TOPOLOGY PROCEEDINGS
Volume 43, 2014
Pages 57–67

http://topology.auburn.edu/tp/

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Electronically published on July 6, 2013

Topology Proceedings

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Mail:	Topology Proceedings
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

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Volume 43 (2014)
Pages 57-67

E-Published on July 6, 2013

RETRACTS OF TOPOLOGICAL GROUPS AND COMPACT MONOIDS

KARL H. HOFMANN AND JOHN R. MARTIN

Abstract. In this note a space which is homeomorphic to a retract of a topological group is called a GR-space and properties which a GR-space must possess are investigated. GR-spaces have earlier been called retral spaces by J. van Mill and G. J. Ridderbos (2006). Every compact space which admits a topological left-loop structure is a GR-space and every GR-space admits an H-space structure. For every positive dimension there are compact connected commutative monoids with zero which fail to be GR-spaces. A characterization is given for the compact GR-spaces which are homeomorphs of n-spheres, real projective n-spaces, compact surfaces, compact bordered surfaces and absolute retracts for the class of compact Hausdorff spaces. In the process, we observe and prove that the Möbius band is both a submonoid and a homotopy retract of the solid torus as compact topological commutative semigroup with identity.

1. Introduction

In this note all spaces are Hausdorff and the term map or mapping shall always mean continuous function.

The concepts of a topological left-loop and an H-space are two generalizations of the notion of a topological group and relationships between these two concepts are discussed in [10]. In this note another generalization of a topological group is introduced which we call a GR-space and

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2010 *Mathematics Subject Classification.* 54H11, 54C15, 54C55, 55P45.

Key words and phrases. Topological group, topological left-loop, H-space, retract, absolute (neighborhood) retract, compact (bordered) surface, compact topological semigroup.

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relationships among GR-spaces, H-spaces, absolute retracts and topological left-loops are discussed. In the case of compact manifolds, GR-spaces often admit an algebraic structure which is richer than that of an H-space. For instance, it is shown that the compact bordered surfaces which are GR-spaces (the closed unit disc, the annulus, and the Möbius band) are all submonoids of the solid torus which is a GR-space. In fact, the noncontractible compact GR-spaces considered in this note which are not topological groups to begin with are either commutative monoids or Moufang loops.

Recall that a space Q has the *fixed point property* (fpp) if every self-mapping of Q has a fixed point. In 2006, J. van Mill and G. J. Ridderbos already investigated GR-spaces under the name of retral spaces. Their results imply that every GR-space continuum with the fixed point property is locally connected, and they observed that the pseudo-arc is not a GR-space.

Definition 1.1. A space X is called a *topological group retract* (or *GR-space*) if X is homeomorphic to a retract of some topological group G.

A space X is said to be an *H-space* if there is a mapping $m\colon X \times X \to X$ and an identity element $e \in X$ such that $m(e, x) = m(x, e) = x$ for all $x \in X$. If the multiplication m is associative, then X is called a *topological monoid*. If a space has the property that for every $a, b \in X$ there is an autohomeomorphism h of X such that $h(a) = b$, then X is said to be *homogeneous*. In [10] it is shown that if X is a retract of a homogeneous H-space, then X admits an H-space structure. Since a topological group is a homogeneous H-space and every topological group is a retract of itself, it follows that

Every topological group is a GR*-space and every* GR*-space admits an H-space structure.*

In [10] we proposed the following definitions:

Definition 1.2. A *left-loop* is a set X with a multiplication $(x, y) \longmapsto xy$ and a right identity e (i.e., $xe = x$ for all $x \in X$) such that for all $a, b \in X$ the equation $ax = b$ is uniquely solvable for x. The solution x is denoted $a\backslash b$.

A *topological left-loop* is a topological space X whose underlying set is a left-loop and is such that the functions $(x, y) \longmapsto xy$ and $(x, y) \longmapsto x\backslash y$ from $X \times X$ into X are continuous.

It was shown in [10] that a compact space carries the structure of a topological left loop if and only if it is homeomorphic to a quotient space of a topological group such that the quotient map has a continuous cross section. In particular, under these circumstances it is a GR-space.

2. Elementary Properties of GR-Spaces

A space X is called an AR(*compact*)-*space* if it is a retract of every compact space Z in which it is imbedded. It is well known that a space X is an AR(compact)-space iff it is homeomorphic to a retract of some Tychonoff cube ([16, Thm. 3.6]).

Proposition 2.1. (i) *Every* GR-*space* X *admits an* H-*space structure. Furthermore, if* X *is homeomorphic to a retract of an abelian topological group, then* X *admits a commutative* H-*space structure.*

(ii) *The fundamental group of an arcwise connected* GR-*space* X *is abelian.*

(iii) *If* X *is a* GR–*space which is homeomorphic to a retract of a connected Lie group, then the universal covering space of* X *is a* GR-*space.*

(iv) *A retract of a* GR-*space is a* GR-*space.*

(v) *A product space* $X = \prod_{\alpha \in J} X_\alpha$ *is a* GR-*space iff each factor space* X_α *is a* GR-*space.*

(vi) *Every* AR*(compact)-space is a* GR-*space. In fact, every* AR *(compact)-space is homeomorphic to a retract of a compact group.*

(vii) *The disjoint sum* $X = \sum_{\alpha \in J} X_\alpha$ *of a nonempty family of spaces is a* GR-*space iff each* X_α *is a* GR-*space.*

(viii) *The class of* H-*spaces* properly *contains the class of* GR-*spaces. In fact, for every positive dimension there is a contractible continuum which is not a* GR-*space but which admits the structure of a commutative monoid.*

Proof. (i) ([10]) Let G be a topological group and let $r\colon G \to X$ be a retraction from G onto X. Since G is a homogeneous space, X is homeomorphic to a retract of G which contains the identity element e of G. Consequently, we may assume that $e \in X$. Define a mapping $m\colon X \times X \to X$ by $m(x, y) = r(xy)$ for all $x, y \in X$. Then, for all $x, y \in X$, we have

(1) $m(e, y) = r(ey) = r(y) = y,$
$m(x, e) = r(xe) = r(x) = x,$

and, if G is abelian,

(2) $m(x, y) = r(xy) = r(yx) = m(y, x).$

(ii) An arcwise connected GR-space admits an H-space structure and hence has an abelian fundamental group.

(iii) Suppose that X is a retract of a connected Lie group G. Since the properties of arcwise connectedness and local contractibility are preserved by retractions, X is semilocally simply connected and therefore has a universal covering space $\widetilde{X}$. Moreover, since X is a retract of G, $\widetilde{X}$ is a retract of the universal covering group $\widetilde{G}$ ([17, Thm. 1.1]).

(iv) If G is a topological group and $r\colon G \to Y$ and $s\colon Y \to X$ are retractions, then the composition mapping $sr\colon G \to X$ is a retraction from G onto X.

(v) Let $X = \prod_{\alpha \in J} X_\alpha$ be a GR-space. Then, since each factor space X_α is homeomorphic to a retract of X, each X_α, $\alpha \in J$, is a GR-space by (iv). Now suppose for each $\alpha \in J$, the mapping $r_\alpha\colon G_\alpha \to X_\alpha$ is a retraction from a topological group G_α onto X_α. Let $G = \prod_{\alpha \in J} G_\alpha$. Then $z \mapsto (r_\alpha(z)) : G \to X$ determines a retraction from G onto X.

(vi) It suffices to show that every retract of a Tychonoff cube is a GR-space. But this follows from (iv) and (v) since the closed unit interval $\mathbb{I} = [0, 1]$ is a retract of the real line $\mathbb{R}$. Since $\mathbb{I}$ is also a retract of the circle group $\mathbb{S}^1$, it follows that every AR(compact)-space is homeomorphic to a retract of a compact group.

(vii) Suppose that the disjoint sum $X = \sum_{\alpha \in J} X_\alpha$ is a GR-space. Let $p \in X_\alpha$ and define a function $r\colon X \to X_\alpha$ from X onto X_α by

$$r(x) = \begin{cases} x & \text{if } x \in X_\alpha, \\ p & \text{if } x \notin X_\alpha. \end{cases}$$

Then, since each X_α is an open (and closed) subset of X, the mapping r is continuous and a retraction from X onto X_α.
For a proof of the converse, now suppose that each X_α, $\alpha \in J$, is a GR-space. Let G be a group such that $\operatorname{card} G = \operatorname{card} J$ and topologize G by endowing it with the discrete topology. Then $X = \sum_{g \in G} X_g$. Let X_g be a retract of a topological group H_g, $g \in G$, and consider the topological group $H = \prod_{g \in G} H_g \times G$. Denote the projection map from H onto the factor G by $p\colon H \to G$. Then $\{p^{-1}(g) | g \in G\}$ is a pairwise disjoint open cover of H. For each $g \in G$, let r_g be a retraction which maps $p^{-1}(g)$ onto a homeomorphic copy of X_g. These retraction mappings determine a retraction from H onto a homeomorphic copy of X as required.

(viii) In [14], three examples are given of contractible plane continua which admit the structures of commutative monoids, namely, the Cantorian swastika (for a picture see [11, p. 270, Figure 8]), two Cantor fans tangent along a segment, and the "closed up" $\sin(1/x)$–curve in the plane together with "its interior", that is, the bounded component of its complement in the plane. The examples all fail to be locally connected. Moreover, since an arcwise connected plane continuum has the fixed point property iff its fundamental group is trivial [6], it follows that each example has the fixed point property. In [15] it is shown that every continuum with the fixed point property that is a retract of a topological group is locally connected. Consequently, none of the three examples is a GR-space. The product of any one of the three examples with a topological power of the closed unit interval $\mathbb{I}$ yields a contractible continuum which admits the structure of a commutative monoid with zero. Such a product space cannot be a GR-space by Proposition 2.1(v). As the first two examples given have dimension one, the assertion follows. □

We remark that COHEN [4] exhibited a compact contractible two-dimensional monoid with zero whose underlying space fails to have the fixed point property (see [11. p. 250, Example 5.1.1], and [13]).

Remark 2.2. There are many examples of continua which are not GR-spaces but which admit the structures of commutative monoids. Recall that $\mathbb{I}$ denotes the unit interval and $\mathbb{D}$ the complex unit disk under multiplication. The monoid

$$C \stackrel{\text{def}}{=} \{(e^{-r}, e^{2\pi i r}) \in \mathbb{I} \times \mathbb{D} : 0 \leq r \in \mathbb{R}\} \cup \{0\} \times \mathbb{D}$$

consisting of a spiral winding down onto the boundary of a disk is not locally connected and has the fixed point property. Consequently, an arbitrary product with factors consisting of C, compact connected Abelian groups, closed unit intervals and Möbius bands would provide examples.

Remark 2.3. In [15] VAN MILL AND RIDDERBOS show that the pseudo-arc is not a GR-space. Since A. L. HUDSON and P. S. MOSTERT have shown in [12] that a finite-dimensional homogeneous continuum that admits a monoid structure must be a group, the pseudo-arc does not admit the structure of a monoid. The the Hudson-Mostert result can also be applied to positive-dimensional Menger curves since they are finite-dimensional homogeneous continua that do not admit the structure of topological left-loops by [10].

3. GR-Spaces and Compact Monoids

A metrizable AR(compact)-space is called an AR-*space* and every such space is homeomorphic to a retract of the Hilbert cube Q. Since Q has the *fixed point property* (fpp) and retractions preserve the fpp, it follows that every AR-space has the fpp. An ANR-*space* is a space which is homeomorphic to a neighbourhood retract of Q.

A self-mapping f of a space X is called a *deformation* if f is homotopic to the identity map 1_X.

Proposition 3.1. *Let X be a* GR*-space which is a connected* ANR*-space. Then either the Euler characteristic $\chi(X)$ vanishes or X is an* AR*-space.*

Proof. Since an ANR-space has the homotopy type of a compact polyhedron ([3, Cor. 44.2]), it follows that X admits an H-space structure and has the homotopy type of a connected compact polyhedron. Then, by the remarks following Corollary IV.2 in [5], $\chi(X) = 0$ or $\chi(X) = 1$, in which case X is contractible. Since a contractible ANR-space is an AR-space ([1, p. 101]), the result follows. □

Corollary 3.2. *Let M be a* GR*-space which is a connected compact n-manifold. Then either M is contractible and has the fixed point property, or $\chi(M) = 0$ and M admits a fixed point free deformation.*

Proof. A finite-dimensional compact metric space is an ANR-space iff it is locally contractible ([1, p. 122]) so M satisfies the hypotheses of Proposition 3.1. It only remains to note (see [2, p. 145]) that a compact n-manifold with $\chi(M) = 0$ admits a fixed point free deformation. □

We note that Corollary 3.2 shows that, if $n > 0$, no $2n$-sphere $\mathbb{S}^{2n}$, real projective $2n$-space $\mathbb{RP}^{2n}$ or complex (or quaternionic) projective space is a GR-space (see [2, pp. 30–33]).
Proposition 3.2 in [10] involves the fixed point sets of a smooth compact n-manifold M having nondegenerate topological left-loops as boundary components. The result depends only on the fact that if X is a nondegenerate compact topological left-loop, then $\chi(X) = 0$. It follows from Poincaré duality (see [7, Ex. 33, p. 260]) that the boundary ∂C of a compact contractible manifold C is a homology sphere and, consequently, C could not serve as a boundary component of M. Thus Corollary 3.2 above applies and the concept of a compact topological left-loop can be replaced by the more general notion of a compact GR-space in Proposition 3.2 in [10] to obtain the following result.

Proposition 3.3. *Every nonempty closed subset of a smooth compact n-manifold M whose boundary components are nondegenerate* GR*-spaces is the fixed point set of an autodiffeomorphism of M. In the case where M is a noncontractible* GR*-space the word nonempty can be deleted.* □

In [10, Prop. 1.3] it is shown that every topological left-loop is a homogeneous space. It follows that no *bordered surface* (a connected 2-manifold with nonempty boundary) admits a topological left-loop structure. However, Proposition 2.1 shows that the closed unit disk $\mathbb{D}$ and the annulus $\mathbb{S}^1 \times \mathbb{I}$ are GR-spaces. In fact, in what follows, it is shown that these two examples, together with the Möbius band $\mathbb{M}$, are the only compact bordered surfaces which are homeomorphic to GR-spaces.

It is interesting, both from an elementary geometry as well as from a topological algebra point of view, to describe the relation between a solid torus $\mathbb{ST}^3$ and the Möbius band $\mathbb{M}$ explicitly as follows.

Examples 3.4. (See Figure 1) (a) Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the additive circle group and notice that the unit interval $\mathbb{I} = [0, 1]$ is a compact topological monoid (that is, semigroup with identity) under multiplication. The complex unit disc $\mathbb{D} = \{re^{2\pi iu} : r \in \mathbb{I}, u \in \mathbb{R}\}$ is likewise a compact monoid. Hence the solid torus $\mathbb{ST}^3 \stackrel{\text{def}}{=} \mathbb{D} \times \mathbb{T}$ is a GR-space which is a compact commutative monoid (and thus, in particular, an H-space).

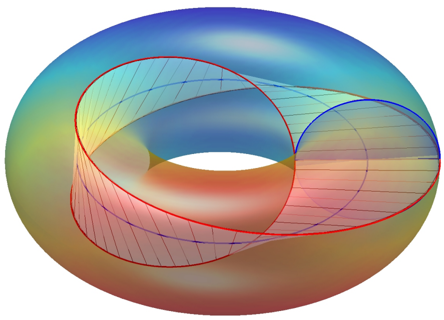

FIGURE 1

(b) The mapping $f : \mathbb{I}\times\mathbb{R}/2\mathbb{Z} \to \mathbb{ST}^3$ given by $f(r, t+2\mathbb{Z}) = (re^{\pi it}, t+\mathbb{Z})$ is a morphism of compact monoids mapping the "annulus" $\mathbb{I} \times \mathbb{R}/2\mathbb{Z}$ in such a fashion that the image in the solid torus $\mathbb{ST}^3$ is naturally isomorphic to the quotient $(\mathbb{I} \times \mathbb{R}/2\mathbb{Z})/R$ modulo the kernel relation R which identifies $(r_1, t_1+2\mathbb{Z})$ and $(r_2, t_2+2\mathbb{Z})$ if and only if $n \stackrel{\text{def}}{=} t_2-t_1 \in \mathbb{Z}$ and $r_1 = r_2e^{\pi in}$; now $n \in \mathbb{Z}$ means $e^{\pi in} \in \{1, -1\}$, and since $r_1,\ r_2 \geq 0$, the relation $n = -1$ occurs precisely when $r_1 = r_2 = 0$, that is, for two different elements in the annulus $\mathbb{I} \times \mathbb{R}/2\mathbb{Z}$ we have $(r_1, t_1 + 2\mathbb{Z})R(r_2, t_2 + 2\mathbb{Z})$ if and only if $r_1 = r_2 = 0$ and $t_2 - t_1 \in \mathbb{Z}$, that is, if they are "opposite" points on the boundary component $\{0\} \times \mathbb{R}/2\mathbb{Z}$ of the annulus. This quotient, however, is the Möbius band (see [11, p. 243, Example 2.3.3.5] and [10, Example 2.8(ii)]). Therefore,

the Möbius band $\mathbb{M} = \operatorname{im} f$ *is a submonoid of the monoid* $\mathbb{ST}^3$*, the solid torus.*

Since $e^{\pi i} = -1$, the Möbius band $\mathbb{M}$ is invariant under the involution $(z, u + \mathbb{Z}) \mapsto (-z, u + \mathbb{Z})$.

(c) For $\tau \in \mathbb{I}$ and $z = x+iy \in \mathbb{D}$ define $h_\tau \colon \mathbb{D} \to \mathbb{D}$ by $h_\tau(x+iy) = x+i\tau y$. Then $\tau \mapsto h_\tau$ defines a homotopy from the identity map to the projection of $\mathbb{D}$ onto the horizontal diameter and thus defines a homotopy retraction. Moreover,

$$h_\tau(-z) = -h_\tau(z).$$

Thus $n \in \mathbb{Z}$ implies $e^{\pi i n} h_\tau(e^{-\pi i n} z) = h_\tau(z)$. Thus we may define $H \colon \mathbb{ST}^3 \times \mathbb{I} \to \mathbb{ST}^3$ by

$$H((z, u+\mathbb{Z}), \tau) = (e^{\pi i u} h_\tau(e^{-\pi i u} z), u+\mathbb{Z})).$$

This is a well defined homotopy of the solid torus into itself. If $z = re^{\pi i u}$, then $H((z, u+\mathbb{Z}), \tau) = (z, u+\mathbb{Z})$, that is,

there is a homotopy retraction H of the solid torus $\mathbb{ST}^3$ onto the Möbius band $\mathbb{M} \subseteq \mathbb{ST}^3$, and therefore $\mathbb{M}$ is a GR-space.

(d) A parametrisation of an embedding of $\mathbb{ST}^3$ into $\mathbb{R}^3$ is obtained as follows: Let

$$R_z(s) = \begin{pmatrix} \cos 2\pi s & -\sin 2\pi s & 0 \\ \sin 2\pi s & \cos 2\pi s & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and}$$

$$R_y(t) = \begin{pmatrix} \cos 2\pi t & 0 & -\sin 2\pi t \\ 0 & 1 & 0 \\ \sin 2\pi t & 0 & \cos 2\pi t \end{pmatrix}$$

be the rotation groups around the z-axis, respectively, the y-axis, and let

$$T \colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2 \\ y \\ z \end{pmatrix}$$

the translation in x-direction by 2 units. Then

$$\mathbb{ST}^3 = \left\{ R_y(t)T\left(r\cdot R_z(s)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) : \ r \in \mathbb{I},\ s, t \in \mathbb{R} \right\}$$

is an embedding of $\mathbb{ST}^3$ into $\mathbb{R}^3$ and the Möbius band

$$\mathbb{M} = \left\{ R_y(2s)TR_z(s)\begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} : \ r \in \mathbb{I},\ s \in [0,1] \right\}$$

embedded into it.

(e) We note in passing, that the morphism f of (b) extends to a morphism $F \colon \mathbb{D} \times \mathbb{R}/2\mathbb{Z} \to \mathbb{ST}^3$ which is a double covering of $\mathbb{ST}^3$. If we denote by $\phi \colon \mathbb{ST}^3 \to \mathbb{D} \times \mathbb{R}/2\mathbb{Z}$ the isomorphism given by $\phi(z, r+\mathbb{Z}) = (z, 2r+2\mathbb{Z})$, then the composition $\Phi \stackrel{\text{def}}{=} F \circ \phi : \mathbb{ST}^3 \to \mathbb{ST}^3$ yields a double covering endomorphism of topological monoids of the solid torus which maps the annulus $\mathbb{I} \times \mathbb{T}$ ("one half of a horizontal section $[-1, 1] \times \mathbb{T} \subseteq \mathbb{D} \times \mathbb{T} = \mathbb{ST}^3$") onto the Möbius submonoid $\mathbb{M}$ of $\mathbb{ST}^3$.

(f) Even though we do not find a reference of the particular embedding of $\mathbb{M}$ and $\mathbb{D}$ into $\mathbb{ST}^3$ discussed here, the class of compact topological semigroups to which they belong is amply discussed in [8, 9], and [11, pp. 83ff., pp. 200ff., notably pp. 209ff., where the bordered n-manifolds which can be endowed with a continuous monoid structure are completely classified].

We remark that the closed unit disk $\mathbb{D}$ and the (so-called) solid Klein bottle $\mathbb{M} \times \mathbb{I}$ show that the boundary of a compact n-manifold which is a GR-space (and indeed even a compact commutative monoid) may or may not be orientable (or be a GR-space) since $\partial\mathbb{D} = \mathbb{S}^1$ and

$\partial(\mathbb{M} \times \mathbb{I})$ *is the Klein bottle which is a nonorientable compact surface that fails (having a nonabelian fundamental group) to admit an H-space structure and fails to be embeddable into* 3*-space* $\mathbb{R}^3$. *In particular,* $\mathbb{M} \times \mathbb{I}$ *is not embeddable into* $\mathbb{R}^3$.

Note in this context that the solid torus $\mathbb{ST}^3$ is a compact tubular neighborhood within $\mathbb{R}^3$ of the Möbius band $\mathbb{M}' \stackrel{\text{def}}{=} \Phi([0, 1/2]) \times \mathbb{T})$ whose boundary is a 2 torus $\cong \mathbb{T}^2$. Semigroup theoretically, $\mathbb{M}'$ is an ideal of $\mathbb{M}$.

Proposition 3.5. *If X is a compact bordered surface, then X admits an H-space structure iff X is a* GR*-space. Moreover, X must be homeomorphic to a space which is the closed unit disk $\mathbb{D}$, the annulus $\mathbb{S}^1 \times \mathbb{I}$ or the Möbius band $\mathbb{M}$.*

Proof. It suffices to show that if X is not one of $\mathbb{D}$, $\mathbb{S}^1 \times \mathbb{I}$ or M, then X does not admit an H-space structure. This follows since the remaining possibilities all contain wedge products of more than one circle as deformation retracts and, consequently, have nonabelian fundamental groups. □

Note that all three surfaces are monoids and indeed submonoids of the solid torus $\mathbb{ST}^3$.

In [10] a space X is called a *quotient retract* (or a *quotient retract of* G) if there is a topological group G with a closed subgroup C such that the quotient map $p\colon G \to G/C$ has a continuous cross section $\sigma\colon G/C \to G$. It follows that

A quotient retract is a GR*-space and a quotient retract of a compact Lie group is an orientable closed manifold.*

Furthermore, in [10] it is shown that a compact space X admits a topological left-loop structure iff X is a quotient retract of some topological group. The results of this note and [10, Prop. 2.2] yield the following result.

Theorem 3.6. *If X is an n-sphere $\mathbb{S}^n$, real projective n-space $\mathbb{RP}^n$, a compact surface, a compact bordered surface or an* AR*(compact)-space, then X admits an H-space structure iff X is a* GR*-space. Moreover, if X is not an* AR*(compact)-space, then X must be homeomorphic to one of the spaces $\mathbb{M}$, $\mathbb{S}^1 \times \mathbb{I}$, $\mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^n$ or $\mathbb{RP}^n$, where $n = 0, 1, 3, 7$.* □

Note that all these spaces are compact monoids with the exception of $n = 7$ in which case they are Moufang loops. All those with $\dim X < 3$ are commutative monoids.

Question 3.7. *Is it true that every compact n-manifold which admits an H-space structure is a* GR*-space?*
A first step towards an answer to this question would be a the answering of the following more special question

Question 3.8. *Is it true that every compact n-manifold which admits the structure of a compact topological monoid is a* GR*-space?*

Acknowledgment. We gratefully acknowledge the kind assistance of Professor Ulrich Reif of the Fachbereich Mathematik of the Technische Universität Darmstadt who programmed the parametrisations provided by Example 3.4.d and who, after a good deal of experimentation with MATHLAB, produced Figure 1 depicting the Möbius band as a submonoid of the solid torus as a compact topological monoid.

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