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ON IDEMPOTENTS IN COMPACT LEFT TOPOLOGICAL UNIVERSAL ALGEBRAS

DENIS I. SAVELIEV

Abstract. A standard fact important for applications is that any compact left topological semigroup has an idempotent. We extend this to certain compact left topological universal algebras.

1. Introduction

A well-known fact is that any compact left topological semigroup has an idempotent, i.e. an element forming a subsemigroup. This firstly was established for compact topological semigroups independently by Numakura [1] and Wallace [2, 3], and in the final form (perhaps) by Ellis in [4]. This fact, despite its easy proof, is fundamental for Ramsey–theoretic applications in number theory, algebra, topological dynamics, and ergodic theory. Hindman's Finite Products Theorem, van der Waerden's and Szemerédy's Arithmetic Progressions Theorems, and Furstenberg's Multiple Recurrence Theorem can be mentioned as widely known examples. Many such applications have no (known) alternative proofs. The crucial fact for all them is the existence of idempotent ultrafilters over semigroups.

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Let us shortly recall what are *idempotent ultrafilters*. The set $\boldsymbol{\beta}X$ of ultrafilters over a set X with a natural topology generated by basic (cl)open sets $\{u \in \boldsymbol{\beta}X : S \in u\}$, $S \subseteq X$, forms the largest (Stone–Čech or Wallman) compactification of the discrete space X. If $\cdot$ is a binary operation on X, it extends to a binary operation on $\boldsymbol{\beta}X$ by letting for all $u, v \in \boldsymbol{\beta}X$

$$uv = \big\{S \subseteq X : \{a \in X : \{b \in X : ab \in S\} \in u\} \in v\big\}.$$

The extended operation is continuous in its second argument with any fixed first, i.e. the groupoid $(\boldsymbol{\beta}X, \cdot\,)$ is left topological; moreover, it is continuous in its first argument whenever the fixed second is in X. Not many algebraic properties are stable under this extension, but associativity is. Hence any semigroup X extends to the compact left topological semigroup $\boldsymbol{\beta}X$, and therefore, there exists an ultrafilter $u \in \boldsymbol{\beta}X$ that is an idempotent of the extended operation. The book [5] is a comprehensive treatise on ultrafilter extensions of semigroups and various applications; it contains also some historical remarks.

The ultrafilter extension actually is a general construction. As was shown in [7], arbitrary first-order model on X, i.e. a set X with some operations and relations on it, canonically extends to the model on $\boldsymbol{\beta}X$ such that its model-theoretic properties are, in a sense, completely analogous to the topological properties of $\boldsymbol{\beta}X$. Certainly, not all extended models contain idempotents; associativity is essential in Ellis' result. In this note, we replace it by other, much wider algebraic conditions thus showing that compact left topological universal algebras satisfying these conditions have single-point subalgebras. We also mention applications using idempotent ultrafilters over such algebras. For simplicity, we consider here only universal algebras with one or two binary operations and give outlines rather than complete proofs; this suffices however to demonstrate related ideas. General results in this direction, with detailed proofs, a number of examples, and a discussion of close concepts, can be found in [8].

2. Basic Concepts

Fix some terminology. An *algebra* is a universal algebra, i.e. a set with arbitrary operations of any arities on it. A *groupoid* is an algebra with one binary operation. As a rule, we call the operation a multiplication and write rather xy than $x \cdot y$. If F is an n-ary operation on X, an *idempotent* of F is an $a \in X$ such that $F(a, \dots, a) = a$. An *idempotent of an algebra* is a common idempotent of all its operations, i.e. an element forming a subalgebra. An *idempotent algebra* consists of idempotents only.

In the sequel, all topological spaces are assumed to be Hausdorff. An algebra endowed with a topology is *left topological* iff for any its operation F, the unary map

$$x \mapsto F(a_1, \ldots, a_n, x)$$

is continuous, for any fixed $a_1, \ldots, a_n \in X$. *Right topological* algebras are defined dually. An algebra is *semitopological* iff all unary maps obtained from any of its operation by fixing all but one argument are continuous, and *topological* iff all its operations are continuous. In particular, a groupoid is semitopological iff it is left and right topological simultaneously, and topological if its multiplication is continuous. This hierarchy does not degenerate, even for compact semigroups (see e.g. [5]).

An algebra is *minimal* iff it includes no proper subalgebras, and *minimal compact* iff it carries a compact topology and includes no proper compact subalgebras. Clearly, an algebra may include no minimal subalgebra (e.g. consider $(\mathbb{N} \setminus \{0\}, +)$, the additive semigroup of positive natural numbers). Contrary to this, any compact algebra *does* include a minimal compact subalgebra (using that the space is Hausdorff, apply Zorn's Lemma to the family of compact subalgebras ordered by the converse inclusion). Of course, idempotents form minimal subalgebras of the least possible size. For the largest possible size, note that any minimal algebra is one-generated, hence its cardinality cannot exceed $\aleph_0$ and the cardinality of its language; e.g. a minimal algebra with at most countably many operations is at most countable. The cardinality of a minimal compact algebra can be larger. As we shall see, certain algebraic properties restrict the possible size of minimal and minimal compact algebras.

An occurrence of a variable x in a term $t(x, \ldots)$ is *right-most* iff for any operation F, whenever

$$F(t_1(x, \ldots), \ldots, t_n(x, \ldots))$$

is a subterm of t, then x occurs in $t_n(x, \ldots)$ but not $t_i(x, \ldots)$ with $i < n$. E.g. in the language of groupoids, all the occurrences of the variable x in the terms x, vx, $v(vx)$, $(v_1v_2)(v_3x)$ are right-most, while all its occurrences in the terms v, xv, $x(vx)$, $(v_1x)(v_2x)$ are not. A *left-most* occurrence is defined dually. Clearly, if the occurrence of x in t is right-most (or left-most), then x occurs there exactly once.

Lemma 2.1. *Let X be a left topological algebra and $t(v_1, \ldots, v_n, x)$ a term with the right-most occurrence of the last argument. Then the map*

$$x \mapsto t(a_1, \ldots, a_n, x)$$

is continuous, for any fixed $a_1, \ldots, a_n \in X$.

Proof. By induction on the construction of t. □

3. One Operation

The following theorem generalizes Ellis' result to groupoids satisfying certain algebraic conditions.

Theorem 3.1. *Let X be a compact left topological groupoid, and let $r(v_1)$, $s(v_1, v_2)$, $t(v_1, v_2, v_3)$ be some terms, where $s(v_1, v_2)$ has the right-most occurrence of the last argument. If X satisfies*

$$s(x, y) \cdot s(x, z) = s(x, t(x, y, z)) \quad \textit{and}$$

$$s(x, y) = s(x, z) = r(x) \ \rightarrow \ s(x, yz) = r(x),$$

then it has an idempotent.

Sketch of proof. The conditions of Theorem 3.1 are universal formulas, so any subgroupoid of X should satisfy them. By using Zorn's Lemma, isolate a minimal compact subgroupoid A and show that A consists of a single point. Pick any $a \in A$. Since the occurrence of x in $s(v, x)$ is right-most, the map $x \mapsto s(a, x)$ is continuous by Lemma 2.1. Hence the first condition implies that $s(a, A) = \{s(a, b) : b \in A\}$ is a compact subgroupoid of A, whence $s(a, A) = A$, and so $B = \{b \in A : s(a, b) = r(a)\}$ is nonempty. Now the second condition implies that B is a compact subgroupoid of A, whence $B = A$, and then $aa = a$ by the first condition. □

Although the conditions of Theorem 3.1 look technical, they follow from various easy particular identities. Thus any compact left topological groupoid satisfying such identities does have an idempotent. Let us give some examples.

First af all, the associativity law implies the conditions, with $r(v) = v$, $s(v_1, v_2) = v_1 v_2$, $t(v_1, v_2, v_3) = v_2 v_1 v_3$. Thus Ellis' result follows from Theorem 3.1.

Next, let us call the following identity

$$x(yz) = (xz)y$$

left skew associativity and groupoids satisfying it *left skew semigroups*. The identity clearly follows from the conjunction of associativity and commutativity but implies neither of them. Left skew associativity also implies the conditions of Theorem 3.1, with $r(v) = v$, $s(v_1, v_2) = v_1 v_2$, $t(v_1, v_2, v_3) = (v_1 v_3) v_2$. Thus we see: *Any compact left topological left skew semigroup has an idempotent.*

There are many identities strictly weaker than associativity that imply the conditions of Theorem 3.1. E.g., so is the identity (of Bol–Moufang type)

$$(xx)(yz) = ((xx)y)z.$$

Here $r(v) = vv$, $s(v_1, v_2) = (v_1v_1)v_2$, $t(v_1, v_2, v_3) = v_2((v_1v_1)v_3)$. Examples of this kind can be easily multiplied.

Using of the conditions of Theorem 3.1 is essential; in general, neither minimal compact left topological groupoids, neither minimal groupoids, when the latter exist, need consist of a single point. E.g. there exist countable minimal commutative quasigroups (see [8] for an example). To mention a topological counterpart, consider the identity

$$x(yz) = (xy)(xz),$$

called *left distributivity*. Thus a groupoid $(X, \cdot\,)$ satisfies it iff the map $x \mapsto ax$ is an endomorphism, for any fixed $a \in X$. *Right distributivity* is defined dually, and *distributivity* is the conjunction of left and right versions.

Such groupoids arise in knot theory, where they usually are idempotents, and also in set theory, where, as Laver has shown, nontrivial elementary embeddings of V_δ into itself with their *application* operation $f \cdot g = \bigcup_{\alpha<\delta} f(g\,|_{V_\alpha})$ form a free left distributive groupoid without minimal subgroupoids. The existence of such embeddings is an extremely large cardinal axiom, and it is still a major open problem whether the axiom is necessary to prove a purely algebraic fact about certain *finite* left distributive groupoids, so-called Laver's tables (the currently best known result is that the fact is unprovable in Primitive Recursive Arithmetic; see [9] and references there).

Ježek and Kepka have shown that in distributive groupoids, terms of the form $(wx)(yz)$ are idempotents (see [10] for an elementary proof), and hence so are all terms with more than two occurrences (of the same or distinct variables). The one-sided case differs; in [11] we have shown: *All minimal left distributive groupoids are finite, and for any finite n there exists exactly one (up to isomorphism) such groupoid of cardinality n. There exists a minimal compact topological left distributive groupoid of cardinality $2^{2^{\aleph_0}}$.* (The proof of the latter fact uses the algebra of ultrafilters.)

Let us now discuss an application of Theorem 3.1. As mentioned, any groupoid uniquely extends to the compact left topological groupoid of ultrafilters over it; moreover, associativity is stable under this extension, so any semigroup extends to the semigroup of ultrafilters over it. One can ask about more stable identities. Elsewhere we prove the following sufficient condition: *Let an identity $s_1 = s_2$ be equivalent to some identity $t_1 = t_2$ such that the common variables of t_1 and t_2 appear in these terms in the same ordering, and any common variable occurs in each of the terms only once. Then the identity $s_1 = s_2$ is stable under $\boldsymbol{\beta}$.*

In particular, identities that follow from associativity are stable under $\boldsymbol{\beta}$ whenever one of its terms is *repeatless* (or *linear*), i.e. each variable occurs in it at most once. (On the other hand, it can be shown that e.g. neither commutativity, nor idempotency is stable. A question arises whether this sufficient condition is also necessary.)

An interesting case is when some identities are stable under $\boldsymbol{\beta}$ and at the same time imply the conditions of Theorem 3.1. If a groupoid satisfies such identities, one can apply Theorem 3.1 to the groupoid of ultrafilters over it, thus obtaining an idempotent ultrafilter. The identity

$$(wx)(yz) = ((wx)y)z$$

is an example. It is stable under $\boldsymbol{\beta}$ (by the condition given above) and implies the weaker identity $(xx)(yz) = ((xx)y)z$, which in turn implies the conditions of Theorem 3.1 (as we have already noted). Therefore, it provides an idempotent ultrafilter. (Note that we could not use the identity $(xx)(yz) = ((xx)y)z$, which is not repeatless and actually is *not* stable under $\boldsymbol{\beta}$.)

As mentioned, such ultrafilters allow us to obtain significant combinatorial results. E.g. the following version of Hindman's Finite Products Theorem holds:

Theorem 3.2. *If a groupoid X satisfies $(wx)(yz) = ((wx)y)z$, then any of its finite partitions has a part containing a countable sequence $a_0, \ldots, a_n, \ldots$ together with finite products*

$$a_{n_0}(a_{n_1} \ldots (a_{n_k} a_{n_{k+1}}) \ldots)$$

for all $n_0 < n_1 < \ldots < n_k < n_{k+1}$. Moreover, if X does not have idempotents or is right cancellative, one can find such a sequence consisting of pairwise distinct elements.

Proof. For the proof and various refinements, see [12]. □

4. More Operations

Theorem 3.1 can be generalized to the case of one operation of arbitrary arity, in the expected way. Let us now pass to algebras with many operations. For simplicity we consider only the case of two binary operations, denoted below as addition and multiplication, although it is possible to establish a general result about arbitrary algebras, for which we refer to [8].

Theorem 4.1. *Let $(X, +, \cdot)$ be a compact left topological algebra such that any compact subgroupoid of its additive groupoid $(X, +)$ has an idempotent. Let $q_1(v)$, $q_2(v)$, $r(v)$, $s(v_1, v_2)$, $t_1(v_1, v_2, v_3)$, $t_2(v_1, v_2, v_3)$ be some* *terms, where* $q_1(v)$, $q_2(v)$ *are additive, and* $s(v_1, v_2)$ *has the right-most occurence of the last argument. If* X *satisfies*

$$x + x = x \quad \to \quad r(x) + r(x) = r(x),$$

$$\begin{aligned} s(x, y) + s(x, z) &= s(x, t_1(x, y, z)), \\ s(x, y) \cdot s(x, z) &= s(x, t_2(x, y, z)), \end{aligned}$$

and

$$\begin{aligned} s(x, y) = s(x, z) = r(x) &\quad \to \quad s(x, y + z) = q_1(r(x)), \\ s(x, y) = s(x, z) = r(x) &\quad \to \quad s(x, yz) = q_2(r(x)), \end{aligned}$$

then it has an idempotent.

Sketch of proof. Let A be a minimal compact subalgebra of X, $a \in A$ an additive idempotent. The map $x \mapsto s(a, x)$ is continuous, hence the conditions imply firstly that $s(a, A)$ is a compact subalgebra of A, and secondly that $B = \{b \in A : s(a, b) = r(a)\}$ is a compact subalgebra of A, so $B = A$. Then $aa = a$ follows. □

Theorem 4.1 extends Theorem 3.1 since any groupoid $(X, \cdot\,)$ satisfying the conditions of Theorem 3.1, with some r, s, and t, can be turned into an algebra $(X, +, \cdot\,)$ satisfying the conditions of Theorem 4.1 by defining an extra operation $+$ as the projection onto the first argument: $x + y = x$ for all $x \in X$. In result, $(X, +)$ is a left-zero semigroup, and one can put $q_1(v) = q_2(v) = v$, $t_1(v_1, v_2, v_3) = v_2$, the same r, s, and t as t_2.

Let us consider some examples of identities implying the conditions of Theorem 4.1. The identity

$$x(y + z) = xy + xz$$

is *left distributivity of* $\cdot$ *w.r.t.* $+$. Thus an algebra $(X, +, \cdot\,)$ satisfies it iff the map $x \mapsto ax$ is an endomorphism of $(X, +)$, for any fixed $a \in X$. If $+$ and $\cdot$ coincide, this gives left distributive groupoids mentioned above. *Right distributivity of* $\cdot$ *w.r.t.* $+$ is defined dually, and *distributivity of* $\cdot$ *w.r.t.* $+$ is the conjunction of left and right versions.

An algebra $(X, +, \cdot)$ is a *left semiring* iff both its groupoids are semigroups and $\cdot$ is left distributive w.r.t. $+$. *Right semirings* are defined dually, and *semirings* are algebras that are left and right semirings simultaneously. E.g. $(\mathbb{N}, +, \cdot\,)$ is a semiring; ordinals with their usual addition and multiplication form a left semiring; if $(X, +)$ is a semigroup then $(X^X, +, \circ)$ is a left semiring where $f \circ g\,(x) = g(f(x))$; and if $(X, +)$ is a compact topological semigroup, then $(C(X), +, \circ)$ is a compact topological left semiring where $C(X)$ is the subset of X^X consisting of continuous maps, or else continuous endomorphisms, with the standard topology.

Left semirings satisfy the conditions of Theorem 4.1, with $q_1(v_1) = v_1 + v_1$, $q_2(v_1) = r(v_1) = v_1$, $s(v_1, v_2) = v_1 v_2$, $t_1(v_1, v_2, v_3) = v_2 + v_3$, $t_2(v_1, v_2, v_3) = v_2 v_1 v_3$. One gets: *Any compact left topological left semiring has an idempotent* (see [6, 13]). Thus if it is minimal compact then it consists of a single point, and an interesting question is about an algebraic counterpart of this, i.e. whether any minimal left semiring consists of a single point. This is indeed the case for *finite* left semirings (since their discrete topology is compact), and we also were able to establish the following result: *Any minimal semiring consists of a single point* (see [13]). The complete answer however seems open.

Extending $(\mathbb{N}, +, \cdot\,)$ to ultrafilters, one gets the algebra $(\boldsymbol{\beta}\mathbb{N}, +, \cdot\,)$ with two semigroups which however satisfies neither left nor right distributivity. The set $\mathbb{N}^* = \boldsymbol{\beta}\mathbb{N} \setminus \mathbb{N}$ of nonprincipal ultrafilters is its compact subalgebra. A long-standing problem is whether some three particular $a, b, c \in \mathbb{N}^*$ satisfy $a(b + c) = ab + ac$ or $(a + b)c = ac + bc$. As van Douwen shown (see [14]), such ultrafilters, if they exist at all, are topologically rare. We can take another step in the negative direction (see [13]): *Neither closed subalgebra of* $(\mathbb{N}^*, +, \cdot\,)$ *is a left semiring.* This follows from the fact that the algebra has no common idempotents (actually, no $a \in \mathbb{N}^*$ with $a + a = aa$, see [5]), despite the existence of additive idempotents as well as multiplicative ones. The complete solution of the problem also remains open. Acually, a much weaker problem whether there exist $a, b, c, d \in \mathbb{N}^*$ satisfying $a + b = cd$ remains open for decades.

As in the case of one operation, it is not difficult to provide other identities that imply the conditions of Theorem 4.1. E.g. let us generalize the concept of left semirings by preserving left distributivity of $\cdot$ w.r.t. $+$ but weakening both associativity laws to

$$\begin{aligned} ((w + x) + y) + z &= (w + x) + (y + z), \\ ((wx)y)z &= (wx)(yz). \end{aligned}$$

These algebras yet satisfy the conditions of Theorem 4.1 (required terms can be obtained from the terms for left semirings if one takes rather xx than x), so any such compact left topological algebra has an idempotent. Both identities are stable under $\boldsymbol{\beta}$, so any algebra satisfying them carries additively idempotent ultrafilters as well as multiplicatively idempotent ones. Furthermore, it can be shown that under left distributivity of $+$ w.r.t. $\cdot$ some multiplicatively idempotent ultrafilters are in the closure of the set of additively idempotent ultrafilters. This fact leads to the following result (established by Hindman and Bergelson for the semiring of natural numbers, see [5]) generalizing Theorem 3.2:

Theorem 4.2. *If* $(X, +, \cdot)$ *satisfies the two identities above and left distributivity of* $\cdot$ *w.r.t.* $+$*, then any of its finite partitions has a part containing countable sequences* $a_0, \ldots, a_n, \ldots$ *and* $b_0, \ldots, b_n, \ldots$ *together with finite sums*

$$a_{n_0} + (a_{n_1} + \ldots + (a_{n_k} + a_{n_{k+1}}) \ldots)$$

and products

$$b_{n_0}(b_{n_1} \ldots (b_{n_k} b_{n_{k+1}}) \ldots)$$

for all $n_0 < n_1 < \ldots < n_k < n_{k+1}$*. Moreover, if each of the operations does not have idempotents or is right cancellative, one can find such sequences consisting of pairwise distinct elements.*

Proof. See [12]. □

Finally, consider algebras $(X, \circ, \cdot)$ satisfying the following identities:

$$\begin{aligned} (x \circ y) \circ z &= x \circ (y \circ z), \\ (x \circ y)z &= x(yz), \\ x(y \circ z) &= xy \circ xz, \\ x \circ y &= xy \circ x. \end{aligned}$$

It follows that $(X, \cdot)$ is a left distributive groupoid (and conversely, it can be shown that any left distributive groupoid extends to such an algebra). As Laver has established, elementary embeddings with their application $\cdot$ and composition $\circ$ form algebras satisfying these identities (see [9]). Unlike the case of one left distributive operation, any such compact left topological algebra *does* have an idempotent: as $(X, \circ)$ is a semigroup, it has an idempotent a, then it easily follows from the identities that aa is a common idempotent. We think that a study of ultrafilter extensions of these algebras could throw light upon the problem of Laver's tables.

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