.

The next question remains open.

Problem 3.1. Suppose that G is a non-locally compact topological group with a first-countable remainder. Furthermore, suppose that the Souslin number of G is countable. Does it follow that G is metrizable?

Remark 3.2 (Remark added September 29, 2012)**.** The authors recently constructed under CH an example of a countable topological group which is not metrizable but has a first-countable remainder. Hence, for countable groups, the question of whether the existence of a first-countable remainder is equivalent to being metrizable is undecidable.

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