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WALLMAN COMPACTIFICATIONS AND TYCHONOFF'S COMPACTNESS THEOREM IN ZF

KYRIAKOS KEREMEDIS AND ELEFTHERIOS TACHTSIS

Abstract. We show that the following statements are pairwise equivalent in ZF:

(1) The axiom of choice (AC).
(2) For every T_1 space (X, T) and every T_1 base $\mathcal{C}$ for X, the set $\mathcal{W}(X, \mathcal{C})$ of all $\mathcal{C}$-ultrafilters when endowed with the Wallman topology $T_{\mathcal{W}(X,\mathcal{C})}$ (definitions are provided in section 1) is a compactification of X.
(3) For every T_1 space (X, T), for every T_1 base $\mathcal{C}$ of X, every filter base $\mathcal{G} \subset \mathcal{C}$ extends to a $\mathcal{C}$-ultrafilter $\mathcal{F}$.
(4) "For every T_1 space (X, T), $\mathcal{W}(X)$ (here $\mathcal{C} = \mathcal{K}(X)$, the family of all closed subsets of X) is a compactification of X" and "For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i\in\omega} X_i)$ and $\prod_{i\in\omega} \mathcal{W}(X_i)$ are topologically homeomorphic."

We also show that " For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" implies that every infinite set has a countably infinite subset.

In addition, we show that " For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" if and only if CFE_1 (= every filter base $\mathcal{G}$ of closed subsets of a T_1 space (X, T) extends to a closed ultrafilter $\mathcal{F}$).

1. Notation and Terminology

Let $\{(X_i, T_i) : i \in I\}$ be a family of topological spaces and let $X = \prod_{i\in I} X_i$ be their Tychonoff product. A closed subset F of X is called *basic closed* in case

$$F = \cup\{\pi_q^{-1}(F_q) : q \in Q\} \text{ where } Q \in [I]^{<\omega} \text{ and for all } q \in Q, F_q^c \in T_q.$$

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We shall denote the collection of all basic closed subsets of X by $\mathcal{C}(X)$. $\mathcal{C}(X)$ is a *base* for the closed subsets of X (every closed subset of X can be expressed as an intersection of members of $\mathcal{C}(X)$). A closed set F of X is called *restricted closed* if there is a finite set $Q \subseteq I$ such that $F = V \times \prod_{i \in Q^c} X_i$, with $V \subset Y_Q = \prod_{i \in Q} X_i$ a closed set. Note that for a given restricted closed set F in X, there is not a unique set Q satisfying the aforementioned condition. (Let F, V, and Q be as in the latter definition, and (for our convenience) suppose that $|I \setminus Q| \geq 2$ and let $i_0 \in I \setminus Q$. Then $F = V' \times \prod_{i \in (Q')^c} X_i$, where $Q' = Q \cup \{i_0\}$ and $V' = V \times X_{i_0}$). However, it is fairly easy to see that to every nonempty proper restricted closed subset F of X, there corresponds a finite set $Q_F \subseteq I$ which is the $\subseteq$-smallest set Q in $[I]^{<\omega}$ with respect to the above condition on Q. (If F is a nonempty (proper) restricted closed set in X, then the $\subseteq$-smallest set $Q_F \subseteq I$ can be defined so that there is a closed set V_F in Y_{Q_F} such that $F = V_F \times \prod_{i \in (Q_F)^c} X_i$, $V_F = \cap\{Z_j : j \in J\}$, where $Z_j = \cup\{\pi_q^{-1}(G_{qj}) : q \in Q_F\} \subset Y_{Q_F}$ and for all $q \in Q_F$, for all $j \in J$, $G_{qj}^c \in T_q$, and *for each $q \in Q_F$, there exists an index $j \in J$ such that* $G_{qj} \neq \varnothing, X_q$. Then any $Q \in [I]^{<\omega}$ for which $F = V \times \prod_{i \in (Q)^c} X_i$ with V closed in Y_Q is such that $Q \supseteq Q_F$; recall that V is expressible as an intersection of basic closed subsets $Z'_j = \cup\{\pi_q^{-1}(G'_{qj}) : q \in Q\}$, $j \in J'$, of Y_Q. Then discard those $q \in Q$, each being such that for every $j \in J'$, G'_{qj} is either empty or all of X_q.) The set Q_F is called the set of *restricted coordinates of* F. The collection of all restricted closed subsets of X shall be denoted by $\mathcal{C}_R(X)$.

Clearly, $\mathcal{C}(X)$ is closed under finite unions but *not* closed under finite intersections. However, the collection $\mathbf{C}(X)$ consisting in all sets of the form

$$G = \cap\{\cup\{\pi_q^{-1}(F_{qi}) : q \in Q_i\} : i \in n\} \tag{1.1}$$

where $n \in \mathbb{N}$ and for all $i \in n$, $Q_i \in [I]^{<\omega}$, and for all $q \in Q_i$, $F_{qi}^c \in T_q$, or, equivalently,

$$G = \cup\{\cap\{f(i) : i \in n\} : f \in \prod_{i \in n}\{\pi_q^{-1}(F_{qi}) : q \in Q_i\}\}, \tag{1.2}$$

is easily seen to be a base for the closed subsets of X, closed under finite unions *and* finite intersections. Likewise, $\mathcal{C}_R(X)$ is a larger base for the closed subsets of X which is also closed under finite unions and finite intersections.

For every $S \subset I$, $p_S : X \to Y_S = \prod_{i \in S} X_i$ will denote the projection of X onto Y_S. In particular, if $S = \{i\}$ for some $i \in I$, then we will denote the projection p_S by π_i. $\mathcal{C}_1(X)$ denotes the collection of all 1-*basic closed* subsets of X, i.e., $\mathcal{C}_1(X) = \{\pi_i^{-1}(F) : i \in I, F^c \in T_i\}$.

Let (X, T) be a topological space. X is *compact* if every open cover $\mathcal{U}$ of X has a finite subcover $\mathcal{V}$. Equivalently, X is compact if and only if for every family $\mathcal{G}$ of closed subsets of X having the finite intersection property (fip) $\bigcap \mathcal{G} \neq \varnothing$.

Let X be a nonempty set and let $\mathcal{E}$ be a collection of subsets of X which is closed under finite intersections. A nonempty subcollection $\mathcal{F}$ of $\mathcal{E} \setminus \{\varnothing\}$ is an $\mathcal{E}$-*filter* if and only if

(i) if F_1, $F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

(ii) if $F \in \mathcal{F}$, $F' \in \mathcal{E}$ and $F \subseteq F'$, then $F' \in \mathcal{F}$.

If $\mathcal{E} = \mathcal{P}(X)$, then an $\mathcal{E}$-filter is called a *filter* on X. If $\mathcal{E}$ is the collection of all closed subsets of a topological space, then we say that $\mathcal{F}$ is a *closed filter*. A nonempty collection $\mathcal{H} \subseteq \mathcal{E} \setminus \{\varnothing\}$ is called an $\mathcal{E}$-*filter base* if, for every $H_1, H_2 \in \mathcal{H}$, there is an $H_3 \in \mathcal{H}$ such that $H_3 \subseteq H_1 \cap H_2$. A filter base $\mathcal{H} \subseteq \mathcal{E} \setminus \{\varnothing\}$ is called *free* if $\bigcap \mathcal{H} = \varnothing$.

A maximal, with respect to inclusion, $\mathcal{E}$-filter is called an $\mathcal{E}$-*ultrafilter*.

An $\mathcal{E}$-filter $\mathcal{F}$ is called *countably closed* if it satisfies the condition: If $\{F_i : i \in \omega\} \subseteq \mathcal{F}$, then $\cap\{F_i : i \in \omega\} \in \mathcal{F}$.

An $\mathcal{E}$-filter $\mathcal{F}$ is called *countably prime* if it satisfies the condition: If $\{F_i : i \in \omega\} \subset \mathcal{E}$ and $\cup\{F_i : i \in \omega\} \in \mathcal{F}$, then there exists $i \in \omega$ such that $F_i \in \mathcal{F}$.

Let (X, T) be a T_1 space and let $\mathcal{B}$ be a base for the closed subsets of X. Then $\mathcal{B}$ is called a T_1 *base* for X if it satisfies the following.

(i) $\varnothing \in \mathcal{B}$.

(ii) If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2 \in \mathcal{B}$ and $B_1 \cup B_2 \in \mathcal{B}$.

(iii) If $x \notin B \in \mathcal{B}$, then there is $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \cap B = \varnothing$.

If $\{(X_i, T_i) : i \in I\}$ is a family of T_1 topological spaces, then $\mathcal{C}_R(X)$ and $\mathbf{C}(X)$ are T_1 bases for the product $X = \prod_{i \in I} X_i$. Note that $\mathcal{C}(X)$ is not a T_1 base for X.

If $X = \prod_{i \in I} X_i$ is a product of topological spaces and $\mathcal{F}$ is an $\mathcal{E}$-ultrafilter, where $\mathcal{C}(X) \subset \mathcal{E} \subseteq \mathcal{K}(X)$ (= the family of all closed subsets of X), then for every $i \in I$, $\mathcal{F}_i$ denotes the family of all closed subsets of X_i whose inverse image under π_i is a member of $\mathcal{F}$, i.e.,

$$\mathcal{F}_i = \{F \subseteq X_i : F^c \in T_i, \pi_i^{-1}(F) \in \mathcal{F}\}. \tag{1.3}$$

Let (X, T) be a T_1 topological space and let $\mathcal{C}$ be a T_1 base for X. $\mathcal{W}(X, \mathcal{C})$ denotes the set $\{\mathcal{F} \subset \mathcal{C} : \mathcal{F}$ is a $\mathcal{C}$-ultrafilter$\}$. In particular, for $\mathcal{C} = \mathcal{K}(X)$, we shall denote $\mathcal{W}(X, \mathcal{K}(X))$ by $\mathcal{W}(X)$.

The topology $T_{\mathcal{W}(X,\mathcal{C})}$ on $\mathcal{W}(X, \mathcal{C})$, having as a base for the closed sets the family

$$\mathcal{B} = \{A^* : A \in \mathcal{C}\},\ A^* = \{\mathcal{F} \in \mathcal{W}(X, \mathcal{C}) : A \in \mathcal{F}\},$$

is called the *Wallman topology corresponding to the base* $\mathcal{C}$. Since for every $A \in \mathcal{C}$, $(A^*)^c = \{\mathcal{F} \in \mathcal{W}(X, \mathcal{C}) : \exists F \in \mathcal{F}$ such that $F \cap A = \varnothing\}$, it follows that

$$\mathcal{U} = \{U^* : U^c \in \mathcal{C}\} \text{ and} \tag{1.4}$$
$$U^* = \{\mathcal{F} \in \mathcal{W}(X, \mathcal{C}) : \exists\, F \in \mathcal{F} \text{ such that } F \subseteq U\}$$

is a base for $\mathcal{W}(X, \mathcal{C})$. In the sequel, $\mathcal{W}(X, \mathcal{C})$ will denote the topological space $(\mathcal{W}(X, \mathcal{C}), T_{\mathcal{W}(X,\mathcal{C})})$. We shall be referring to $\mathcal{W}(X, \mathcal{C})$ as the *Wallman space of* X *corresponding to the base* $\mathcal{C}$.

A T_1-*compactification* of X is a compact T_1 space (Y, Q) such that X embeds in Y as a dense subspace. We recall (See [14, Theorem IV.3, p. 138]) that, in ZFC, a *Wallman compactification* of (X, T) is the topological space $\mathcal{W}(X, \mathcal{C})$.

If (X, T) and (Y, P) are topological spaces and $f : X \to Y$ is a $1 : 1$, onto continuous and open mapping, then we will write $X \simeq Y$ and say that X and Y are *homeomorphic*.

BPI: Every Boolean algebra has a prime ideal. (It is known that BPI is equivalent in ZF to each one of the statements: "The Tychonoff product of compact T_2 spaces is compact" (see [15]) and "2^X is compact for every infinite set X" (see [13]).)

CFE (Axiom of Closed Filter Extendability): For every topological space (X, T), every filter base $\mathcal{G}$ of closed subsets of X extends to a closed ultrafilter $\mathcal{F}$.

CFE_1: CFE restricted to T_1 spaces.

TCT (Tychonoff Compactness Theorem): For every family $\{(X_i, T_i) : i \in I\}$ of compact topological spaces, the Tychonoff product $X = \prod_{i \in I} X_i$ is compact. (It is known that TCT if and only if TCT restricted to T_1 spaces (see [8]).)

$\mathsf{AC}(I)$: For every family $\mathcal{A} = \{A_i : i \in I\}$ of nonempty sets there exists a function $f : \mathcal{A} \to \bigcup \mathcal{A}$ such that $f(A_i) \in A_i$ for all $i \in I$.

AC (Axiom of Choice): For every set I, $\mathsf{AC}(I)$.

$\mathsf{AC}_{\mathrm{fin}}$: AC restricted to families of nonempty finite sets.

CAC (Countable Axiom of Choice): $\mathsf{AC}(\omega)$. It is known (see [7]) that CAC is equivalent to its partial version PCAC; i.e., every countably infinite family of nonempty sets has an infinite subfamily with a choice function.

2. **Introduction and Some Known Results**

If (X, T) is a topological space, then one cannot prove the existence of a closed ultrafilter of X in ZF. In fact, Horst Herrlich [3] has shown the following.

Theorem 2.1. *Every topological space has a closed ultrafilter if and only if* AC.

However, if (X, T) is a T_1 space, then one can always prove in ZF the existence of a closed ultrafilter. Indeed, for every $x \in X$,

$$\mathcal{F}(x) = \{F \subseteq X : x \in F, F^c \in T\} \tag{2.1}$$

is easily seen to be a closed ultrafilter of X. More generally, if $\mathcal{C}$ is a T_1 base for X, then $\mathcal{F}(x) = \{F \in \mathcal{C} : x \in F\}$ is a $\mathcal{C}$-ultrafilter. (This follows from condition (iii) in the definition of a T_1 base; see section 1.) It follows that we can rephrase CFE_1 as "every free filter base $\mathcal{G}$ of closed subsets of a T_1 space (X, T) extends to a closed ultrafilter $\mathcal{F}$" and "For every T_1 space (X, T), for every T_1 base $\mathcal{C}$ of X, every filter base $\mathcal{G} \subset \mathcal{C}$ extends to a $\mathcal{C}$-ultrafilter $\mathcal{F}$" as " for every T_1 space (X, T) for every T_1 base $\mathcal{C}$ of X, every free filter base $\mathcal{G} \subset \mathcal{C}$ extends to a $\mathcal{C}$-ultrafilter $\mathcal{F}$."

CFE was introduced in [6] where Herrlich and Juris Steprāns established the following.

Theorem 2.2. AC *if and only if* CFE + CAC.

They asked whether CFE implies CAC . In [10, Theorem 2], it was shown that the answer to this question is in the affirmative. If the readers go through the proof of Theorem 2.2 as is given in [6, Proposition 2, p. 700], they will realize that they can use the same proof in order to establish the following.

Theorem 2.3. AC *if and only if* CFE_1 + CAC.

Unlike the case of Theorem 2.2, one cannot use the proof of Theorem 2 in [10] to show that CFE_1 implies CAC. The space involved in that proof satisfies the T_0 axiom but not the T_1. In fact, this is an intriguing open problem we shall be concerned with in this paper.

Theorem 2.4. (i) AC *implies "For every T_1 space (X, T) and every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X, \mathcal{C})$ is a compactification of X" implies "For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X."*

(ii) *For every T_1 space (X, T) and for every T_1 base $\mathcal{C}$ of X, every filter base $\mathcal{G} \subset \mathcal{C}$ extends to a $\mathcal{C}$-ultrafilter $\mathcal{F}$ if and only if for every T_1 space (X, T) and every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X, \mathcal{C})$ is a compactification of X.*

(iii) CFE_1 *if and only if "For every* T_1 *space* (X,T), $\mathcal{W}(X)$ *is a compactification of* X*."*

(iv) CFE_1 *implies* BPI *implies* AC_{fin}.

Proof. (i) For the first implication, see the proof of Theorem IV.3, p. 138, in [14]. The proof of the second implication is straightforward.

(ii) ($\rightarrow$) Let (X,T) be a T_1 space and let $\mathcal{C}$ be a T_1 base for X. To prove that $\mathcal{W}(X,\mathcal{C})$ is compact, we start with a family $\mathcal{G}=\{G_i^*: i\in I\}$, where for all $i\in I$, $G_i\in\mathcal{C}$, having the fip. Then $\mathcal{H}=\{G_i: i\in I\}\subset\mathcal{C}$ is a family with the fip. By our hypothesis, $\mathcal{H}$ extends to a $\mathcal{C}$-ultrafilter $\mathcal{F}$. Clearly, $\mathcal{F}\in\cap\{\mathcal{G}_i^*: i\in I\}\neq\varnothing$, and consequently $\mathcal{W}(X,\mathcal{C})$ is compact.

In addition to being compact, see [14, Theorem IV.3, p. 138], $\mathcal{W}(X,\mathcal{C})$ is T_1 and the mapping

$$\varphi: X\rightarrow\mathcal{W}(X,\mathcal{C}),\ \varphi(x)=\mathcal{F}(x), \tag{2.2}$$

where $\mathcal{F}(x)=\{F\in\mathcal{C}: x\in F\}$, is a topological embedding such that $\overline{\varphi(X)}=\mathcal{W}(X,\mathcal{C})$. Thus, $\mathcal{W}(X,\mathcal{C})$ is a T_1 compactification of X.

(ii) ($\leftarrow$) Let (X,T) be a T_1 space, $\mathcal{C}$ a T_1 base for X, and $\mathcal{H}$ a free filter base of $\mathcal{C}$. Clearly, $\{H^*: H\in\mathcal{H}\}$ is a family of closed subsets of $\mathcal{W}(X,C)$ having the fip. Hence, by our hypothesis, $K=\cap\{H^*: H\in\mathcal{H}\}\neq\varnothing$. Clearly, any $\mathcal{F}\in K$ extends $\mathcal{H}$.

(iii) Follow the proof of (i) with $\mathcal{K}(X)$ in place of $\mathcal{C}$.

(iv) The first implication is straightforward and the second is well known; see [7]. $\square$

At this point, one may ask the following questions.

Question 2.5. Does " For every T_1 space (X,T), $\mathcal{W}(X)$ is a compactification of X" imply AC ?

Question 2.6. Does " For every T_1 space (X,T) and every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X,\mathcal{C})$ is a compactification of X" imply AC ?

Question 2.7. Does " For every T_1 space (X,T), $\mathcal{W}(X)$ is a compactification of X" imply " For every T_1 space (X,T) and every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X,\mathcal{C})$ is a compactification of X" ?

Taking into consideration Theorem 2.4 and Theorem 2.3, Question 2.5 reduces to the one introduced after the statement of Theorem 2.3. Namely,

Question 2.8. Does CFE_1 imply CAC ?

With regard to Question 2.6, we prove in Theorem 4.1 that the answer is in the affirmative. Hence, Question 2.7 is simply a rewording of Question 2.5.

Regarding Question 2.8, the best we could get is stated in Theorem 4.3. Namely, CFE_1 implies *every infinite set has a countably infinite subset.* We conjecture that the answer to Question 2.8 is in the negative.

Before we set out proving our results, let us list a few results which we are going to use in the sequel.

Proposition 2.9 ([9]). *Let $\{(X_i, T_i) : i \in I\}$ be a family of topological spaces and let $X = \prod_{i \in I} X_i$ be their product. Then the following hold.*

(i) *A closed subset F of X is restricted closed if and only if there exists a finite set $Q \subseteq I$ such that $F = \cap\{\cup\{\pi_q^{-1}(F_{jq}) : q \in Q\} : j \in J\}$, where for $j \in J$ and $q \in Q$, $F_{jq}^c \in T_q$.*

(ii) *The union (intersection) of finitely many restricted closed sets is a restricted closed set. In particular, if each X_i is T_1, then $\mathcal{C}_R(X)$ is a T_1 base for X.*

(iii) *Every closed subset of X can be expressed as an intersection of fewer than $|[I]^{<\omega}|$-many restricted closed sets.*

Proposition 2.10 (ZF). (i) *The product of finitely many compact spaces is compact.*

(ii) *Let (X, T) be a topological space and let (Y, Q) be a compact space. Then the projection $\pi_X : X \times Y \to X$ of $X \times Y$ onto X is a closed map.*

(iii) *Let (X, T) be a topological space and $\mathcal{B}$ a base for X. Then X is compact if and only if every open cover $\mathcal{U} \subset \mathcal{B}$ has a finite subcover if and only if every family $\mathcal{H} \subset \{B^c : B \in \mathcal{B}\}$ having the fip has a nonempty intersection.*

Proof. The proof of (i) is well known and it is demonstrated in any standard textbook, such as [12] and [18]. For the proof of (ii), see [12, Exercise 8, p. 172]. Finally, the proof of (iii) is well known and straightforward. □

Proposition 2.11. (i) AC *if and only if* TCT.

(ii) TCT *restricted to countable families of compact topological spaces implies* CAC.

Proof. (i) $\mathsf{TCT} \to \mathsf{AC}$ has been proved in [8]. There is a plethora of proofs of the implication $\mathsf{AC} \to \mathsf{TCT}$; see [9, Theorem 2.1] for a recent one. In fact, in every book of general topology there is such a proof. The following proof is the shortest one attributing the necessary appreciation to CFE_1.

Fix $\{(X_i, T_i) : i \in I\}$ a family of compact T_1 spaces. We show that $X = \prod_{i \in I} X_i$ is compact. Fix, by Proposition 2.10(iii),

$$\mathcal{H} = \{\cup\{\pi_q^{-1}(F_{jq}) : q \in Q_j\} : j \in J\} \subset \mathcal{C}(X), \tag{2.3}$$

where for all $j \in J$, $Q_j \in [I]^{<\omega}$, and for all $j \in J$ and $q \in Q_j$, $F_{jq}^c \in T_q$, a family with the fip and, by AC, let $\mathcal{F}$ be a closed ultrafilter extending $\mathcal{H}$.

For every $i \in I$, let $A_i = \bigcap \mathcal{F}_i$ where $\mathcal{F}_i$ is given by (1.3). Since $X_i \in \mathcal{F}_i$ and X_i is compact, it follows that $A_i \neq \varnothing$. By AC, fix $a_i \in A_i$, $i \in I$, and let $a \in X$ satisfy $a(i) = a_i$ for all $i \in I$. Since $\mathcal{F}$ is maximal, it follows that, for every $j \in J$, there is $q_j \in Q_j$ with $\pi_{q_j}^{-1}(F_{jq}) \in \mathcal{F}$. Hence, for every $j \in J$, $a \in \pi_{q_j}^{-1}(F_{jq})$, and consequently $a \in \bigcap \mathcal{H}$ and X is compact as required.

(ii) Use the proof of TCT $\to$ AC given in [8]. □

Remark 2.12. Regarding Question 2.8, we may conclude from the proof of Proposition 2.11 that if CFE_1 implies for every $i \in I$, $\mathcal{F}_i$ is a closed ultrafilter, as is the case when $\mathcal{F}$ is a $\mathbf{C}(X)$-ultrafilter or a $\mathcal{C}_R(X)$-ultrafilter (see [9, Theorem 3.2]), then CFE_1 is equivalent to AC.

In the spirit of Proposition 2.11 we could not resist giving a very simple and notably short proof of the following characterization of the Boolean Prime Ideal Theorem (BPI) provided by Eric Schechter [16].

Proposition 2.13. BPI *if and only if the Tychonoff product of spaces each endowed with the cofinite topology is compact.*

Proof. ($\to$) Let $\{(X_i, T_i) : i \in I\}$ be a family of spaces such that T_i is the cofinite topology on X_i for every $i \in I$. Let X be its Tychonoff product. If $X = \emptyset$, then there is nothing to show. So let $x = (x_i)_{i \in I} \in X$. Let $\mathcal{H}$ be given by (2.3), and, by BPI, let $\mathcal{F}$ be an ultrafilter of X including $\mathcal{H}$. For every $i \in I$, let $A_i = \bigcap \mathcal{F}_i$, where $\mathcal{F}_i$ is given by (1.3). Clearly, $A_i \neq \varnothing$ for all $i \in I$. Furthermore, as A_i is closed in X_i and T_i is the cofinite topology, we have that either $A_i = X_i$ or A_i is finite. In the second case, it easily follows from the maximality of $\mathcal{F}$ that A_i is a singleton, say $A_i = \{a_i\}$. Now define a function $f \in X$ by requiring $f(i) = x_i$ if $A_i = X_i$, and $f(i) = a_i$ otherwise. As in the proof of Proposition 2.11, one may verify that $f \in \bigcap \mathcal{H} \neq \varnothing$. Thus, X is compact as required.

($\leftarrow$) The hypothesis readily implies the equivalent form of BPI "for every infinite set X, 2^X is compact." □

3. Some Preliminary Results

Proposition 3.1. *Let (X, T) be a T_1 space and let $\mathcal{C}$ be a T_1 base for X. If X has no free $\mathcal{C}$-ultrafilters, then $X \simeq \mathcal{W}(X, \mathcal{C})$. In particular, if X is compact, then $X \simeq \mathcal{W}(X, \mathcal{C}) \simeq \mathcal{W}(X)$.*

Proof. The proof follows at once from the observation that $\mathcal{W}(X, \mathcal{C}) = \{\mathcal{F}(x) : x \in X\}$, where $\mathcal{F}(x) = \{F \in \mathcal{C} : x \in F\}$, and the proof of Theorem IV.3, p. 138, in [14]. (The function $\varphi : X \to \mathcal{W}(X, \mathcal{C})$, $\varphi(x) = \mathcal{F}(x)$, is an onto topological embedding.) □

Remark 3.2. (i) If T is the discrete topology on the infinite set X and $X \simeq \mathcal{W}(X)$, then X has no free ultrafilters. (If $h : X \to \mathcal{W}(X)$ is a homeomorphism and $\mathcal{F}$ is a free ultrafilter of X, then $\mathcal{F} = h(x)$ for some $x \in X$. Since $\{x\} \in T$, it follows that $\{\mathcal{F}\}$ is open in $\mathcal{W}(X)$. Hence, there exists $A \subset X$ with $(A^*)^c = \{\mathcal{F}\}$. Clearly, $A \neq X$, so let $y \in A^c$. Since $\mathcal{F}$ is free, $\{y\} \notin \mathcal{F}$. Consider the ultrafilter $\mathcal{G} = \{G \subset X : y \in G\}$ of X. Since $\{y\} \cap A = \varnothing$, $\mathcal{G} \in (A^*)^c$ and $\mathcal{G} \neq \mathcal{F}$. This is a contradiction; hence, X has no free ultrafilters.)

(ii) Let (X, T) be a T_1 space and $h : X \to \mathcal{W}(X)$ be a homeomorphism. If $\mathcal{F}$ is a free closed ultrafilter of X, then X has a countably infinite subset. Indeed, $\mathcal{F} = h(x_1)$ for some unique $x_1 \in X$ and by a straightforward induction, there exists (unique) $x_{n+1} \in X \backslash \{x_1, ..., x_n\}$ such that $h(x_{n+1}) = \mathcal{F}_{x_n} = \{F \subset X : F^c \in T \text{ and } x_n \in F\}$.

Clearly, in products $X = \prod_{i \in I} X_i$, projections of members of $\mathcal{C}(X)$ are closed sets. We show next, as expected, that the members of $\mathbf{C}(X)$ share the same property. This property of the base $\mathbf{C}(X)$ turns out to be very useful in the sequel.

Proposition 3.3. *Let $\{(X_i, T_i) : i \in I\}$ be a family of topological spaces and let $X = \prod_{i \in I} X_i$ be their Tychonoff product. For every $F \in \mathbf{C}(X)$ and for every $i \in I$, $\pi_i(F)$ is a closed set.*

Proof. Fix $F \in \mathbf{C}(X)$. By (1.2), express F as

$$F = \cup\{\cap\{f(v) : v \leq n\} : f \in Y = \prod_{v \leq n} \{\pi_q^{-1}(F_{qv}) : q \in Q_v\}\}.$$

Since $|Y| < \aleph_0$ and for all $f \in Y$, $\pi_i(\cap\{f(v) : v \leq n\})$ is a closed set, it follows that $\pi_i(F)$ is a closed set as required. □

Proposition 3.4. *Let $\{(X_i, T_i) : i \in I\}$ be a family of topological spaces with Tychonoff product $X \neq \varnothing$ and let $\mathcal{F}$ and $\mathcal{H}$ be $\mathbf{C}(X)$-ultrafilters. Then*

(i) *$\mathbf{F} = \mathcal{F} \cap \mathcal{C}_1(X)$ is a maximal subfamily of $\mathcal{C}_1(X)$ with the fip, and for every $i \in I$, $\mathcal{F}_i = \pi_i(\mathbf{F}) = \{\pi_i(K) : K \in \mathbf{F}\}$ is a closed ultrafilter, where $\mathcal{F}_i$ is given by (1.3). Furthermore, the $\mathbf{C}(X)$-filter $\mathcal{G}$ generated by $\mathbf{F}$ coincides with $\mathcal{F}$. In particular, if $\mathcal{F} \neq \mathcal{H}$, then there exists $i \in I$ with $\mathcal{F}_i \neq \mathcal{H}_i$, where for all $i \in I$, $\mathcal{F}_i$ and $\mathcal{H}_i$ are given by (1.3).*

(ii) *If, for every $i \in I$, $\mathcal{F}_i$ is a closed ultrafilter of X_i, then the $\mathbf{C}(X)$-filter $\mathcal{F}$ generated by $\{\pi_i^{-1}(F) : i \in I, F \in \mathcal{F}_i\}$ is a $\mathbf{C}(X)$-ultrafilter.*

Proof. (i) $\mathbf{F}$ is maximal with the fip and $\pi_i(\mathbf{F}) = \{\pi_i(K) : K \in \mathbf{F}\}$ is a closed ultrafilter follow at once from [9, Theorem 3.2(i)]. (Note that the compactness of the spaces (X_i, T_i) is not used in Theorem 3.2(i); $\mathbf{F} = (\mathcal{F} \cap \mathcal{C}(X)) \cap \mathcal{C}_1(X))$ and $\mathcal{F} \cap \mathcal{C}(X)$ is a maximal subfamily of $\mathcal{C}(X)$ with the fip.) $\mathcal{F}_i = \pi_i(\mathbf{F})$ follows from Proposition 3.3 and the fact that $\mathcal{F}$ is a $\mathbf{C}(X)$-filter.

Regarding the equality $\mathcal{F} = \mathcal{G}$, the inclusion $\mathcal{G} \subseteq \mathcal{F}$ is obvious. To see $\mathcal{F} \subseteq \mathcal{G}$, fix $F \in \mathcal{F}$ and, by (1.1), express F as

$$F = \cap\{Z_i : i \in n\}, \tag{3.1}$$

where $n \in \mathbb{N}$; for all $i \in n$, $Z_i = \cup\{\pi_q^{-1}(F_{qi}) : q \in Q_i\}$, $Q_i \in [I]^{<\omega}$; and for all $q \in Q_i$, $F_{qi}^c \in T_q$. By the maximality of $\mathcal{F}$, fix, for every $i \in n$, a $q_i \in Q_i$ with $\pi_{q_i}^{-1}(F_{qi}) \in \mathcal{F}$ and let $G = \cap\{\pi_{q_i}^{-1}(F_{qi}) : i \in n\}$. As $F \supseteq G \in \mathcal{G}$, we see that $F \in \mathcal{G}$ and $\mathcal{F} \subseteq \mathcal{G}$ as required.

(ii) Fix $F \in \mathbf{C}(X)$ such that $\{F\} \cup \mathcal{F}$ has the fip. We show that $F \in \mathcal{F}$. Express F in the form of (3.1). Clearly, for every $i \in n$, $\{Z_i\} \cup \mathcal{F}$ has the fip. Hence, for every $i \in n$, there is a $q_i \in Q_i$ such that $\pi_{q_i}^{-1}(F_{qi}) \in \mathcal{F}_i$. (If not, then there is an $i \in n$ such that for every $q \in Q_i$, there exists, by the maximality of $\mathcal{F}_i$, a closed subset B_q of X_q such that $B_q \in \mathcal{F}_q$ and $\pi_q^{-1}(B_q) \cap \pi_q^{-1}(F_{qi}) = \varnothing$. Hence, $K \cap Z_i = \varnothing, K = \cap\{\pi_q^{-1}(B_q) : q \in Q_i\} \in \mathcal{F}$, contradicting the fip of $\{Z_i\} \cup \mathcal{F}$). Since $F \supseteq \cap\{\pi_{q_i}^{-1}(F_{qi}) : i \in n\} \in \mathcal{F}$, it follows that $F \in \mathcal{F}$ as required. □

In the next proposition we extract the essence of the proof of Proposition 2.11(i) in order to use it in the rest of the paper.

Proposition 3.5. (i) *Let $\{(X_i, T_i) : i \in I\}$ be a family of topological spaces with Tychonoff product X. Assume that $\mathcal{F}$ is an $\mathcal{E}$-ultrafilter, where $\mathcal{C}(X) \subset \mathcal{E} \subseteq \mathcal{K}(X)$. Then $\prod_{i \in I}(\bigcap \mathcal{F}_i) \subseteq \bigcap \mathcal{F}$, where $\mathcal{F}_i$ is given by (1.3). Hence, if $\prod_{i \in I}(\bigcap \mathcal{F}_i) \neq \varnothing$, then $\mathcal{F}$ is not free. In particular, if each X_i is compact and T_1, then X has no free $\mathbf{C}(X)$-ultrafilters. Hence,* TCT *if and only if "For every non-compact T_1 space (X, T) and for every T_1 base $\mathcal{C}$ of X, there is a free $\mathcal{C}$-ultrafilter $\mathcal{F}$."*

(ii) *Every non-compact T_1 topological space has a free closed ultrafilter implies* BPI.

Proof. (i) Let $x = (x_i)_{i \in I} \in \prod_{i \in I}(\bigcap \mathcal{F}_i)$. We show that $x \in \bigcap \mathcal{F}$. To this end, fix $F \in \mathcal{F}$. Since $\mathcal{C}(X)$ is a base for the closed sets of X, we express F as

$$F = \cap\{Z_j : j \in J\},$$

where for all $j \in J$, $Z_j = \cup\{\pi_q^{-1}(F_{qj}) : q \in Q_j\}$, $Q_j \in [I]^{<\omega}$, and for all $q \in Q_j$, $F_{qj}^c \in T_q$. Since $\mathcal{F}$ is a filter and $F \in \mathcal{F}$, it follows that for every $j \in J$, $Z_j \in \mathcal{F}$.

We shall show that $x \in Z_j$ for all $j \in I$; hence, $x \in F$. To this end, fix an index $j_0 \in J$. As Q_{j_0} is finite and $\mathcal{F}$ is an $\mathcal{E}$-ultrafilter, it follows that there is a $q_* \in Q_{j_0}$ such that $\pi_{q_*}^{-1}(F_{q_* j_0}) \in \mathcal{F}$. Then

$$F_{q_* j_0} = \pi_{q_*}(\pi_{q_*}^{-1}(F_{q_* j_0})) \in \mathcal{F}_{q_*}.$$

Thus, $\bigcap \mathcal{F}_{q_*} \subseteq F_{q_* j_0}$ and since $x(q_*) \in \bigcap \mathcal{F}_{q_*}$, we have that $x(q_*) \in F_{q_* j_0}$. Hence,

$$x \in \pi_{q_*}^{-1}(F_{q_* j_0}) \subset \cup\{\pi_q^{-1}(F_{q j_0}) : q \in Q_{j_0}, F_{q j_0}^c \in T_q\} = Z_{j_0}.$$

Thus, $x \in Z_{j_0}$ and since j_0 was arbitrary, it follows that $x \in \cap\{Z_j : j \in J\} = F$. Therefore, $x \in \bigcap \mathcal{F}$ as required.

To see the last assertion, fix $\mathcal{F}$ a $\mathbf{C}(X)$-ultrafilter. Then, by Proposition 3.4, each $\mathcal{F}_i$ is a closed ultrafilter of X_i. For all $i \in I$, let x_i be the unique element of $\cap \mathcal{F}_i$. (Recall that X_i is compact. Thus, all closed ultrafilters of X_i are not free and singletons are closed since X_i is a T_1 space.) By (i), the element $x \in X$, satisfying for all $i \in I$, $x(i) = x_i$, is in $\bigcap \mathcal{F}$. Hence, $\mathcal{F}$ is not free as required.

(ii) If for some X, 2^X is not compact, then, by our hypothesis, 2^X has a free closed ultrafilter $\mathcal{F}$. As each $\mathcal{F}_i$ is clearly a closed ultrafilter (canonical projections of closed subsets of 2^X are closed), it follows, similarly to the proof of (i), that $\mathcal{F}$ is not free. This is a contradiction showing that 2^X is compact. □

Proposition 3.6. *Let $\{(X_i, T_i) : i \in I\}$ be a family of compact topological spaces with Tychonoff product X.*

(i) *If $\mathcal{F}$ is a $\mathcal{C}_R(X)$-ultrafilter, then $\mathbf{F} = \mathcal{F} \cap \mathbf{C}(X)$ is a $\mathbf{C}(X)$-ultrafilter.*

(ii) *If $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{F} \neq \mathcal{H}$ are $\mathcal{C}_R(X)$-ultrafilters, then $\mathbf{F} \neq \mathbf{H} = \mathcal{H} \cap \mathbf{C}(X)$.*

(iii) *If $\mathbf{F}$ is a $\mathbf{C}(X)$-ultrafilter, then the filter $\mathcal{F}$ generated by the collection $\mathcal{S} = \{\cap \mathcal{F}_Q : Q \in [I]^{<\omega}$ and Q is a set of restricted coordinates of some member of $\mathbf{F}\}$, where $\mathcal{F}_Q = \{F \in \mathbf{F} : Q_F = Q\}$, is a $\mathcal{C}_R(X)$-ultrafilter. (Note that $\cap \mathcal{F}_Q \neq \varnothing$ by the compactness of $\prod_{i \in Q} X_i$ for a finite subset Q of I.)*

Proof. (i) This follows at once from [9, Proposition 3.2(ii)].

(ii) Since $\mathcal{F} \neq \mathcal{H}$, there exist $F \in \mathcal{F}$ and $H \in \mathcal{H}$ with $F \cap H = \varnothing$. By Proposition 2.9, express F and H as

$$F = \cap\{\cup\{\pi_q^{-1}(F_{jq}) : q \in Q_F\} : j \in J\} \text{ and}$$
$$H = \cap\{\cup\{\pi_q^{-1}(F_{vq}) : q \in Q_H\} : v \in V\}, \tag{3.2}$$

where for each $j \in J$ and $q \in Q_F$, $F_{jq}^c \in T_q$, and for each $v \in V$ and $q \in Q_H$, $F_{vq}^c \in T_q$. Let $Q = Q_F \cup Q_H$. By the compactness of $Y_Q = \prod_{q \in Q} X_q$, there exist finite subsets $\mathcal{K}$ of $\{\cup\{\pi_q^{-1}(F_{jq}) : q \in Q_F\} : j \in J\}$ and $\mathcal{W}$ of $\{\cup\{\pi_q^{-1}(F_{vq}) : q \in Q_H\} : v \in V\}$ such that $\cap(\mathcal{K} \cup \mathcal{W}) = (\cap\mathcal{K}) \cup (\cap\mathcal{W}) = \varnothing$. Otherwise, $F \cap H \neq \varnothing$, contradicting the hypothesis $F \cap H = \varnothing$. Since $\cap\mathcal{K} \in \mathbf{F}$ and $\cap\mathcal{W} \in \mathbf{H}$, we see that $\mathbf{F} \neq \mathbf{H}$ as required.

(iii) We note, by the compactness of $Y_Q = \prod_{q \in Q} X_q$ for finite Q, that $\mathcal{S}$ has the fip. Assume that $F \in \mathcal{C}_R(X)$ meets non-trivially each member of $\mathcal{F}$. We show that $F \in \mathcal{F}$. By Proposition 2.9, express F in the form of (3.3). Since $\mathbf{F}$ is a $\mathbf{C}(X)$-ultrafilter, it follows that $\{\cup\{\pi_q^{-1}(F_{jq}) : q \in Q_F\} : j \in J\} \subset \mathcal{F}_{Q_F}$, and consequently $F = \cap\{\cup\{\pi_q^{-1}(F_{jq}) : q \in Q_F\} : j \in J\} \supseteq \cap\mathcal{F}_{Q_F}$. Hence, $F \in \mathcal{F}$ and $\mathcal{F}$ is a $\mathcal{C}_R(X)$-ultrafilter as required. □

4. On the Existence of Wallman Compactifications of $\mathbf{T}_1$ Spaces and the Axiom of Choice

Our first result in this section shows that the existence of Wallman compactifications of T_1 spaces is equivalent to AC.

Theorem 4.1. *In* ZF*, the following statements are pairwise equivalent.*

(i) AC.
(ii) *For every T_1 space (X,T) and for every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X,\mathcal{C})$ is a compactification of X.*
(iii) *For every T_1 space (X,T) and for every T_1 base $\mathcal{C}$ of X, every filter base $\mathcal{G} \subset \mathcal{C}$ extends to a $\mathcal{C}$-ultrafilter $\mathcal{F}$.*
(iv) TCT.
(v) *For every non-compact T_1 space (X,T) and for every T_1 base $\mathcal{C}$ of X, there is a free $\mathcal{C}$-ultrafilter $\mathcal{F}$.*

Proof. The implications (i) $\to$ (ii) $\leftrightarrow$ (iii) are established in Theorem 2.4.

(iii) $\to$ (iv) has been established in [9, Theorem 5.1].

(iv) $\leftrightarrow$ (i) is Proposition 2.11(i) and (iv).

(iv) $\to$ (v) is straightforward.

Finally, by Proposition 3.5(i), we have (v) $\leftrightarrow$ (iv). □

Combining Theorem 2.3 and Theorem 4.1, we get the following straightforward corollary.

Corollary 4.2. *In* ZF*, the following statements are pairwise equivalent.*

(i) AC.

(ii) *For every T_1 space (X, T) and every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X, \mathcal{C})$ is a compactification of X.*
(iii) *"For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" +* CAC*.*
(iv) CFE_1 + CAC*.*
(v) *"Every non-compact T_1 topological space has a free closed ultrafilter" +* CAC*.*

In particular, under CAC*, the statements* CFE_1*, "Every non-compact T_1 topological space has a free closed ultrafilter," "For every T_1 space (X, T) and every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X, \mathcal{C})$ is a compactification of X," and "For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" are pairwise equivalent.*

Proof. We only show that (v) implies (i). Fix $\mathcal{A} = \{A_i : i \in I\}$ a pairwise disjoint family of nonempty sets. For every $i \in I$, let $X_i = A_i \cup \{*_i\}, *_i \notin A_i$ and $T_i = \{\{*_i\}, \varnothing\} \cup \{Z \subseteq X_i : |Z^c| < \aleph_0\}$. Clearly, for every $i \in I$, (X_i, T_i) is a compact T_1 space. Put $X = \prod_{i \in I} X_i$. Follow now the proof of Proposition 2 in [6] in order to verify that canonical projections of closed subsets of X are closed. (We indicate here that this is the only point where CAC is needed in the proof of (v) $\to$ (i).) It follows, by Proposition 3.5 and our hypothesis, that X is compact. Since $\mathcal{B} = \{\pi_i^{-1}(A_i) : i \in I\}$ is a family of closed subsets of X with the fip, it follows that any $f \in \bigcap \mathcal{B}$ is a choice function of $\mathcal{A}$, finishing the proof of the corollary. □

In the next theorem we show that "Every non-compact T_1 topological space has a free closed ultrafilter" implies " every infinite set has a countably infinite subset" which is known to be a consequence of CAC ; see [7].

Theorem 4.3. *"Every non-compact T_1 topological space has a free closed ultrafilter"; hence,* CFE_1*, implies every infinite set has a countably infinite subset.*

Proof. Assume the contrary and let A be an infinite set without countably infinite subsets. For every $i \in \mathbb{N}$, let $Y_i = Z_i \cup \{*\}$, where $Z_i = \{f \in A^i : f$ is $1:1\}$ and $* \notin \cup\{Z_i : i \in \mathbb{N}\}$. Clearly, for each $i \in \mathbb{N}$, (Y_i, Q_i), where

$$Q_i = \{\{*\}, \varnothing\} \cup \{Z \subseteq Y_i : |Z^c| < \aleph_0\},$$

is a compact T_1 space. Put $Y = \prod_{i \in \mathbb{N}} Y_i$. If Y is compact, then $\mathcal{S} = \{\pi_i^{-1}(Z_i) : i \in \mathbb{N}\}$ is a family of closed subsets of Y with the fip; hence, $\bigcap \mathcal{S} \neq \varnothing$ and any element in the latter intersection easily yield an injection from $\mathbb{N}$ into A, which contradicts our assumption. So we may assume that Y is not compact. Furthermore, as A has no countably infinite subsets, any infinite subfamily of $\mathcal{S}$ has empty intersection. By our hypothesis, let $\mathcal{V}$ be a free closed ultrafilter of Y. We assert that canonical projections of elements of $\mathcal{V}$ are closed sets. Assume not. Then there is an $i \in \mathbb{N}$ and a $V \in \mathcal{V}$ such that $\pi_i(V)$ is infinite and $Z_i \backslash \pi_i(V) \neq \varnothing$. Express V as

$$V = \cup\{V_n : n \in \mathbb{N}\}, \quad V_n = \{x \in V : \forall m \geq n,\ x(m) = *\}$$

Note that for every $n \in \mathbb{N}$, V_n is a closed subset of Y. Indeed, let $n \in \mathbb{N}$ and let $x \in V_n^c$. If $x \in V^c$, then V^c is a neighborhood of x avoiding V_n. If $x \in V \backslash V_n$, then there is $m \geq n$ such that $x(m) \in Z_m$. Clearly, $\pi_m^{-1}(Z_m)$ is a neighborhood of x missing V_n.

For every $n \in \mathbb{N}$, $\pi_i(V_n)$ is a closed subset of Y_i. Indeed, fix $n \in \mathbb{N}$. If $i \geq n$, then $\pi_i(V_n) = \{*\}$ which is closed in Y_i. Assume that $i < n$ and let $S = \{1, 2, ..., n-1\}$. Since V_n is closed in Y, it follows that V_n is closed in the subspace $Y_1 \times Y_2 \times \cdots \times Y_{n-1} \times \prod_{j \geq n}\{*\}$ of Y which is topologically homeomorphic to the compact space $W_S = \prod_{j \in S} Y_j$. Thus, the projection $p_S(V_n)$ of V_n onto W_S is a closed set in W_S. As $\prod_{j \in S \backslash \{i\}} Y_j$ is compact, it follows that the projection π_i of $Y_i \times \prod_{j \in S \backslash \{i\}} Y_j$ (which is homeomorphic to W_S) onto Y_i is a closed map (see Proposition 2.10 (ii)). Hence, $\pi_i(p_S(V_n))$ is closed in Y_i. Since $\pi_i(V_n) = \pi_i(p_S(V_n))$, we have that $\pi_i(V_n)$ is closed in Y_i as asserted.

Since for all $n \in \mathbb{N}$, $Z_i \neq \pi_i(V_n)$ and $\pi_i(V_n)$ is closed in Y_i, it follows that $\pi_i(V_n) \cap Z_i$ is finite. Therefore,

$$\pi_i(V) \cap Z_i = \pi_i(\cup\{V_n : n \in \mathbb{N}\}) \cap Z_i = \cup\{\pi_i(V_n) \cap Z_i : n \in \mathbb{N}\}$$

is an infinite subset of Z_i which is expressible as a countable union of finite sets. Since our hypothesis implies $\mathsf{AC_{fin}}$ (see Proposition 3.5 and Theorem 2.4), we have that $\pi_i(V) \cap Z_i$ is a countably infinite subset of Z_i, say $\pi_i(V) \cap Z_i = \{f_n : n \in \mathbb{N}\}$. Clearly, $|\cup\{Ran(f_n) : n \in \mathbb{N}\}| = \aleph_0$, and consequently A has a countably infinite subset, contradicting our assumption on A.

Thus, canonical projections of elements of $\mathcal{V}$ are closed sets and we may follow the proof of Corollary 4.2 in order to verify that Y is compact. This contradicts our assumption that Y is not compact. Therefore, A has a countably infinite subset, finishing the proof of the theorem. □

Remark 4.4. From Theorem 4.3, we conclude that in ZF, BPI implies neither CFE_1 nor "Every non-compact T_1 topological space has a free closed ultrafilter." Indeed, in the basic Cohen model $\mathcal{M}1$ in [7], BPI is true, whereas the set of the added Cohen reals has no countably infinite subsets. Thus, "Every non-compact T_1 topological space has a free closed ultrafilter"; hence, CFE_1 also fails in $\mathcal{M}1$.

5. Wallman Compactifications of Tychonoff Products

In [9, Theorem 5.1] it is shown that the statement "for every family $\{(X_i, T_i) : i \in I\}$ of compact topological spaces, every family $\mathcal{H} \subset \mathcal{C}(X), X = \prod_{i \in I} X_i$, with the fip extends to a maximal family $\mathcal{F} \subset \mathcal{C}(X)$ with the fip" (or, equivalently, "for every family $\{(X_i, T_i) : i \in I\}$ of compact topological spaces, every $\mathbf{C}(X)$-filter $\mathcal{H}$ of $X = \prod_{i \in I} X_i$ extends to a $\mathbf{C}(X)$-ultrafilter") and the proposition "for every family $\{(X_i, T_i) : i \in I\}$ of compact topological spaces, every $\mathcal{C}_R(X)$-filter $\mathcal{H}$ of $X = \prod_{i \in I} X_i$ extends to a $\mathcal{C}_R(X)$-ultrafilter" are both equivalent to TCT. Hence, for every family $\{(X_i, T_i) : i \in I\}$ of compact T_1 spaces, $\mathcal{W}(X, \mathbf{C}(X))$, $X = \prod_{i \in I} X_i$, is compact if and only if $\prod_{i \in I} X_i$ is compact if and only if $\mathcal{W}(X, \mathcal{C}_R(X))$ is compact. It is straightforward to see that TCT implies that for every family $\{(X_i, T_i) : i \in I\}$ of compact T_1 spaces with product X,

$$\mathcal{W}(X, \mathbf{C}(X)) \simeq \mathcal{W}(X, \mathcal{C}_R(X)) \simeq \prod_{i \in I} \mathcal{W}(X_i) \simeq \prod_{i \in I} X_i. \tag{5.1}$$

We show in the forthcoming Theorem 5.2 that for a Tychonoff product $\prod_{i \in I} X_i$ of compact T_1 spaces, (5.1) is actually a ZF-result.

Our first result in this section shows that "For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i \in \omega} X_i) \simeq \prod_{i \in \omega} \mathcal{W}(X_i)$" is a consequence of the countable axiom of choice CAC.

Theorem 5.1. (i) *For any index set I, AC(I) implies "For every family $\{(X_i, T_i) : i \in I\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i \in I} X_i) \simeq \prod_{i \in I} \mathcal{W}(X_i)$."*

(ii) *"For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i \in \omega} X_i) \simeq \prod_{i \in \omega} \mathcal{W}(X_i)$" does not imply "For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" in* ZF.

(iii) CAC + *"In a countable product of compact T_1 spaces, every filter of closed sets extends to a closed ultrafilter" if and only* TCT *restricted to countable families of compact topological spaces.*

(iv) *"For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" + "For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i \in \omega} X_i) \simeq \prod_{i \in \omega} \mathcal{W}(X_i)$" can be added to the list of Corollary 4.2.*

Proof. (i) Fix a set I and let $\{(X_i, T_i) : i \in I\}$ be a family of compact T_1 spaces with product $X = \prod_{i \in I} X_i$. Let $\mathcal{F}$ be a closed ultrafilter of X. For every $i \in I$, let $A_i = \bigcap \mathcal{F}_i$ where $\mathcal{F}_i$ is given by (1.3). Since X_i is compact, it follows that $A_i \neq \varnothing$. Put $\mathcal{A} = \{A_i : i \in I\}$ and, by $\mathsf{AC}(I)$, let x be a choice function of $\mathcal{A}$. By Proposition 3.5, $x \in \bigcap \mathcal{F}$, and consequently $\mathcal{F}$ is not free. Hence, X has no free closed ultrafilters and, by Proposition 3.1, $\mathcal{W}(X) \simeq X$.

Since for every $i \in I$, X_i is compact, the function $\varphi_i : X_i \to \mathcal{W}(X_i)$, $\varphi_i(x) = \mathcal{F}(x)$ where $\mathcal{F}(X)$ is given by (2.6), is a homeomorphism. Thus, $X = \prod_{i\in\omega} X_i \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$ under the mapping $\phi : \prod_{i\in\omega} X_i \to \prod_{i\in\omega} \mathcal{W}(X_i)$, $\phi(x)(i) = \varphi_i(x(i))$. Hence, $\mathcal{W}(X) \simeq X = \prod_{i\in I} X_i \simeq \prod_{i\in I} \mathcal{W}(X_i)$ as required.

(ii) It is known that in Model $\mathcal{M}$47(n,M) in [7], there exists a family $\{(X_i, T_i) : i \in I\}$ of compact T_1 spaces whose product X is not compact (it is known that BPI, and hence by Theorem 2.4 "For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" also, fails in $\mathcal{M}$47(n,M)) but CAC holds. However, by (i), "For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i\in\omega} X_i) \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$" holds in $\mathcal{M}$47(n,M).

(iii) It suffices to show ($\to$) as the other implication is straightforward. Let $\{(X_i, T_i) : i \in \omega\}$ and X be as in (i). Let $\mathcal{G}$ be a family of closed subsets of X with the fip. Let $\mathcal{F}$ be a closed ultrafilter which includes $\mathcal{G}$. As in the proof of (i), we may show that $\cap\mathcal{F} \neq \varnothing$; hence, $\cap\mathcal{G} \neq \varnothing$ and X is compact as required.

(iv) First note that our hypothesis implies TCT restricted to countable families of compact topological spaces ($X = \prod_{i\in\omega} X_i \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$; hence, by our hypothesis, $\mathcal{W}(X) \simeq X$ and by "For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X," $\mathcal{W}(X)$, hence X, is compact). The conclusion now follows from Theorem 2.4(iii) and Corollary 4.2. □

The next theorem indicates a link between Wallman compactifications and Tychonoff's compactness theorem.

Theorem 5.2. (i) *(ZF) For every family $\{(X_i, T_i) : i \in I\}$ of compact T_1 spaces with product X, $\mathcal{W}(X, \mathbf{C}(X)) \simeq \prod_{i\in I} \mathcal{W}(X_i) \simeq \prod_{i\in I} X_i$. In particular, "For every T_1 space (X, T) and every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X, \mathcal{C})$ is a compactification of X" if and only if TCT.*

(ii) *(ZF) For every family $\{(X_i, T_i) : i \in I\}$ of compact T_1 spaces with product X, $\mathcal{W}(X, \mathcal{C}_R(X)) \simeq \mathcal{W}(X, \mathbf{C}(X))$.*

(iii) *Assume that for every family $\{(X_i, T_i) : i \in I\}$ of compact T_1 topological spaces, for every closed ultrafilter $\mathcal{F}$ of $X = \prod_{i\in I} X_i$, and for every $i \in I$, $\mathcal{F}_i$, given by (1.3), is a closed ultrafilter of X_i. Then $\mathcal{W}(X) \simeq$*

$\prod_{i\in I} \mathcal{W}(X_i) \simeq \prod_{i\in I} X_i$. *In addition, if we add to our hypothesis, "Every non-compact* T_1 *topological space has a free closed ultrafilter," then* TCT *holds.*

(iv) AC *if and only if "Every non-compact* T_1 *topological space has a free closed ultrafilter" + "for every family* $\{(X_i, T_i) : i \in I\}$ *of infinite compact* T_1 *topological spaces, for every closed ultrafilter* $\mathcal{F}$ *of* $X = \prod_{i\in I} X_i$*, and for every* $i \in I$*,* $|\mathcal{F}_i| > 2$*, where* $\mathcal{F}_i$ *is given by (1.3)."*

Proof. (i) Fix $\{(X_i, T_i) : i \in I\}$ a family of compact T_1 spaces and let $X = \prod_{i\in I} X_i$ be their product. As in the proof of Theorem 5.1(i), $\prod_{i\in I} \mathcal{W}(X_i) \simeq \prod_{i\in I} X_i$. Let $\mathcal{F}$ be a $\mathbf{C}(X)$-ultrafilter of X. By Proposition 3.4, it follows that for all $i \in I$, the collection $\mathcal{F}_i$ given by (1.3) is a closed ultrafilter and since X_i is a compact T_1 space, $\cap\mathcal{F}_i$ is a singleton, say $\{x_i\}$, $i \in I$. By Proposition 3.5, $(x_i)_{i\in I} \in \cap\mathcal{F}$. Thus, X has no free $\mathbf{C}(X)$-ultrafilters, and by Proposition 3.1, $\mathcal{W}(X, \mathbf{C}(X)) \simeq \prod_{i\in I} X_i$.

(ii) Let $h : \mathcal{W}(X, \mathcal{C}_R(X)) \to \mathcal{W}(X, \mathbf{C}(X))$ be the mapping given by $h(\mathcal{F}) = \mathbf{F} = \mathcal{F} \cap \mathbf{C}(X)$. By (i), (ii), and (iii) of Proposition 3.6, h is well defined, $1:1$, and onto. We show now that h is a closed mapping. Let $A \in \mathcal{C}_R(X)$. Then by (i) of Proposition 2.9, there is a $Q \in [I]^{<\omega}$ such that $A = \cap\{U_j : j \in J\}$, where for all $j \in J$, $U_j = \cup\{\pi_q^{-1}(F_{jq}) : F_{jq}^c \in T_q, q \in Q\}$. It follows that $h(A^*) = h(\{\mathcal{F} \in \mathcal{W}(X, \mathcal{C}_R(X)) : A \in \mathcal{F}\}) = \cap\{\mathcal{F} \in \mathcal{W}(X, \mathbf{C}(X)) : U_j \in \mathcal{F}\} = \cap\{U_j^* : j \in J\}$. Thus, $h(A^*)$ is closed in $\mathcal{W}(X, \mathbf{C}(X))$, being an intersection of closed subsets of $\mathcal{W}(X, \mathbf{C}(X))$, and consequently h is closed as required.

Similarly, one may show that h is continuous. Hence, h is a homeomorphism.

(iii) As in part (i), our hypothesis implies that X has no free closed ultrafilters. Hence, by Proposition 3.1, $\mathcal{W}(X) \simeq \prod_{i\in I} \mathcal{W}(X_i) \simeq \prod_{i\in I} X_i$. Since X has no free closed ultrafilters, our second hypothesis implies that X is compact.

(iv) It suffices to show $(\leftarrow)$ as the other implication is straightforward. To this end, it suffices, in view of Corollary 4.2(v), to show that CAC holds. Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a family of pairwise disjoint nonempty sets. for every $i \in \omega$, let X_i, T_i, X, and $\mathcal{B}$ be as in the proof of Corollary 4.2. If X is compact, then any $f \in \cap\mathcal{B}$ is a choice function of $\mathcal{A}$. If X is not compact, then X has a free closed ultrafilter $\mathcal{F}$. For every $i \in \omega$, let $\mathcal{F}_i$ be given by (1.3). Clearly, for every $i \in \omega$, $V_i = \bigcap \mathcal{F}_i \neq \varnothing$. If $|\mathcal{B} \cap \mathcal{F}| < \aleph_0$, then for all ${}^{\infty}i \in \omega$, $V_i = \{*_i\}$ ($X_i = A_i \cup \{*_i\}$ and by the maximality of $\mathcal{F}$, either $\pi_i^{-1}(A_i) \in \mathcal{F}$ or $\pi_i^{-1}(\{*_i\}) \in \mathcal{F}$). For every $i \in \omega$ with $V_i \neq \{*_i\}$, fix $a_i \in V_i$ and let $a \in X$ satisfy $a(i) = a_i$ if $V_i \neq \{*_i\}$

and $a(i) = *_i$ otherwise. As in Proposition 3.5, it follows that $a \in \cap\mathcal{F}$. Hence, $\mathcal{F}$ is not free. This is a contradiction showing that $|\mathcal{B}\cap\mathcal{F}| = \aleph_0$. In view of the equivalence between PCAC and CAC, we may assume, for our convenience, that $\mathcal{B} \subset \mathcal{F}$. Since $|\mathcal{F}_i| > 2$ for all $i \in \omega$, it follows easily that each V_i is a finite nonempty subset of A_i. Hence, by "Every non-compact T_1 topological space has a free closed ultrafilter," Proposition 3.5(ii), and Theorem 2.4(iv), $\{V_i : i \in \omega\}$ has a choice function f. Clearly, f is also a choice function of $\mathcal{A}$, finishing the proof of (iv) and the proof of the theorem. □

The question which arises at this point is whether the compactness requirement in Theorem 5.2(i) can be dropped. We show next that this can be done.

Theorem 5.3 (ZF). *For every family $\{(X_i, T_i) : i \in I\}$ of T_1 spaces, their product X satisfies $\mathcal{W}(X, \mathbf{C}(X)) \simeq \prod_{i\in I} \mathcal{W}(X_i)$. In particular,* TCT *implies "for every family $\{(X_i, T_i) : i \in I\}$ of T_1 spaces, $\prod_{i\in I} \mathcal{W}(X_i)$ is a Wallman compactification of the product $\prod_{i\in I} X_i$," a special case of a more general* ZFC *result established in* [11].

Proof. We show that the mapping $h : \mathcal{W}(X, \mathbf{C}(X)) \to \prod_{i\in I} \mathcal{W}(X_i)$ (given by $h(\mathcal{F}) = (\mathcal{F}_i)_{i\in I}$, where for all $i \in I$, $\mathcal{F}_i$ is given by (1.3)) is a homeomorphism.

By Proposition 3.4, h is well defined, $1:1$, and onto.

Additionally, h is continuous. Fix $\pi_i^{-1}(A^*), A^c \in T_i$ a subbasic closed set of $\prod_{i\in I} \mathcal{W}(X_i)$. Since $\pi_i^{-1}(A^*) = \{(\mathcal{F}_q)_{q\in I} \in \prod_{q\in I} \mathcal{W}(X_q) : A \in \mathcal{F}_i\}$, we see that $h^{-1}(\pi_i^{-1}(A^*)) = \{\mathcal{F} \in \mathcal{W}(X, \mathbf{C}(X)) : \pi_i^{-1}(A) \in \mathcal{F}\} = (\pi_i^{-1}(A))^*$ is a closed subset of $\mathcal{W}(X, \mathbf{C}(X))$.

To complete the proof of the theorem, it suffices to show that h is closed. Fix $(\pi_i^{-1}(A))^*$ and $A^c \in T_i$ a subbasic closed set of $\mathcal{W}(X, \mathbf{C}(X))$. Since $(\pi_i^{-1}(A))^* = \{\mathcal{F} \in \mathcal{W}(X, \mathbf{C}(X)) : \pi_i^{-1}(A) \in \mathcal{F}\}$, we see that $h((\pi_i^{-1}(A))^*) = \{(\mathcal{F}_q)_{q\in I} \in \prod_{q\in I} \mathcal{W}(X_q) : A \in \mathcal{F}_i\} = \pi_i^{-1}(A^*)$ is a closed subset of $\prod_{i\in I} \mathcal{W}(X_i)$ as required. □

Question 5.4. (i) Do " For every T_1 space (X, T)" and "$\mathcal{W}(X)$ is a compactification of X" imply "For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i\in\omega} X_i) \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$" ?

(ii) Do " For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces" and "$\mathcal{W}(\prod_{i\in\omega} X_i) \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$" imply CAC ?

(iii) Is " For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i\in\omega} X_i) \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$" provable in ZF ?

(iv) Does CAC imply the statement, " In a countable product of compact T_1 spaces, every filter of closed sets extends to a closed ultrafilter" ?

6. **Countably Closed and Countably Prime Closed Ultrafilters in T_1 Spaces**

The proof of Theorem 4.3 indicates that if the closed ultrafilter $\mathcal{V}$ is countably prime, then there exists $V \in \mathcal{V}$ which has finite projections on some coordinate spaces. Hence, the existence of countably prime closed ultrafilters in products of compact T_1 spaces might lead to compactness of their product. We prove in this section that this is the case.

Proposition 6.1. (i) *Let (X, T) be a compact topological space. Then every closed ultrafilter $\mathcal{F}$ of X is countably closed and countably prime.*

(ii) *Assume that for every family $\mathcal{A} = \{A_i : i \in \omega\}$ of nonempty sets, there exists a family $\mathcal{B} = \{B_i : i \in \omega\}$ of countable nonempty sets such that $B_i \subseteq A_i$ for all $i \in \omega$. Let (X, T) be a topological space and let $\mathcal{F}$ be a closed ultrafilter of X which is countably closed. Then $\mathcal{F}$ is countably prime.*

Proof. (i) Let $\mathcal{F}$ be a closed ultrafilter of X. Since X is compact, $\bigcap \mathcal{F}$ is a nonempty closed subset of X and, as $\mathcal{F}$ is a closed ultrafilter, $\bigcap \mathcal{F} \in \mathcal{F}$. It follows that $\mathcal{F}$ is countably closed.

To see that $\mathcal{F}$ is countably prime, fix $\{G_i : i \in \omega\}$ a countable family of nonempty closed subsets of X such that $G = \cup\{G_i : i \in \omega\} \in \mathcal{F}$. We show that $G_i \in \mathcal{F}$ for some $i \in \omega$. Assume the contrary. Then, for all $i \in \omega$, $\{F \in \mathcal{F} : F \cap G_i = \emptyset\} \neq \emptyset$. For every $i \in \omega$, put

$$F_i = \cap\{F \in \mathcal{F} : F \cap G_i = \varnothing\}.$$

Since $F_i \supseteq \bigcap \mathcal{F} \in \mathcal{F}$, it follows that for every $i \in \omega$, $F_i \in \mathcal{F}$. Hence, $F = \bigcap\{F_i : i \in \omega\} \in \mathcal{F}$ and $F \cap G \in \mathcal{F}$. But

$$F \cap G = F \cap (\cup\{G_i : i \in \omega\}) = \\ \cup\{F \cap G_i : i \in \omega\} \subseteq \cup\{F_i \cap G_i : i \in \omega\} = \varnothing$$

and $F \cap G \notin \mathcal{F}$. This is a contradiction finishing the proof of (i).

(ii) Fix $\{G_i : i \in \omega\}$ a family of nonempty closed subsets of X satisfying $G = \cup\{G_i : i \in \omega\} \in \mathcal{F}$. We show that there exists $i \in \omega$ with $G_i \in \mathcal{F}$. Assume on the contrary that for every $i \in \omega$, $G_i \notin \mathcal{F}$. It follows, by the maximality of $\mathcal{F}$, that for every $i \in \omega$, $\varnothing \neq A_i = \{F \in \mathcal{F} : G_i \cap F = \varnothing\}$. By our hypothesis and the fact that $\mathcal{F}$ is countably closed, it is easy to see that there exists a choice function f of the family $\{A_i : i \in \omega\}$. Put $F = \cap\{f(i) : i \in \omega\}$. Since $\mathcal{F}$ is countably closed, it follows that $F \in \mathcal{F}$, and consequently $F \cap G \neq \varnothing$. Similarly to the proof of (i), we may show that $F \cap G = \emptyset$, a contradiction. This completes the proof of (ii) and the proof of the proposition. □

Remark 6.2. Clearly, in ZFC, $T = \{(x, +\infty) : x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is a topology on the real line $\mathbb{R}$ and $\mathcal{F} = \{(-\infty, x] : x \in \mathbb{R}\} \cup \{\mathbb{R}\}$ is a countably prime closed ultrafilter of $(\mathbb{R}, T)$ which fails to be countably closed. Thus, in ZFC, $\mathcal{F}$ is countably prime $\nrightarrow$ $\mathcal{F}$ is countably closed.

Theorem 6.3. *In* ZF*, the following are equivalent.*

(i) AC.

(ii) *For every family $\{(X_i, T_i) : i \in I\}$ of compact T_1 spaces, every closed filter $\mathcal{H}$ of the product $X = \prod_{i \in I} X_i$ extends to a closed ultrafilter $\mathcal{F}$ which is countably closed and countably prime.*

(iii) *For every family $\{(X_i, T_i) : i \in I\}$ of compact T_1 spaces, every closed filter $\mathcal{H}$ of the product $X = \prod_{i \in I} X_i$ extends to a closed ultrafilter $\mathcal{F}$ which is countably closed.*

Proof. (i) → (ii). Let $X = \prod_{i \in I} X_i$ be the product of the compact T_1 spaces X_i. By AC, X is compact and every closed filter is extended to a closed ultrafilter. The conclusion follows from Proposition 6.1.

(ii) → (iii) is straightforward.

(iii) → (i). We show first that CAC holds. Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a family of nonempty sets. For every $i \in \omega$, let $X_i = A_i \cup \{*_i\}$, $*_i \notin A_i$, and $T_i = \{\{*_i\}, \varnothing\} \cup \{Z \subseteq X_i : |Z^c| < \aleph_0\}$. Clearly, for every $i \in \omega$, (X_i, T_i) is a compact T_1 space. Put $X = \prod_{i \in \omega} X_i$ and let $\mathcal{H}$ be the closed filter generated by the family $\mathcal{S} = \{\pi_i^{-1}(A_i) : i \in \omega\}$. Let $\mathcal{F}$ be a closed, countably closed ultrafilter extending $\mathcal{H}$. Since $\mathcal{S} \subset \mathcal{F}$, it follows that $\cap\mathcal{S} \in \mathcal{F}$. Hence, $\cap\mathcal{S} \neq \varnothing$, and consequently $\mathcal{A}$ has a choice function as required.

We show now that AC holds. Fix $\mathcal{A} = \{A_i : i \in I\}$ a family of nonempty sets. For every $i \in I$, let X_i, T_i, X, $\mathcal{S}$, and $\mathcal{F}$ be as in the first part of this proof. In view of the proof of Proposition 2 in [6], it suffices, to show that canonical projections of closed sets are closed. Since CAC holds, the proof of Proposition 2 in [6] shows that we further need to establish that there is a free ultrafilter on ω. However, following the proof of Proposition 2.13, we have that (iii) of the present theorem implies BPI, which in turn implies that there is a free ultrafilter on ω. Now, as in the proof of Proposition 2 of [6], we may define a choice function of $\mathcal{A}$. This completes the proof of the theorem. □

Corollary 6.4. *The following are equivalent.*

(i) AC.

(ii) *For every family* $\{(X_i, T_i) : i \in I\}$ *of compact* T_1 *spaces, every filter* $\mathcal{H}$ *of closed subsets of the product* $X = \prod_{i\in\omega} X_i$ *extends to a closed ultrafilter* $\mathcal{F}$ *which is countably closed and countably prime.*

(iii) *For every family* $\{(X_i, T_i) : i \in I\}$ *of compact* T_1 *spaces, every filter* $\mathcal{H}$ *of closed subsets of the product* $X = \prod_{i\in\omega} X_i$ *extends to a closed ultrafilter* $\mathcal{F}$ *which is countably prime.*

Proof. Since (i) and (ii) of the corollary coincide with (i) and (ii), respectively, of Theorem 6.3, it suffices to show that (iii) → (i). Furthermore, since (iii) implies BPI (as in the proof of Theorem 6.3), we only show that CAC holds. To see this, fix $\mathcal{A} = \{A_i : i \in \omega\}$ a disjoint family of infinite sets. Assume that $\mathcal{A}$ has no partial choice sets. For every $i \in \omega$, let X_i, T_i, and X be as in the proof of Theorem 6.3 and let $\mathcal{F}$ be a countably prime closed ultrafilter of X which extends the closed filter $\mathcal{H}$ generated by the family $\mathcal{S} = \{\pi_i^{-1}(A_i) : i \in \omega\}$.

Claim. For every $i \in \omega$, there is a unique $a_i \in A_i$ such that $\pi_i^{-1}(\{a_i\}) \in \mathcal{F}$.

Proof of the claim. Fix $i \in \omega$ and let $G_i = \{F \in \mathcal{F} : A_i \backslash \pi_i(F) \neq \varnothing\}$. Clearly, $G_i \neq \varnothing$. (If $G_i = \varnothing$, then for every $a \in A_i$, $\mathcal{F} \cup \{\pi_i^{-1}(\{a\})\}$ has the fip, and consequently, by the maximality of $\mathcal{F}$, $\pi_i^{-1}(\{a\}) \in \mathcal{F}$, which is a contradiction.) Fix $F \in G_i$ and express F as $F = \cup\{F_n : n \in \omega\}$,

$$F_n = \{f \in F : \forall m \geq n, f(m) = *_m\}. \tag{6.1}$$

Since each F_n is closed (as in the proof of Theorem 4.3), $F \in \mathcal{F}$, and $\mathcal{F}$ is countably prime, it follows that for some $n \in \omega$, $F_n \in \mathcal{F}$. Since $K_i = \pi_i(F_n) \cap A_i$ is a nonempty closed proper subset of A_i, it follows that K_i is a finite subset of A_i and $\pi_i^{-1}(K_i) \in \mathcal{F}$. Thus, by the maximality of $\mathcal{F}$, there exists a unique $a_i \in A_i$ with $\pi_i^{-1}(\{a_i\}) \in \mathcal{F}$, finishing the proof of the claim.

For every $i \in \omega$, let a_i be the unique element of A_i which is guaranteed by the claim. Clearly, $\{a_i : i \in \omega\}$ is a choice set of $\mathcal{A}$, contradicting our hypothesis on $\mathcal{A}$ having no partial choice sets. Therefore, CAC holds and the proof of the corollary is complete. □

7. Open Questions and Directions for Further Study

Below, we summarize the open problems mentioned throughout the paper.

(1) Does "For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" imply AC? Equivalently (see Theorem 2.4), does CFE_1 imply AC? Equivalently (see Theorem 4.1), does "For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" imply "For every T_1 space (X, T) and every T_1 base $\mathcal{C}$ for X, $\mathcal{W}(X, \mathcal{C})$ is a compactification of X"?

(2) Does "For every T_1 space (X, T), $\mathcal{W}(X)$ is a compactification of X" imply "For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i\in\omega} X_i) \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$"?

(3) Does "For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i\in\omega} X_i) \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$" imply CAC?

(4) Is "For every family $\{(X_i, T_i) : i \in \omega\}$ of compact T_1 spaces, $\mathcal{W}(\prod_{i\in\omega} X_i) \simeq \prod_{i\in\omega} \mathcal{W}(X_i)$" provable in ZF?

(5) Does CAC imply the statement, "In a countable product of compact T_1 spaces, every filter of closed sets extends to a closed ultrafilter"?

For Wallman-type compactifications and their relationship to AC or to certain weak forms of AC, the reader is referred to the following related papers: [1], [2], [4], [5].

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