TOPOLOGY PROCEEDINGS
Volume 43, 2014
Pages 47–56

http://topology.auburn.edu/tp/

Another Jordan curve theorem in the topological space $(\mathbb{Z}^2, w)$

by

Josef Šlapal

Electronically published on June 7, 2013

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 43 (2014)
Pages 47-56

E-Published on June 7, 2013

ANOTHER JORDAN CURVE THEOREM IN THE TOPOLOGICAL SPACE $(\mathbb{Z}^2, w)$

JOSEF ŠLAPAL

Abstract. As an alternative to the Khalimsky topology, the topology w on the digital plane $\mathbb{Z}^2$ was introduced by the author of this note who also proved a Jordan curve theorem for it. In the present paper, another Jordan curve theorem for the topology w is proved determining a large variety of Jordan curves in the topological space $(\mathbb{Z}^2, w)$.

1. Introduction

Until late 1980's, only the classical, graph-theoretic approach to digital topology was used utilizing 4-adjacency and 8-adjacency relations for structuring the digital plane (see [9] and [10]). A disadvantage of this approach is that neither 4-adjacency nor 8-adjacency itself allows for an analogue of the Jordan curve theorem so that a combination of the two binary relations has to be used. To overcome this disadvantage, Khalimsky, Kopperman and Meyer proposed in their pioneering paper [3] a new, purely topological approach to the problem of finding a structure for the digital plane $\mathbb{Z}^2$ convenient for applications in digital image processing. They showed that the Khalimsky topology introduced in [2] provides such a convenient structure. At present, the Khalimsky topology is one of the most important concepts of digital topology and it has been studied and used by many authors, e.g., [4]-[7]. The possibility of structuring $\mathbb{Z}^2$ by closure operators more general than the Kuratowski ones is discussed in [11] and [12]. In [13], the author of this note introduced and studied another convenient topology on $\mathbb{Z}^2$ denoting it by w.

2010 *Mathematics Subject Classification.* Primary 54D05, 54A05, 54B15; Secondary 52C99, 05C40.

Key words and phrases. Alexandroff topology, quotient topology, digital plane, Jordan curve.

The author acknowledges partial support from the Grant Agency of Brno University of Technology, project no. FSI-S-11-3.

He showed, by proving a Jordan curve theorem for the topology w, that the topology has some advantages over the Khalimsky one. The topology w was then further investigated in [14] where it was shown that its quotient topologies include the Khalimsky and Marcus-Wyse (cf. [8]) topologies as well as some other interesting topologies and pretopologies on $\mathbb{Z}^2$. In the present note, we continue the study of the topology w and prove a new Jordan curve theorem for it. This Jordan curve theorem, together with the one proved in [13], shows that the topology w possesses a rich variety of Jordan curves and may, therefore, be used as a convenient structure on the digital plane for solving problems of digital image processing.

2. Preliminaries

For the topological terminology used we refer to [1]. Throughout the note, all topologies dealt with are thought of as being (given by) Kuratowski closure operators. By a *graph* on a set V we always mean an undirected simple graph without loops whose vertex set is V. Recall that a *path* in a graph is a finite (nonempty) sequence $x_0, x_1, ..., x_n$ of pairwise different vertices such that x_{i-1} and x_i are adjacent (i.e., joined by an edge) whenever $i \in \{1, 2, ...n\}$. By a *cycle* in a graph we understand any finite set of at least three vertices which can be ordered into a path whose first and last members are adjacent. The *connectedness graph* of a topology p on X is the graph on X in which a pair of vertices x, y is adjacent if and only if $x \neq y$ and $\{x, y\}$ is a connected subset of (X, p). Let p be an Alexandroff topology on a set X. Then a subset $A \subseteq X$ is connected in (X, p) if and only if each pair of (different) points of A may be joined by a path in the connectedness graph of (X, p) contained in A. Clearly, p is given by its connectedness graph provided that every edge of the graph is adjacent to a point which is known to be closed or to a point which is known to be open (in which case p is T_0). Indeed, the closure of a closed point consists of just this point, the closure of an open point consists of this point and all points adjacent to it and the closure of a mixed point (i.e., a point that is neither closed nor open) consists of this point and all closed points adjacent to it. In the sequel, only the connectedness graphs of some connected Alexandroff topologies on $\mathbb{Z}^2$ will be considered in which the closed points will be ringed and the mixed ones boxed (so that the points neither ringed nor boxed will be open - note that no point of $\mathbb{Z}^2$ may be both closed and open).

By a *(discrete) closed curve* in a topological space (X, p) we mean a cycle in the connectedness graph of p. Thus, every closed curve is a nonempty, finite and connected set. In accordance with [14], a closed curve $C \subseteq X$ in (X, p) is said to be *simple* if, for each point $x \in C$, there are exactly two points of C adjacent to x in the connectedness graph of p.

A simple closed curve C in (X,p) is said to be a *(discrete) Jordan curve* if it separates (X,p) into precisely two components (i.e., if the subspace $X-C$ of (X,p) consists of precisely two components).

Since quotient topologies of Alexandroff topologies will play an important role in this note, we will start with presenting some general facts concerning their behavior. It may easily be seen that, given an Alexandroff space (X,p), a topology q on a set Y is the quotient topology of p generated by a surjection $e: X\to Y$ if and only if q is an Alexandroff topology on Y with the property that, for every pair of points $x,y\in Y$, $x\in q\{y\}$ if and only if there are $a\in e^{-1}(x)$ and $b\in e^{-1}(y)$ such that $a\in p\{b\}$. Using this fact, we will easily prove the following

Lemma 2.1. *Let (X,p) be an Alexandroff space, $e: X\to Y$ be a surjection and let q be the quotient topology of p on Y generated by e. Let e have connected fibres (i.e., let $e^{-1}(\{y\})$ be connected in (X,p) for every point $y\in Y$) and let $B\subseteq Y$ be a subset. Then B is connected in (Y,q) if and only if $e^{-1}(B)$ is connected in (X,p).*

Proof. If $e^{-1}(B)$ is connected, then so is B because $B=e(e^{-1}(B))$. Conversely, let B be connected and let $x,y\in e^{-1}(B)$ be an arbitrary pair of different points. Then there is a path $e(x)=z_0,z_1,...,z_n=e(y)$ in the connectedness graph of p contained in B. Consequently, we have $z_i\in q\{z_{i-1}\}$ or $z_{i-1}\in q\{z_i\}$ for each $i\in\{1,2,...,n\}$. Thus, for each $i\in\{1,2,...,n\}$, there are points $y_i\in e^{-1}(z_i)$ and $x_{i-1}\in e^{-1}(z_{i-1})$ such that $y_i\in p\{x_{i-1}\}$ or $x_{i-1}\in p\{y_i\}$. It means that, for each $i\in\{1,2,...,n\}$, there is a point in $e^{-1}(z_i)$ adjacent to a point of $e^{-1}(z_{i-1})$ in the connectedness graph of p. Since $x\in e^{-1}(z_0)$, $y\in e^{-1}(z_n))$ and $e^{-1}(z_i)$ is connected for each $i\in\{1,2,...,n\}$, there is a path in the connectedness graph of p contained in $e^{-1}(B)$ and connecting x and y. Thus, $e^{-1}(B)$ is connected in (X,p). □

Let us note that, in general, for a topological space (X,p) and a quotient topology of p, the statement of the previous Lemma need not be true. It is true if, for example, B is closed or open in (Y,q).

Proposition 2.2. *Let (X,p) be an Alexandroff space and let q be the quotient topology of p on a set Y generated by a surjection $e: X\to Y$. Let e have connected fibres and let $C\subseteq X$ be a simple closed curve in (X,p). Then C is a Jordan curve in (X,p) if the following two conditions are satisfied:*

1. *$e(C)$ is a Jordan curve in (Y,q), i.e., separates (Y,q) into exactly two components D_1 and D_2.*
2. *For every component E of the subspace $e^{-1}(e(C))-C$ of (X,p), there are $x\in E$ and $i_E\in\{1,2\}$ such that x is adjacent to a point of $e^{-1}(D_{i_E})$ but no point of E is adjacent to a point of $e^{-1}(D_{3-i_E})$ (in the connectedness graph of p).*

Proof. Let the conditions (1) and (2) of the statement be fulfilled and, for every $i \in \{1, 2\}$, put

$$C_i = e^{-1}(D_i) \cup \bigcup\{E;\ E \text{ is a component of } e^{-1}(e(C)) - C \text{ with } i_E = i\}.$$

Since $e^{-1}(D_i)$ is connected for $i \in \{1, 2\}$ by Lemma 2.1, C_1 and C_2 are connected. Clearly, $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = X - C$ is not connected. Therefore, C_1 and C_2 are the only components of the subspace $X - C$ of (X, p). Thus, C is a Jordan curve in (X, p). □

Let $z = (x, y) \in \mathbb{Z}^2$ be a point. We put
$H_2(z) = \{(x + k, y);\ k \in \{-1, 1\}\}$,
$V_2(z) = \{(x, y + l);\ l \in \{-1, 1\}\}$,
$D_5(z) = H_2(z) \cup \{(x + k, y - 1);\ k \in \{-1, 0, 1\}\}$,
$U_5(z) = H_2(z) \cup \{(x + k, y + 1);\ k \in \{-1, 0, 1\}\}$,
$L_5(z) = V_2(z) \cup \{(x - 1, y + l);\ l \in \{-1, 0, 1\}\}$,
$R_5(z) = V_2(z) \cup \{(x + 1, y + l);\ l \in \{-1, 0, 1\}\}$.
Next, we put
$A_4(z) = H_2(z) \cup V_2(z)$,
$A_8(z) = L_5(z) \cup R_5(z) (= D_5(z) \cup U_5(z))$, and
$D_4(z) = A_8(z) - A_4(z)$.
Thus, the number of points of each of the nine sets introduced above equals the index of the symbol denoting this set. In the literature, the points of $A_4(z)$ and $A_8(z)$ are said to be *4-adjacent* and *8-adjacent* to z, respectively. It is natural to call the points of $H_2(z)$, $V_2(z)$, $D_5(z)$, $U_5(z)$, $L_5(z)$, $R_5(z)$ and $D_4(z)$ *horizontally 2-adjacent, vertically 2-adjacent, down 5-adjacent, up 5-adjacent, left 5-adjacent, right 5-adjacent* and *diagonally 4-adjacent* to z, respectively. Clearly, each of these adjacencies implies 8-adjacency.

The union of each of the above nine sets $H_2(z)$, $V_2(z)$... with the singleton $\{z\}$ is denoted by the corresponding barred symbols, i.e., by $\bar{H}_2(z)$, $\bar{V}_2(z)$....

Recall [2] that the Khalimsky topology on $\mathbb{Z}^2$ is the Alexandroff topology t given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$t\{z\} = \begin{cases} \bar{A}_8(z) \text{ if } x, y \text{ are even,} \\ \bar{H}_2(z) \text{ if } x \text{ is even and } y \text{ is odd,} \\ \bar{V}_2(z) \text{ if } x \text{ is odd and } y \text{ is even,} \\ \{z\} \text{ otherwise.} \end{cases}$$

The Khalimsky topology is connected and T_0; a portion of its connectedness graph is shown in Figure 1. In the literature, the Khalimsky topology on $\mathbb{Z}^2$ is usually defined as the product topology obtained from two copies of the topology on $\mathbb{Z}$ given by the subbase $\{\{2n, 2n+1, 2n+2\};\ n \in \mathbb{Z}\}$, which is also called Khalimsky (it is easy to see that this topology on $\mathbb{Z}^2$ coincides with t). The topology $\overline{t}$ dual to t is called Khalimsky, too.

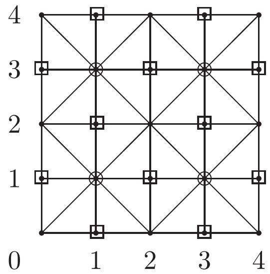


FIGURE 1. A portion of the connectedness graph of the Khalimsky topology.

The topological space $(\mathbb{Z}^2, t)$ is called the *Khalimsky plane*. The following Jordan curve theorem in the Khalimsky plane was proved by Khalimsky, Kopperman and Meyer in [3]:

Proposition 2.3. *In the Khalimsky plane, any simple closed curve having at least four points is a Jordan curve.*

3. Topology w

We denote by w the Alexandroff topology on $\mathbb{Z}^2$ given as follows: For any point $z = (x, y) \in \mathbb{Z}^2$,

$$w\{z\} = \begin{cases} \bar{A}_8(z) \text{ if } x = 4k,\ y = 4l,\ k, l \in \mathbb{Z}, \\ \bar{D}_4(z) \text{ if } x = 2 + 4k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \bar{D}_5(z) \text{ if } x = 2 + 4k,\ y = 1 + 4l,\ k, l \in \mathbb{Z}, \\ \bar{U}_5(z) \text{ if } x = 2 + 4k,\ y = 3 + 4l,\ k, l \in \mathbb{Z}, \\ \bar{L}_5(z) \text{ if } x = 1 + 4k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \bar{R}_5(z) \text{ if } x = 3 + 4k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \bar{H}_2(z) \text{ if } x = 2 + 4k,\ y = 4l,\ k, l \in \mathbb{Z}, \\ \bar{V}_2(z) \text{ if } x = 4k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \text{ otherwise.} \end{cases}$$

Clearly, w is connected and T_0. A portion of the connectedness graph of w is shown in Figure 2. Observe that the connectedness graph of w is a subgraph of the 8-adjacency graph. Evidently, such a property is necessary for a topology on $\mathbb{Z}^2$ to be useful for applications in digital topology.

We will need the following immediate consequence of [14], Theorem 10:

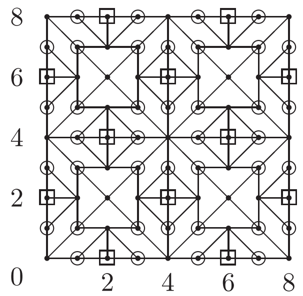


FIGURE 2. A portion of the connectedness graph of w.

Theorem 3.1. *The Khalimsky topology is the quotient topology of w generated by the surjection $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ given as follows:*

$$f(x,y) = \begin{cases} (2k, 2l) \text{ if } (x,y) = (4k, 4l), \ k,l \in \mathbb{Z}, \\ (2k, 2l+1) \text{ if } (x,y) \in \bar{A}_4(4k, 4l+2), \ k,l \in \mathbb{Z}, \\ (2k+1, 2l) \text{ if } (x,y) \in \bar{A}_4(4k+2, 4l), \ k,l \in \mathbb{Z}, \\ (2k+1, 2l+1) \text{ if } (x,y) \in \bar{D}_4(4k+2, 4l+2), \ k,l \in \mathbb{Z}. \end{cases}$$

The surjection f is demonstrated in Figure 3 where the corresponding decomposition of the topological space $(\mathbb{Z}^2, w)$ is marked by the dashed lines. All points of a class of the decomposition are mapped by f to the center point of the class expressed in the bold coordinates.

The following Jordan curve theorem in $(\mathbb{Z}^2, w)$ immediately follows from [13], Theorem 11:

Theorem 3.2. *Every cycle in the graph as partly shown by Figure 4 is a Jordan curve in $(\mathbb{Z}^2, w)$.*

As the main result of this note, we will prove another Jordan curve theorem in $(\mathbb{Z}^2, w)$.

Consider the following two conditions for a cycle C in the topological space $(\mathbb{Z}^2, w)$:

(1) $\bar{H}_2(z) \not\subseteq C$ whenever $z = (4k+2, 2l+1)$ for some $k, l \in \mathbb{Z}$ and $\bar{V}_2(z) \not\subseteq C$ whenever $z = (2k+1, 4l+2)$ for some $k, l \in \mathbb{Z}$.

(2) If $z = (4k, 4l)$ for some $k, l \in \mathbb{Z}$, then $z \notin C$.

As the main result of this note, we get the following digital analogue of the Jordan curve theorem:

Theorem 3.3. *In the topological space $(\mathbb{Z}^2, w)$, every simple closed curve C with at least eight points satisfying conditions (1) and (2) is a Jordan curve.*

Proof. Let C be a simple closed curve in $(\mathbb{Z}^2, w)$ with at least eight points satisfying conditions (1) and (2) and let f be the surjection from Theorem 3.1. Then it is easy to see, with the help of Figure 3, that $f(C)$ is a simple

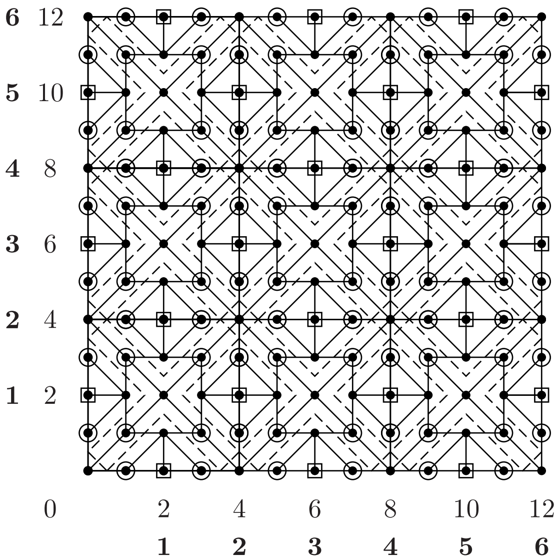


FIGURE 3. Decomposition of $(\mathbb{Z}^2, w)$ given by the surjection f.

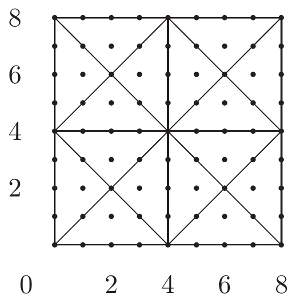


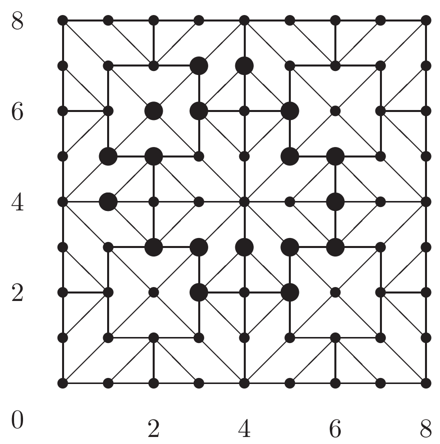
FIGURE 4. A portion of a subgraph of the connectedness graph of w.

closed curve in the Khalimsky plane having at least four points. Thus, by Theorem 2.3, $f(C)$ is a Jordan curve.

Clearly, we have $\mathbb{Z}^2 - \{(4k, 4l);\ k, l \in \mathbb{Z}\} = \bigcup\{\bar{A}_4(z);\ z = (4k, 4l+2)$ or $z = (4k+2, 4l),\ k, l \in \mathbb{Z}\} \cup \bigcup\{\bar{D}_4(4k+2, 4l+2);\ k, l \in \mathbb{Z}\}$. Let $z \in \bar{A}_4(4k, 4l+2)$ or $z \in \bar{A}_4(4k+2, 4l)$ for some $k, l \in \mathbb{Z}$, i.e., let $z \in \bar{A}_4(x, y)$ where (x, y) is a mixed point in $(\mathbb{Z}^2, w)$. Then $f^{-1}(f(z)) = A_4(x, y)$ and $f^{-1}(f(z)) - C$ consists of one or two components. In the case of two components, they are singletons lying in different components of $f^{-1}(f(C)) - C$ and each of

them is adjacent, in the connectedness graph of w, to exactly one point of $\mathbb{Z}^2 - f^{-1}(f(z))$, namely, a point $(4m, 4n)$, $m, n \in \mathbb{Z}$. On the other hand, in the case of one component, there is exactly one point in this component adjacent, in the connectedness graph of w, to a point of $\mathbb{Z}^2 - f^{-1}(f(C))$, namely, a point $(2m, 2n)$, $m, n \in \mathbb{Z}$. Further, let $z \in \bar{D}_4(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$. Then $f^{-1}(f(z)) - C$ consists of two singleton components. If C does not turn at the point z, then the components are contained in different components of $f^{-1}(f(C)) - C$ and each of them is adjacent to exactly one point of $\mathbb{Z}^2 - f^{-1}(f(C))$, namely, a point $(4m, 4n)$, $m, n \in \mathbb{Z}$. If C turns at z, then both components of $f^{-1}(f(z)) - C$ are contained in the same component of $f^{-1}(f(C)) - C$ and each of them is adjacent to exactly two points - a point $(4m, 4n)$, $m, n \in \mathbb{Z}$, and the point lying between these components (indeed, there is a point $(x, y) \in \mathbb{Z}^2$ such that the two (singleton) components of $f^{-1}(f(z)) - C$ coincide with the two points of $V_2(x, y)$ or $H_2(x, y)$, respectively, and, therefore, each of them is adjacent to (x, y)). It follows from (1) and (2) that all points adjacent to any of the two components of $f^{-1}(f(z)) - C$ lie in exactly one of the sets $f^{-1}(D_1)$ and $f^{-1}(D_2)$ where D_1 and D_2 denote the two components of the subspace $\mathbb{Z}^2 - f(C)$ of the Khalimsky plane.
Since $f^{-1}(f(C)) - C = \bigcup_{z \in C}(f^{-1}(f(z)) - C)$, we have shown that, in every component of $f^{-1}(f(C)) - C$ of $(\mathbb{Z}^2, w)$, there is a point adjacent to a point of exactly one of the sets $f^{-1}(D_1)$ and $f^{-1}(D_2)$. By Proposition 2.2, C is a Jordan curve in $(\mathbb{Z}^2, w)$. □

Example 3.4. By Theorem 3.3, in the following figure, the points denoted by bold dots constitute a Jordan curve in $(\mathbb{Z}^2, w)$:



Remark 3.5. Clearly, the set of Jordan curves identified in Theorem 3.2 is disjoint from that one identified in Theorem 3.3 and the two Theorems show that the topology w possesses a rich variety of Jordan curves and may, therefore, be used for solving problems of digital image processing. Of course, there are Jordan curves in $(\mathbb{Z}^2, w)$ that neither Theorem 3.2 nor Theorem 3.3 has identified. These are, for example, the simple closed curves $A_4(4k+2, 4l)$, $A_4(4k, 4l+2)$ and $A_8(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$.

We determined Jordan curves in the topological space $(\mathbb{Z}^2, w)$ by using Jordan curves in a quotient space of $(\mathbb{Z}^2, w)$, the Khalimsky plane. We may also apply the converse procedure, namely, having determined Jordan curves in the topological space $(\mathbb{Z}^2, w)$, use them to determine Jordan curves in quotient topologies of w on $\mathbb{Z}^2$ (different from the Khalimsky one). This was done in [15] and [16] where Theorem 3.2 was used to prove Jordan curve theorems for certain quotient topologies and pretopologies of w on the digital plane. Analogously, we may use Theorem 3.3 to determine further Jordan curves for quotient topologies (and pretopologies) of w on the digital plane.

Acknowledgement. The author acknowledges partial support from Grant Agency of the Brno University of Technology, project no. FSI-S-11-3.

References

[1] Engelking, R., General Topology. Państwowe Wydawnictwo Naukowe, Warszawa, (1977).

[2] Khalimsky, E.D., *On topologies of generalized segments*, Soviet Math. Dokl., **10** (1999), 1508–1511.

[3] Khalimsky, E.D., Kopperman, R., Meyer, P.R., *Computer graphics and connected topologies on finite ordered sets*, Topology Appl., **36** (1990), 1–17.

[4] Khalimsky, E.D., Kopperman R., Meyer, P.R., *Boundaries in digital planes*, Jour. of Appl. Math. and Stoch. Anal., **3** (1990), 27–55.

[5] Kiselman, C.O., *Digital Jordan curve theorems*, Lect. Notes Comput. Sc., **1953** (2000), 46–56.

[6] Kong, T.Y., Kopperman, R., Meyer, P.R., *A topological approach to digital topology*, Amer. Math. Monthly, **98** (1991), 902–917.

[7] Kopperman, R., Meyer, P.R., Wilson, R.G., *A Jordan surface theorem for three-dimensional digital spaces*, Discr. and Comput. Geom., **6** (1991), 155–161.

[8] Marcus, D. at al., *A special topology for the integers (Problem 5712)*, Amer. Math. Monthly, **77** (1970), 1119.

[9] Rosenfeld, A., *Digital topology*, Amer. Math. Monthly, **86** (1979), 621–630.

[10] ———, *Picture Languages*, Academic Press, New York (1979).

[11] Šlapal, J., *Closure operations for digital topology*, Theor. Comp. Sci., **305** (2003), 457–471.

[12] ———, *A digital analogue of the Jordan curve theorem*, Discr. Appl. Math., **139** (2004), 231–251.

[13] ———, *Digital Jordan curves*, Top. Appl., **153** (2006), 3255–3264.

[14] ———, *A quotient-universal digital topology*, Theor. Comp. Sci., **405** (2008), 164–175.
[15] ———, *Jordan curve theorems with respect to certain pretopologies on* $\mathbb{Z}^2$, Lect. Notes Comput. Sc., **5852** (2009), 425–436.
[16] ———, *A Jordan curve theorem in the digital plane*, Lect. Notes Comput. Sc., **6636** (2011), 120–131.

Brno University of Technology, Department of Mathematics, 616 69 Brno, Czech Republic
E-mail address: slapal@fme.vutbr.cz