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A NEW VERSION OF HOMOTOPICAL HAUSDORFF

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ABSTRACT. It is known that shape injectivity implies homotopical Hausdorff and that the converse does not hold, even if the space is required to be a Peano continuum. This paper gives an alternative definition of homotopical Hausdorff inspired by a new topology on the set of fixed endpoint homotopy classes of paths. This version is equivalent to shape injectivity for Peano spaces.

1. INTRODUCTION

Homotopical Hausdorff is a homotopical criterion that detects if a Hausdorff space X has \tilde{X} Hausdorff where \tilde{X} is the set of fixed endpoint homotopy classes of paths in X starting at some basepoint with a standard topology (see Definition 3.1). Homotopical Hausdorff was discussed in [14] and [1] and given its present name in [4]. In [9] the authors show that shape injectivity implies homotopical Hausdorff. A space X is shape injective if the homomorphism $\pi_1(X) \rightarrow \tilde{\pi}_1(X)$ is injective. Spaces that are known to be shape injective include one dimensional Hausdorff compacta [7] and subsets of closed surfaces [8]. Thus, these spaces have nice spaces of path homotopy classes.

The authors in [5] give examples of two Peano continua that are homotopically Hausdorff but not shape injective. The present paper notes that if the definition of homotopical Hausdorff is modified to reflect a new topology on \tilde{X} (Definition 3.3), then the two concepts are equivalent for Peano spaces (Theorem 3.7).

Section 2 gives a treatment of shape injectivity that mirrors the theory of generalized paths in [2]. This viewpoint relates paths in a space X to

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chains of points in X . It is quite geometric and lends itself to the proof of Theorem 3.7.

2. SHAPE INJECTIVITY

Generalized paths and the uniform shape group are defined for uniform spaces in [2]. We introduce an analogous construction for all topological spaces. Let us first recall the definition of the classical shape group.

We will consider only normal open covers. Recall an open cover \mathcal{U} of X is *normal* if it admits a partition of unity $\{\phi_U : X \rightarrow [0, 1]\}_{U \in \mathcal{U}}$ with $\phi_U^{-1}(0, 1) \subset U$ for each $U \in \mathcal{U}$. We say that the partition of unity is *subordinate to* \mathcal{U} . The partition of unity can be chosen to be locally finite. The *nerve* $N(\mathcal{U})$ of \mathcal{U} is the simplicial complex whose vertices are elements of \mathcal{U} and whose simplices are finite subsets $\mathcal{A} \subset \mathcal{U}$ such that the intersection of the elements of \mathcal{A} is nonempty.

Fix an $x_0 \in X$ and, for each normal cover \mathcal{U} , fix a $U_0 \in \mathcal{U}$ with $x_0 \in U_0$. Given two normal covers \mathcal{U} and \mathcal{V} , define $\mathcal{U} \leq \mathcal{V}$ if (\mathcal{V}, V_0) refines (\mathcal{U}, U_0) , that is, \mathcal{V} refines \mathcal{U} and $V_0 \subset U_0$. In this case, choose a bonding map $p : N(\mathcal{V}) \rightarrow N(\mathcal{U})$ such that each $V \in \mathcal{V}$ gets mapped to a $U \in \mathcal{U}$ with $V \subset U$, making sure to send V_0 to U_0 . The *shape group* $\tilde{\pi}_1(X, x_0)$ is the inverse limit $\varprojlim \pi_1(N(\mathcal{U}), U_0)$ [10].

Given a cover \mathcal{U} of X define a *\mathcal{U} -chain* to be a finite list x_1, \dots, x_n of points in X such that for each $i < n$, $x_i, x_{i+1} \in U$ for some $U \in \mathcal{U}$. Let $R(X, \mathcal{U})$ be the simplicial complex whose vertices are points in X and $A \subset X$ is a simplex if it is a finite subset of some $U \in \mathcal{U}$. It is the *Rips complex* of X with respect to \mathcal{U} .

We identify a \mathcal{U} -chain x_1, \dots, x_n with the concatenation of the edge paths $[x_1, x_2], \dots, [x_{n-1}, x_n]$ in $R(X, \mathcal{U})$. We define two \mathcal{U} -chains to be *\mathcal{U} -homotopic* if the corresponding paths in $R(X, \mathcal{U})$ are fixed endpoint homotopic. This homotopy can be chosen to be simplicial. Thus, two \mathcal{U} -chains are \mathcal{U} -homotopic if one can move from one to the other by a finite sequence of vertex additions and deletions.

Define a *generalized path* in X to be a collection $\alpha = \{[\alpha_{\mathcal{U}}]_{\mathcal{U}}\}$ of equivalence classes of \mathcal{U} -chains in X , where \mathcal{U} runs over all normal open covers of X such that, if \mathcal{V} refines \mathcal{U} , $\alpha_{\mathcal{V}}$ is \mathcal{U} -homotopic to $\alpha_{\mathcal{U}}$. We define the *covering shape group* $\tilde{\pi}_1^{\text{cov}}(X, x_0)$ to be the group of generalized loops in X based at x_0 under the operation of concatenation. It is isomorphic to $\varprojlim \pi_1(R(X, \mathcal{U}), x_0)$.

We will show that $\tilde{\pi}_1^{\text{cov}}(X, x_0)$ is isomorphic to the classical shape group. In order to do so, let us recall the following definition. Given an open cover \mathcal{U} of X , the *star* of a point $x \in X$ in \mathcal{U} is the union of all $U \in \mathcal{U}$ containing x . We say that a cover \mathcal{V} is a *star refinement* of a cover

\mathcal{U} if the cover $\{\text{St}(x, \mathcal{V}) : x \in X\}$ refines \mathcal{U} . Any normal open cover has a normal star refinement [6, Proposition 5.3].

It is more convenient to use the following notion of a star of a cover. Given an open cover \mathcal{U} and a $U \in \mathcal{U}$, let $\text{St}U$ be the union of all $V \in \mathcal{U}$ that meet U . Let $\text{St}\mathcal{U}$ be the set of all $\text{St}U$ for $U \in \mathcal{U}$. Notice the similarity between the open set $\text{St}U$ in X and the open star $\text{St}U$ of the vertex U in $N(\mathcal{U})$. They are both defined in terms of all $V \in \mathcal{U}$ that meet U .

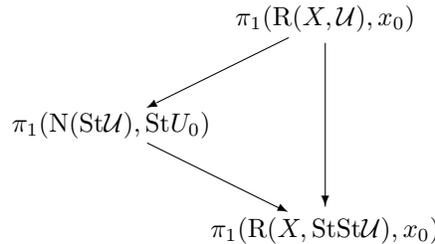
Lemma 2.1. *Suppose \mathcal{W} is a star refinement of \mathcal{V} and that \mathcal{V} is a star refinement of \mathcal{U} . Then $\text{St}\mathcal{W}$ refines \mathcal{U} .*

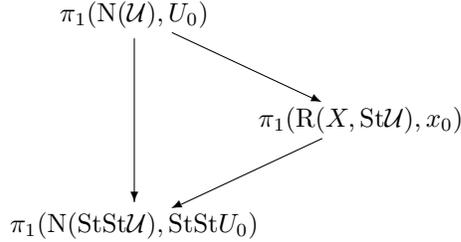
Proof. Given $W \in \mathcal{W}$, let $x \in W$. We will show that $\text{St}(W, \mathcal{W}) \subset \text{St}(x, \mathcal{V})$. Suppose $y \in \text{St}(W, \mathcal{W})$. Then $y \in W'$ where $W' \in \mathcal{W}$ meets W , say at a point z . Then $x, y \in \text{St}(z, \mathcal{W})$ which is contained in some $V \in \mathcal{V}$. Thus, $y \in \text{St}(x, \mathcal{V})$. \square

Proposition 2.2. *$\tilde{\pi}_1(X, x_0)$ is isomorphic to $\tilde{\pi}_1^{\text{cov}}(X, x_0)$.*

Proof. Fix a basepoint $x_0 \in X$ and, for each normal cover \mathcal{U} of X , fix a “basepoint” $U_0 \in \mathcal{U}$ with $x_0 \in U_0$. Define a pointed map $(R(X, \mathcal{U}), x_0) \rightarrow (N(\text{St}\mathcal{U}), \text{St}U_0)$ to send a vertex $x \in X$ to $\text{St}U$ for some $U \in \mathcal{U}$ with $x \in U$. Note, we can assume $\text{St}U_0$ is the basepoint of $\text{St}\mathcal{U}$ since any other $\text{St}U$ that contains x_0 meets $\text{St}U_0$. Let us see that this map on vertices extends to a simplicial map. Suppose $[x_1, \dots, x_n] \in R(X, \mathcal{U})$. If $x_i \mapsto \text{St}U_i$, then $x_1 \in \text{St}U_i$ for each $i \leq n$, so $[\text{St}U_1, \dots, \text{St}U_n] \in N(\text{St}\mathcal{U})$.

Now define a pointed map $(N(\mathcal{U}), U_0) \rightarrow (R(X, \text{St}\mathcal{U}), x_0)$ to send a vertex $U \in \mathcal{U}$ to some x with $x \in U$. Let us see that this map extends to a simplicial map. Suppose $[U_1, \dots, U_n] \in N(\mathcal{U})$. If $U_i \mapsto x_i$, then $x_1, \dots, x_n \in \text{St}U_1$, so $[x_1, \dots, x_n] \in R(X, \text{St}\mathcal{U})$. These maps induce homomorphisms $\pi_1(R(X, \mathcal{U}), x_0) \rightarrow \pi_1(N(\text{St}\mathcal{U}), \text{St}U_0)$ and $\pi_1(N(\mathcal{U}), U_0) \rightarrow \pi_1(R(X, \text{St}\mathcal{U}), x_0)$. By the lemma and the fact that any normal open cover has a normal star refinement, it suffices to check that the following two diagrams commute.





Suppose x_0, \dots, x_n is a \mathcal{U} -chain in X representing a loop in $R(X, \mathcal{U})$ based at x_0 . Suppose x_0, \dots, x_n is sent to $\text{St}U_0, \dots, \text{St}U_n$ which in turn is sent to y_0, \dots, y_n . Now $y_0 = y_n = x_0$ by assumption. To see that y_0, \dots, y_n is $\text{StSt}\mathcal{U}$ -homotopic to x_0, \dots, x_n , notice that for any $i < n$, $x_i, y_i, x_{i+1}, y_{i+1} \in \text{StSt}U$ where $U \in \mathcal{U}$ contains x_i and x_{i+1} .

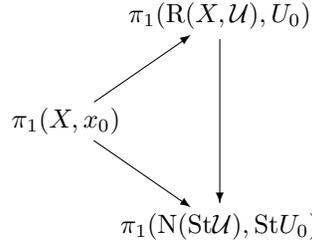
Now suppose U_0, \dots, U_n is a sequence of vertices in $N(\mathcal{U})$ that represents a loop in $N(\mathcal{U})$ based at U_0 . Suppose U_0, \dots, U_n is sent to x_0, \dots, x_n which in turn is sent to $\text{StSt}V_0, \dots, \text{StSt}V_n$ where $V_i \in \mathcal{U}$. Now $V_0 = V_n = U_0$ by assumption. To see that the loop represented by $\text{StSt}V_0, \dots, \text{StSt}V_n$ is homotopic to the one represented by U_0, \dots, U_n in $N(\text{StSt}\mathcal{U})$, notice that for any $i < n$, $x \in \text{StSt}U_i \cap \text{StSt}V_i \cap \text{StSt}U_{i+1} \cap \text{StSt}V_{i+1}$ where x is an element in $U_i \cap U_{i+1}$. \square

There is a natural homomorphism $\pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$ from the fundamental group to the shape group. Suppose α is a path in X and \mathcal{U} is an open cover of X . Choose $\delta > 0$ so that any subinterval of $[0, 1]$ of length δ is sent by α to some $U \in \mathcal{U}$. Define a \mathcal{U} -chain $\varphi_{\mathcal{U}}(\alpha) = \alpha(0), \alpha(\delta), \alpha(2\delta), \dots, \alpha(1)$. This definition is independent of the choice of δ ; given $\delta_1 < \delta_2$, the corresponding \mathcal{U} -chains will be \mathcal{U} -homotopic. Simply add the two chains together according to the order on $[0, 1]$ to get another \mathcal{U} -chain which is \mathcal{U} -homotopic to both.

A similar argument shows that if \mathcal{V} refines \mathcal{U} , $\varphi_{\mathcal{V}}(\alpha)$ will be \mathcal{U} -homotopic to $\varphi_{\mathcal{U}}(\alpha)$. Thus, we have a generalized path $\varphi(\alpha) = \{\varphi_{\mathcal{U}}(\alpha)\}$. We show that φ induces a well-defined homomorphism $\pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$. Suppose α is fixed endpoint homotopic to β . Let \mathcal{U} be a cover of X and $\delta > 0$ be such that any square $I \times I \subset [0, 1] \times [0, 1]$ of side length δ is sent by the homotopy to an element of \mathcal{U} . We have a sequence of paths $\alpha_0, \alpha_{\delta}, \alpha_{2\delta}, \dots, \beta$ given by the homotopy. To see that $\varphi_{\mathcal{U}}(\alpha_{i\delta})$ is \mathcal{U} -homotopic to $\varphi_{\mathcal{U}}(\alpha_{(i+1)\delta})$, notice the following chain is \mathcal{U} -homotopic to both.

$$\alpha_{i\delta}(0) \ \alpha_{i\delta}(\delta) \ \alpha_{(i+1)\delta}(\delta) \ \alpha_{(i+1)\delta}(2\delta) \ \alpha_{i\delta}(2\delta) \ \alpha_{i\delta}(3\delta) \ \alpha_{(i+1)\delta}(3\delta) \ \dots$$

We end the section by showing that this homomorphism is identical to the classical homomorphism. Given a cover \mathcal{U} , there is a map $\phi : X \rightarrow N(\mathcal{U})$ given by a partition of unity subordinate to \mathcal{U} , $\phi(x) = \sum \phi_U(x)U$. This map enjoys the property that $\phi^{-1}(\text{St}U) \subset U$ where $\text{St}U$ is the open star of the vertex U in $N(\mathcal{U})$. ϕ induces a homomorphism $\pi_1(X) \rightarrow \tilde{\pi}_1(X)$. We show the following diagram commutes.



Let $\phi : X \rightarrow N(\text{St}\mathcal{U})$ be a map given by a partition of unity subordinate to $\text{St}\mathcal{U}$. Let α be a path in X . Let $\delta > 0$ so that if I is a subinterval of $[0, 1]$ of length δ , $\alpha(I)$ is sent by ϕ to some $\text{St}U$, the open star of a vertex U of $N(\text{St}\mathcal{U})$. Let δ also be such that $\alpha(I)$ lies in an element of \mathcal{U} .

Now α is sent to the chain $\alpha(0), \alpha(\delta), \alpha(2\delta), \dots$ in $\mathbb{R}(X, \mathcal{U})$ which in turn is sent to the chain $\text{St}U_0, \text{St}U_\delta, \text{St}U_{2\delta}, \dots$ in $N(\text{St}\mathcal{U})$ where $\alpha(k\delta) \in U_{k\delta}$ for $k \geq 0$. On the other hand, α is sent to the path $\phi\alpha$ in $N(\text{St}\mathcal{U})$. We need to show that $\phi\alpha$ is fixed endpoint homotopic to the concatenation of edge paths associated with the chain $\text{St}U_0, \text{St}U_\delta, \text{St}U_{2\delta}, \dots$. We proceed by induction on the number of terms in this chain.

Now $\phi\alpha[0, \delta] \subset \text{St}U$ and $\phi\alpha[\delta, 2\delta] \subset \text{St}U_1$ for some $U, U_1 \in \mathcal{U}$. Since $\phi\alpha(\delta) \in \text{St}U \cap \text{St}U_1$, it is in an open simplex having $\text{St}U$ and $\text{St}U_1$ as vertices. Now $\alpha[0, \delta] \subset \text{St}U$ and $\alpha[\delta, 2\delta] \subset \text{St}U_1$, so $\alpha(\delta) \in \text{St}U \cap \text{St}U_1$ and $[\text{St}U, \text{St}U_1, \text{St}U_\delta]$ is a simplex. The open simplex $[\text{St}U, \text{St}U_1, \text{St}U_\delta]$ is contained in $\text{St}U \cap \text{St}U_1$, so we can join $\phi\alpha(\delta)$ to $\text{St}U_\delta$ by a path γ with $\gamma[0, 1] \subset \text{St}U \cap \text{St}U_1$. Then, since $\phi\alpha[0, \delta] \subset \text{St}U$, we can find a homotopy from $\phi\alpha[0, \delta]$ to the edge path $[\text{St}U_0, \text{St}U_\delta]$ that fixes $\alpha(0) = \text{St}U_0$ and follows γ from $\phi\alpha(\delta)$ to $\text{St}U_\delta$.

For each $i > 1$, $\phi\alpha[i\delta, (i+1)\delta] \subset \text{St}U_i$ for some $U_i \in \mathcal{U}$. Suppose that $\phi\alpha[0, k\delta]$ is homotopic to the concatenation of edge paths associated with $\text{St}U_0, \text{St}U_\delta, \dots, \text{St}U_{k\delta}$ where the homotopy fixes $\alpha(0) = \text{St}U_0$ and follows a path γ_k from $\phi\alpha(k\delta)$ to $\text{St}U_{k\delta}$ with $\gamma_k[0, 1] \subset \text{St}U_{k-1} \cap \text{St}U_k$. We follow the same procedure as above to find a path γ_{k+1} from $\phi\alpha((k+1)\delta)$ to $\text{St}U_{(k+1)\delta}$ that is contained in $\text{St}U_k \cap \text{St}U_{k+1}$. Since $\phi\alpha[k\delta, (k+1)\delta] \subset \text{St}U_k$, we can find a homotopy from $\phi\alpha[k\delta, (k+1)\delta]$ to the edge

path $[\text{St}U_{k\delta}, \text{St}U_{(k+1)\delta}]$ that follows γ_k from $\phi\alpha(k\delta)$ to $\text{St}U_{k\delta}$ and γ_{k+1} from $\phi\alpha((k+1)\delta)$ to $\text{St}U_{(k+1)\delta}$.

3. HOMOTOPICAL HAUSDORFF

We recall a standard topology on \tilde{X} , the set of fixed endpoint homotopy classes of paths in X starting at some basepoint x_0 .

Definition 3.1. Given $[\alpha] \in \tilde{X}$ with terminal point x and a neighborhood U of x in X , $B([\alpha], U)$ is the set of all $[\beta] \in \tilde{X}$ such that $\alpha^{-1}\beta$ is fixed endpoint homotopic to a path in U . We will call the topology generated by these sets the *whisker topology* on \tilde{X} following [3].

This topology is used by Edwin H. Spanier in [13] and James R. Munkres in [11] for the classic construction of covering spaces. It is equivalent to the quotient topology inherited from $(X, x_0)^{(I, 0)}$ under the compact open topology for locally path connected and semilocally simply connected spaces [9, Lemma 2.1].

Investigations into the structure of \tilde{X} lead one to realize that it can fail to be Hausdorff. The harmonic archipelago in [1] is a standard example of a Hausdorff space whose space of path homotopy classes is not Hausdorff. This situation motivates the following definition (see [4]).

Definition 3.2. A space X is *homotopically Hausdorff* if for each $x \in X$ and each essential loop γ based at x , there is a neighborhood U of x such that γ is not fixed endpoint homotopic to a path in U .

Notice that a space X is homotopically Hausdorff if and only if for all $x \in X$, $\cap \pi_1(U, x) = 1$ where U runs over all neighborhoods of x . Also, for a Hausdorff space X , X is homotopically Hausdorff if and only if \tilde{X} is Hausdorff for any basepoint.

It is shown in [9] that if a space is shape injective then it is homotopically Hausdorff. The space A in [5] is an example of a Peano continuum that is homotopically Hausdorff but not shape injective (see Example 3.5).

Investigation into the structure of $\pi_1(X)$ as a subspace of \tilde{X} in [3] leads to the definition of a new topology on \tilde{X} . The new topology is based on the following definition. Given an open cover \mathcal{U} , let $\pi(\mathcal{U}, x)$ be the subgroup of $\pi_1(X, x)$ generated by elements of the form $[\alpha\gamma\alpha^{-1}]$ where α is a path starting at x and γ is a loop in some $U \in \mathcal{U}$. These groups are used in [13] to detect when a fibration with unique path lifting is a covering map.

Definition 3.3. Given $[\alpha] \in \tilde{X}$ with terminal point x , a normal open cover \mathcal{U} of X , and a neighborhood V of x in X , $B([\alpha], \mathcal{U}, V)$ is the set of all $[\beta] \in \tilde{X}$ such that $\alpha^{-1}\beta$ is fixed endpoint homotopic to a loop in $\pi(\mathcal{U}, x)$

concatenated with a path in V . We will call the topology generated by these basic sets the *lasso topology*.

There are slight differences between the above definition and the one that appears in [3]. As in the definition of the shape group, here we restrict our attention to normal covers. Also, in [3] it is required that $V \in \mathcal{U}$. This requirement does not affect the topology generated by the sets.

We now define an analogous version of homotopical Hausdorff for the lasso topology.

Definition 3.4. A space X is *lasso homotopically Hausdorff* if for each $x \in X$ and each essential loop γ based at x , there is a normal open cover \mathcal{U} of X such that $[\gamma] \notin \pi_1(\mathcal{U}, x)$.

Notice that a space X is lasso homotopically Hausdorff if for all $x \in X$, $\cap \pi_1(\mathcal{U}, x) = 1$ where \mathcal{U} runs over all normal open covers of X . Also, for a Hausdorff space X , X is lasso homotopically Hausdorff if and only if \tilde{X} is Hausdorff for any basepoint under the lasso topology.

This concept was investigated in [9] where it is shown that if $\cap \pi_1(\mathcal{U}) = 1$ for some collection of open covers of X , then the endpoint map $\tilde{X} \rightarrow X$ has unique path lifting (where \tilde{X} is given the whisker topology).

Example 3.5. We show that the space A from [5] is not lasso homotopically Hausdorff. Let A' be the topologist's sine curve $\{(x, \sin(1/x)) : x \in (0, 1]\} \cup \{0\} \times [-1, 1]$ rotated about its limiting arc. It is the surface and central limit arc portions of A . Connecting arcs are added to form the Peano continuum A and the authors show that if a loop in A' is null-homotopic in A , then it is null-homotopic in A' (Lemma 4.3). Choose a basepoint x on the surface portion of A and a loop β that goes once around the surface portion. Since β is essential in A' it must be essential in A as well. Given any neighborhood U of a point on the central limit arc, β is fixed endpoint homotopic to a loop of the form $\alpha\gamma\alpha^{-1}$ where γ is contained in U . Thus, $[\beta] \in \pi_1(\mathcal{U}, x)$ for all open covers \mathcal{U} of A . Note that A is not shape injective.

We now see that this version of homotopical Hausdorff is equivalent to shape injectivity for Peano spaces.

Lemma 3.6. *Suppose X is locally path connected. Any normal open cover \mathcal{U} of X has a normal open refinement composed of path connected sets.*

Proof. Let \mathcal{U} be an open cover with associated partition of unity $\{\phi_U\}$. Given $U \in \mathcal{U}$, decompose it into its path components $\{V_\alpha\}$. Since X is locally path connected, these components are open. Given $x \in X$, define

$\psi_{V_\alpha}(x) = \phi_U(x)$ if $x \in V_\alpha$ and $\psi_{V_\alpha}(x) = 0$ otherwise. Then $\{\psi_{V_\alpha}\}_{U \in \mathcal{U}}$ is a partition of unity subordinate to $\{V_\alpha\}_{U \in \mathcal{U}}$. \square

Theorem 3.7. *Suppose X is path connected and locally path connected. Then X is lasso homotopically Hausdorff if and only if it is shape injective.*

Proof. The reverse direction is essentially [9, Proposition 4.8]. We provide a proof here. Suppose X is shape injective and that $[\beta] \in \cap \pi_1(\mathcal{U}, x)$. Given \mathcal{U} , let λ be a path that is fixed endpoint homotopic to β so that $\lambda = \lambda_1 \cdots \lambda_n$ where each $\lambda_i = \alpha_i \gamma_i \alpha_i^{-1}$, α_i is a path starting at x , and γ_i is a loop in some $U \in \mathcal{U}$. Send λ to $\pi_1(R(X, \mathcal{U}), x)$. Then the image of γ_i is \mathcal{U} -homotopic to the constant chain at the terminal point of α_i , so the image of λ_i is \mathcal{U} -homotopic to the constant chain at x . Thus, the image of λ is trivial and the image of λ in $\tilde{\pi}_1(X, x)$ is trivial. Given that X is shape injective, we have that λ is trivial.

Now suppose that X is lasso homotopically Hausdorff. Suppose β is a loop in X based at x whose image in $\tilde{\pi}_1(X, x)$ is trivial. Given a cover \mathcal{U} , we wish to show $[\beta] \in \pi_1(\mathcal{U}, x)$. Let \mathcal{V} be a cover so that $\text{St}\mathcal{V}$ refines \mathcal{U} and let \mathcal{W} be a refinement of \mathcal{V} composed of path connected sets. The image of β in $R(X, \mathcal{W})$ is \mathcal{W} -homotopic to the trivial chain at x . We proceed by induction on the number of steps in this simplicial homotopy.

The image of β in $R(X, \mathcal{W})$ is represented by a \mathcal{W} -chain x_0, \dots, x_n , i.e., $\beta = \beta_0 \cdots \beta_{n-1}$ where each β_i is a path in some element of \mathcal{W} from x_i to x_{i+1} .

Suppose a step of the simplicial homotopy starts at the \mathcal{W} -chain y_0, \dots, y_m . Suppose there is a $[\lambda] \in \pi_1(\mathcal{U}, x)$ and a path $\alpha = \alpha_0 \cdots \alpha_{m-1}$ associated with y_0, \dots, y_m (that means each α_i is a path in some element of \mathcal{W} from y_i to y_{i+1}) such that β is fixed endpoint homotopic to $\lambda\alpha$. We show the same thing can be said about the next chain in the simplicial homotopy.

Suppose the next step of the simplicial homotopy is obtained by vertex addition, say $\dots, y_i, y_{i+1}, \dots$ to $\dots, y_i, y, y_{i+1}, \dots$. Now $y_i, y, y_{i+1} \in W$ for some $W \in \mathcal{W}$. Join y_i to y by a path λ_1 in W and join y to y_{i+1} by a path λ_2 in W . Let $\lambda = \alpha_0 \cdots \alpha_{i+1} \lambda_2^{-1} \lambda_1^{-1} \alpha_i^{-1} \cdots \alpha_0^{-1}$ and $\alpha = \alpha_0 \cdots \alpha_{i-1} \lambda_1 \lambda_2 \alpha_{i+1} \cdots \alpha_{m-1}$. Then β is fixed endpoint homotopic to $\lambda\alpha$, $[\lambda] \in \pi_1(\mathcal{U}, x)$, and α is associated with the new \mathcal{W} -chain.

Now suppose the next step of the simplicial homotopy is obtained by vertex deletion, say $\dots, y_i, y_{i+1}, y_{i+2}, \dots$ to $\dots, y_i, y_{i+2}, \dots$. Then $y_i, y_{i+2} \in W$ for some $W \in \mathcal{W}$. Join y_i to y_{i+2} by a path λ in W . Let $\lambda = \alpha_0 \cdots \alpha_{i+1} \lambda^{-1} \alpha_{i-1}^{-1} \cdots \alpha_0^{-1}$ and $\alpha = \alpha_0 \cdots \alpha_{i-1} \lambda \alpha_{i+2} \cdots \alpha_{m-1}$. Then β is fixed endpoint homotopic to $\lambda\alpha$, $[\lambda] \in \pi_1(\mathcal{U}, x)$, and α is associated with the new \mathcal{W} -chain.

At the end of the simplicial homotopy the \mathcal{W} -chain is the trivial chain at x . Thus, we have β is fixed endpoint homotopic to $\lambda\alpha$ where $[\lambda] \in \pi_1(\mathcal{U}, x)$ and α is associated with the trivial chain. Thus, α is a loop based at x in some element of \mathcal{W} , so $[\lambda\alpha] \in \pi_1(\mathcal{U}, x)$. \square

Remark 3.8. The requirement of local path connectivity cannot be removed. In [9, Remark 4.9], the authors give an example of a path connected space that is lasso homotopically Hausdorff but not shape injective. The space is related to the space B in [5] (see Example 3.9). In fact, the authors in [9] cite [12] which became [5].

Example 3.9. Let B' be the topologist's sine curve $\{(x, \sin(1/(1-x))) : x \in (0, 1]\} \cup \{1\} \times [-1, 1]$ rotated about a vertical axis at the point $(0, \sin(1))$. This space is lasso homotopically Hausdorff (it is locally simply connected). Connecting arcs are used in [5] to obtain the Peano continuum B . Since B is not shape injective, it cannot be lasso homotopically Hausdorff. A loop that goes around the outer annulus can be pulled in to the surface portion creating lassos.

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