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PERIODS OF ENDOMORPHISMS ON AN ABELIAN GROUP

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Abstract. Given a group G and an endomorphism f of G, a point $x \in G$ is a periodic point of f of period $n \in \mathbb{N}$ if $f^n(x) = x$ and $f^k(x) \neq x$ for all $1 \leq k < n$. The set of periods of f is the set $Per(f) = \{n \in \mathbb{N} : f$ has a periodic point of period $n\}$. In this paper, we characterize the sets of periods of an endomorphism of an abelian group. We prove that $\{A \subset \mathbb{N} : 1 \in A$ and A is closed under lcm$\}$ is the family of period sets of endomorphisms of an abelian group.

1. Introduction

A dynamical system is a pair (X, f) consisting of a topological space X and a continuous self map f on it. We denote f^n the composition of f with itself $n-1$ times. For $x \in X$, the sequence x, $f(x)$, $f^2(x)$,..., $f^n(x)$,... is called the *forward f-trajectory* of x and the set $\{f^n(x) : n = 0, 1, 2,...\}$ is called the *forward f-orbit* of x. A point $x \in X$ is a *periodic point* of f of period (f-period) $p \in \mathbb{N}$ if $f^p(x) = x$ and $f^m(x) \neq x$ for all $1 \leq m \leq p-1$. The periodic points of period 1 are called *fixed* points of f. Let $Per(f)=$ $\{n \in \mathbb{N}$ such that f has a point of period $n\}$ and we call this the *set of periods* (or simply *period set*) of f. All these notions make sense for any set not necessarily a topological space and for any self map not necessarily continuous. For $m, n \in \mathbb{N}$, $m \vee n$ denotes the least common multiple (lcm) of m and n; for $A, B \subset \mathbb{N}$, $A \bigvee B = \{m \vee n : m \in A, n \in B\}$; and a triple $(k_1, k_2, k_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ satisfies *property P* if each number divides the

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lcm of the other two. A subset $A \subset \mathbb{N}$ is said to be closed under lcm if $m \vee n \in A$ for $m, n \in A$.

There have been a lot of papers that characterize the sets of periods for various classes of self maps, such as (i) continuous self maps of the real line $\mathbb{R}$ (see [6], [8]); (ii) polynomials on $\mathbb{C}$ (see [3]); (iii) toral automorphisms (see [7]); (iv) totally transitive maps on I (see [4]); (v) different classes of subshifts (see [1]); and (vi) (a) linear operators on $\mathbb{C}^n$ and $\mathbb{R}^n$ and on the Hilbert space l^2 and (b) isometries of Hilbert spaces (see [2]). In this paper, we determine the same for endomorphisms of an abelian group. Many important examples of dynamical systems arise as endomorphisms of topological groups (see [5]). We now state the main result of this paper.

Theorem 1.1 (Main Theorem)**.** *The following are equivalent for a subset A of $\mathbb{N}$.*

(1) $1 \in A$ *and A is closed under lcm.*
(2) *There is an abelian group G and an endomorphism T of G such that $Per(T) = A$.*

Our proofs involve some basic properties of endomorphisms of an abelian group and follow because of the decomposition of torsion groups into their prime parts. For each $m, n \in Per(T)$, the period set of an endomorphism T, we produce an element of T-period $m \vee n$, and for each $1 \in A \subset \mathbb{N}$ which is closed under lcm, we produce an automorphism T on an abelian group G such that $Per(T) = A$. We discuss the cases of torsion free abelian groups and torsion abelian groups separately. In this paper, we extend our results in [2] to a more general setting and provide a more general proof.

2. Preliminaries and Some Basic Results

An abelian group G is called a *torsion group* if every element of G has finite order and is called *torsion-free* if every element of G except the identity is of infinite order. A *homomorphism* of a group to itself is usually called an *endomorphism*; an invertible endomorphism is an *isomorphism*.

Lemma 2.1. *Let T_i be endomorphisms on abelian groups G_i for $i = 1, 2$. Let $T = T_1 \times T_2 : G_1 \times G_2 \to G_1 \times G_2$ be defined by $T(x, y) = (T_1(x), T_2(y))$. Then $Per(T)$ is the smallest subset of $\mathbb{N}$ which is closed under lcm and contains both $Per(T_1)$ and $Per(T_2)$.*

Proof. Let $x \in G_1$ such that $T_1^n(x) = x$ and $T_1^m(x) \neq x$ for all $m < n$. Then $T^n(x, 0) = (x, 0)$ and $T^m(x, 0) \neq (x, 0)$ for all $m < n$. Then $Per(T_1) \subset Per(T)$. Similarly, $Per(T_2) \subset Per(T)$.

Let $(p, q) \in G_1 \times G_2$ be T-periodic. Then $T^n(p, q) = (T_1^n(p), T_2^n(q)) = (p, q)$ for some $n \in \mathbb{N}$. It follows that p is T_1-periodic and q is T_2-periodic.

Let m_1 be the T_1-period and m_2 be the T_2-period. Let $l = m_1 \vee m_2$. Then $T^l(p,q) = (T_1^l(p), T_2^l(q)) = (p,q)$. It is noted that l is the T-period of (p,q). Then every element of $Per(T)$ is the lcm of some element of $Per(T_1)$ and of some element of $Per(T_2)$, and the lcm of every such pair of element is in $Per(T)$. Observe that $Per(T)$ is closed under lcm. Hence, the proof is complete. □

Corollary 2.2. *If we assume that $Per(T_1)$ and $Per(T_2)$ are closed under lcm in Lemma 2.1, then $Per(T) = Per(T_1) \bigvee Per(T_2)$.*

Remark 2.3. Let T_i be endomorphisms on abelian groups G_i for $i = 1,2,...,n$. Define $T = T_1 \times T_2 \times ... \times T_n : G_1 \times G_2 \times .... \times G_n \to G_1 \times G_2 \times .... \times G_n$ such that $T((x_1,x_2,...,x_n)) = (T_1(x_1), T_2(x_2),...,T_n(x_n))$. Then $\mathrm{Per}(T)$ is the smallest subset of $\mathbb{N}$ which is closed under lcm and contains $Per(T_i)$ for all $1 \le i \le n$. If we assume that $\mathrm{Per}(T_i)$'s are closed under lcm, then $\mathrm{Per}(T) = \mathrm{Per}(T_1) \vee \mathrm{Per}(T_2) \vee ... \vee \mathrm{Per}(T_n)$.

Proof. A proof similar to Lemma 2.1 and Corollary 2.2 will work. □

3. Main Results

First, we ask: Which subsets of $\mathbb{N}$ arise as sets of periods of an endomorphism on a torsion free abelian group?

Theorem 3.1. *The following are equivalent for a subset A of $\mathbb{N}$.*

(1) *$1 \in A$ and A is closed under lcm.*
(2) *There is a torsion free abelian group G and an endomorphism T of G such that $Per(T) = A$.*

Proof. (1) $\implies$ (2)

Consider $(\mathbb{C}, +)$, which is a torsion free abelian group. Let $n_0 \in \mathbb{N}$. Define $T_{n_0} : \mathbb{C} \to \mathbb{C}$ by $T_{n_0}(z) = ze^{\frac{2\pi i}{n_0}}$. Then $T_{n_0}(z)$ is an endomorphism and $Per(T_{n_0}) = \{1, n_0\}$.

Let $1 \in A \subset \mathbb{N}$, and A is closed under lcm. There are two cases to consider: (1) A is finite, (2) A is infinite.

Case 1: Suppose that A is finite, say $\{1, a_1, a_2, ..., a_n\}$.

Let $G_A = \mathbb{C}^n$. Then G_A is a torsion free abelian group with the additive structure. Define $T_A(z_1, z_2, ..., z_n) = (T_{a_1}(z_1), ..., T_{a_n}(z_n))$. Then T_A is an automorphism of G_A and $Per(T_A)$ is the smallest subset of $\mathbb{N}$ which is closed under lcm and contains $\{1, a_1, ..., a_n\} = A$. Hence, $Per(T_A) = A$ since A is closed under lcm.

Case 2: Suppose that A is infinite.

Let $G_A = \{x \in \prod_{a \in A \setminus \{1\}} C_a : x = (x_a)_{a \in A \setminus \{1\}},\ x_a = 0$ for all but finitely many $a\}$ and let $C_a = \mathbb{C}$ for all $a \in A$. Then G_A is a torsion free abelian group. Define $T_A : G_A \to G_A$ such that $T_A(x) = (y_a)_{a\in A\setminus\{1\}}$ where $y_a = T_a(x_a)$ for all $a \in A \setminus \{1\}$; i.e., $T_A = \prod_{a\in A\setminus\{1\}} T_a$. Then $Per(T_A)$ is the smallest set closed under lcm containing $\{1, a\}$ for all $a \in A \setminus \{1\}$, i.e., containing A. Hence, $Per(T_A) = A$ since A is closed under lcm.

(2) $\implies$ (1)

Let G be a torsion free abelian group and let $T : G \to G$ be an endomorphism with $Per(T) = A$.

Note that $T(0) = 0$. Hence, $1 \in Per(T)$. Next, let $x \in G$ have T-period m and let $y \in G$ have T-period n. We shall find a $k \in \mathbb{N}$ such that $x + ky$ has T-period $m \vee n$.

Step 1: The triple (T-period of x, T-period of y, T-period of $x + y$) satisfies property P.

If $T^p(a) = a$ and $T^q(b) = b$, then $T^{p\vee q}(a + b) = T^{p\vee q}(a) + T^{p\vee q}(b) = a+b$. Next, let $c, d \in G$ be any two periodic points. Taking $a = c+d$ and $b = -d$, we obtain that the T-period of c divides the lcm of the T-period of $c + d$ and the T-period of d. Hence, the result follows.

Step 2: If n is a non-zero integer, then a and na have the same T-period because if $T^p(a) = a$, then $T^p(na) = nT^p(a) = na$. Conversely, if $T^q(na) = na$, then $n(T^q(a) - a) = 0$ since G is torsion free. This implies $T^q(a) = a$.

Step 3: Let $k_1 = T$-period of $x + y$ and let $k_2 = T$-period of $x + 2y$. Then $k_1 \vee k_2 = m \vee n$.

Write $x + 2y = x + y + y$. Then the triple (k_1, k_2, n) satisfies property P by Step 1. Therefore, n divides $k_1 \vee k_2$. So, if a prime power p^r divides n, then p^r has to divide either k_1 or k_2. Now suppose that another prime power q^s divides m but not n. Then, again by Step 1, the triple (m, n, k_1) satisfies property P, and we have to have that q^s divides k_1. Then any prime power that divides m or n, should divide k_1 or k_2. Therefore, $m\vee n$ divides $k_1 \vee k_2$. Hence, $m \vee n = k_1 \vee k_2$.

Step 4: If $m, n \in \mathrm{Per}(T)$, then $m \vee n \in \mathrm{Per}(T)$.

Let k_t be the T-period of $x + ty$ where t varies over $\mathbb{Z}$. Then, by Step 3, we have $m\vee n = k_{t_1} \vee k_{t_2}$ for all $t_1 \neq t_2$. This implies that every prime power divisor of m or n should divide k_t for all $t \in \mathbb{Z}$ except at most one.

It follows that, barring finitely many elements of $\mathbb{Z}$, for all other $t \in \mathbb{Z}$, we have $m \vee n$ divides k_t. Thus, there are infinitely many elements in G whose T-period is $m \vee n$. This is more than we claimed. □

Remark 3.2. The following are equivalent for a subset A of $\mathbb{N}$.

(1) $1 \in A$ and A is closed under lcm.

(2) There are a vector space $\mathbb{V}$ over a scalar field $\mathbb{K}$ and a linear operator $T : \mathbb{V} \to \mathbb{V}$ such that $Per(T) = A$.

A *topological group* is a topological space G with a group structure such that the group multiplication $(g, h) \to gh$ and the inverse $g \to g^{-1}$ are continuous maps.

Remark 3.3. The following are equivalent for any subset A of $\mathbb{N}$.

(1) $1 \in A$ and A is closed under lcm.

(2) There is a torsion free topological abelian group G and a continuous endomorphism T of G such that $Per(T) = A$.

Now, we let S be a set and let $\phi : S \to S$ be a bijection. Let G_S be the set of all functions $f : S \to \{0, 1\}$ such that $f^{-1}(1)$ is finite. Then G_S is a group under pointwise addition modulo 1. Any element is of order 2 since $f + f = \mathbf{0}$ where $\mathbf{0}$ denotes the zero function on S. Hence, G_S is a torsion group. Define $T_\phi : G_S \to G_S$ by $T_\phi(f)(s) = f(\phi(s))$ for all $s \in S$ and $f \in G_S$. Then T_ϕ is an automorphism. The first part of the following proposition gives a relation between the period set of the bijection ϕ and the corresponding automorphism T_ϕ. The second part says that the inclusion in the first part can be strict.

Proposition 3.4. (a) *$Per(T_\phi)$ contains the smallest subset of $\mathbb{N}$ which is closed under lcm and contains $Per(\phi) \cup \{1\}$.*

(b) *Let $Per(\phi) = \{1, m\}$, and all points of S are ϕ-periodic. Then $Per(T_\phi)$ is the set of all divisors of m.*

Proof. (a) In order to prove $Per(\phi) \subset Per(T_\phi)$, let $n \in Per(\phi)$; i.e., suppose there exists $s \in S$ such that $\phi^n(s) = s$ and $\phi^m(s) \neq s$ for all $m < n$. By induction hypothesis, we can prove that $T_\phi^m f = f \circ \phi^m$ for every m. Therefore, $(T_\phi^n f)(s) = f(s)$ since $\phi^n(s) = s$. Take $f = \chi_{\{s\}}$, the characteristic function on $\{s\}$. Then $T_\phi^n f(t) = \chi_{\{s\}}(t) = 1$ if $t = s$, and $\chi_{\{s\}}\phi^n(t) = 0$ if $t \neq s$ (since ϕ^n is a bijection). Then $\chi_{\{s\}}$ is T_ϕ-periodic. For $m < n$, $T_\phi^m(f(s)) = \chi_{\{s\}}(\phi^m(s)) = 0$, but $f(s) = \chi_{\{s\}}(s) = 1$ for $s \in S$. Therefore, $T_\phi^m(f) \neq f$. Hence, the T_ϕ period of f is n. Therefore, $\mathrm{Per}(T_\phi) \supset \mathrm{Per}(\phi)$. Next, to see that $\mathrm{Per}(T_\phi)$ is closed under lcm, let s_1 be a periodic point of ϕ with period m and let s_2 be a periodic point of ϕ with period n. Take $f = \chi_{\{s_1,s_2\}}$. Then $T_\phi^{m\vee n} f(t) = \chi_{\{s_1,s_2\}}(t) = 1$ if $t \in \{s_1, s_2\}$, and $\chi_{\{s_1,s_2\}}\phi^{m\vee n}(t) = 0$ if $t \notin \{s_1, s_2\}$. If $k < m \vee n$ and $k \notin \{m, n\}$, then $T_\phi^k(f(t)) = 0$ if $t \in \{s_1, s_2\}$, and $f(t) = 1$ if $t \in \{s_1, s_2\}$. If $k = m$ or n, then $T_\phi^k f(t) = \chi_{\{s_1,s_2\}}(\phi^k(t)) = 1$ if $t = s_1$ and is equal to 0 if $t = s_2$. Therefore, $m \vee n \in \mathrm{Per}(T_\phi)$. Note that $T_\phi(0) = 0$. So $1 \in Per(T_\phi)$. This proves (a).

(b) By (a), $m \in Per(T_\phi)$. Then for some f, there are only finitely many x such that $f(\phi^d(x)) \neq f(x)$ for each divisor d of m since $f^{-1}(1)$ is finite. Define $f' : S \to \{0,1\}$ such that $f'(x) = f'(\phi^d(x))$ for each such x and $f'(x) = f(x)$ for all other x. Then $T_\phi^d(f') = f'$. If all points are ϕ-periodic and $Per(\phi) = \{1, m\}$, then $T^m f(x) = f(\phi^m(x)) = f(x)$ for all x. Therefore, if $T_\phi^k f = f$ (k is the period of T_ϕ) for some f, then k divides m. Hence, in this case, $Per(T_\phi)$ is the set of all divisors of m. □

A torsion abelian group can always be decomposed into its p-parts. That is, given an abelian group G and a prime p, let $G_p = \{a \in G :$ the order of a is a power of $p\}$. Then G_p is a subgroup of G and $G = \bigoplus_p G_p$ (the direct sum of G_p's) where p ranges over all primes. Two torsion abelian groups are isomorphic if and only if their p-components G_p are isomorphic. Therefore, by Remark 2.3, it is enough to consider the set of all endomorphisms on the abelian p-groups G_p because of the following argument. Let $T : G_p \to G_p$ be an endomorphism. If n is a non-zero integer which is not a multiple of p and a is T-periodic, then a and na have the same period because if $n(T^q(a) - a) = 0$, then $T^q(a) = a$. Then, by Proposition 3.4 and a similar argument as in Theorem 3.1 (step 3 and step 4), $Per(T)$ is closed under lcm; and, for each subset A of $\mathbb{N}$ which is closed under lcm and closed under divisors, there exists an endomorphism T on a torsion group such that $Per(T) = A$. Now we are ready to prove the main theorem.

Proof of the Main Theorem: Let G be an abelian group, tG be its torsion part, and T be an endomorphism on G. Then $T|_{tG} : tG \to tG$ is an endomorphism on the subgroup tG of G. We already proved that if $tG = \{0\}$, then $Per(T)$ is closed under lcm containing 1, and that for every such a subset of $\mathbb{N}$, there is an endomorphism T on G such that $Per(T)$ is equal to that set. Also, we proved that if $G = tG$, then $Per(T)$ is closed under lcm. Now, let $x \in G$ be a periodic point of T-period m and let $y \in G$ be a periodic point of T-period n. If both x and y are either torsion free or torsion elements, then there is a T-periodic point of period $m \vee n$. If x is a torsion element and y is a torsion free element (or vice versa), then, as in the proof of Theorem 3.1, the main theorem follows. □

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