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by

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FUNCTION SPACES AND D-PROPERTY

R. ROJAS-HERNÁNDEZ

Abstract. In this article we introduce the notion of monotonically retractable space and we show that: (1) $C_p(X)$ is Lindelöf and a D-space whenever X is monotonically retractable. (2) If X is monotonically retractable then $C_{p,2n}(X)$ is monotonically retractable for any $n \in \omega$. (3) Any first countable countably compact subspace of an ordinal is monotonically retractable. (4) Every closed subspace of a Σ-product of cosmic spaces is monotonically retractable.

As a consequence of these results we conclude that $C_{p,2n+1}(X)$ is Lindelöf and has the D-property for any $n \in \omega$, whenever X is a first countable countably compact subspace of an ordinal; this answers a question posed by Tkachuk in [15].

1. Introduction

The notion of D-space is due to van Douwen, first studied with Pfeffer in [3]. The question whether every regular Lindelöf space is D has been attributed to van Douwen [6]. There are no consistency results in either direction even for hereditarily Lindelöf spaces. In [12], assuming $\diamond$, an example of a hereditarily Lindelöf T_2-space that is not a D-space was constructed. The example also has the property that any of its finite powers is Lindelöf, but is not known if it can be made regular. The concept of a D-space was studied a great deal ever since in almost every context and C_p-theory was not an exception. However, the C_p-version of the question of van Douwen and Pfeffer remains open. Indeed, it is not known if $C_p(X)$ is a D-space whenever it is Lindelöf [4].

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It is easy to see that the ordinal space ω_1 embeds in a Σ-product of real lines as a closed subspace, so the results in [7] are applicable to prove that $C_p(\omega_1)$ is Lindelöf. Buzyakova generalized this result in [2] establishing that, for any first countable countably compact subspace X of an ordinal, the space $C_p(X)$ is Lindelöf. In the same paper Buzyakova asked whether $C_p(X)$ has to be a D-space if X is a countably compact first countable subspace of an ordinal. Peng showed in [8] that the answer to Buzyakova's question is positive. On the other hand, it was established by Tkachuk in [15] that if X is a first countable countably compact subspace of an ordinal then X is a Sokolov space. It follows from general properties of Sokolov spaces that for any first countable countably compact subspace X of an ordinal, the spaces $(C_p(X))^\omega$ and $C_p(X^\omega)$ are Lindelöf and the iterated function space $C_{p,2n+1}(X)$ is Lindelöf for any $n \in \omega$. Also, using the same technique, Tkachuk gave a new method of proof of Peng's result which answered the question of Buzyakova.

It follows from the above results that: if X is a first countable countably compact subspace of an ordinal, then $C_p(X)$ is a D-space and $Y = C_p(C_p(C_p(X)))$ is a Lindelöf space and a Sokolov space. Tkachuk asked in [15] whether Y is a D-space.

In this paper we introduce the notion of monotone retractability and it is proved that this property is preserved under retracts, countable free topological sums, σ-products, Σ-products, and Σ_s-products. Our main results are the following: $C_p(X)$ is Lindelöf and has the D-property whenever X is monotonically retractable, and if X is monotonically retractable then $C_p(C_p(X))$ is monotonically retractable. We also prove that any first countable countably compact subspace of an ordinal is monotonically retractable. As a consequence, if X is a first countable countably compact subspace of an ordinal, then the spaces $(C_p(X))^\omega$ and $C_p(X^\omega)$ are Lindelöf and D-spaces and the iterated function space $C_{p,2n+1}(X)$ is Lindelöf and has the D-property for any $n \in \omega$. This answers the above question of Tkachuk in the affirmative. Furthermore, we prove that any closed subspace of a Σ-product of cosmic spaces is monotonically retractable. In particular, any Corson compact space is monotonically retractable. Finally, we will deal with the class of Sokolov spaces. The Sokolov spaces constitute a wide class systematically studied in [13]. It is known that: if X is a Sokolov space with $t^*(X) \leq \omega$ then $C_{p,2n+1}(X)$ is Lindelöf for any $n \in \omega$, if X is a first countable countably compact subspace of an ordinal then X is a Sokolov space, and any closed subspace of a Σ-product of cosmic spaces is Sokolov. So, it will be interesting to clarify the relationship between monotonically retractable spaces and Sokolov spaces. We give an example of a compact scattered Sokolov space of countable tightness which is not monotonically retractable.

2. Notation and Terminology

The letters α, β and γ represent ordinal numbers and the letters κ and λ represent infinite cardinal numbers; ω is the first infinite cardinal. All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology; if $x \in X$, then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$; besides, for any set $A \subset X$, we will need the family $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. We denote by $\mathbb{R}$ the real line with its natural topology. From now on we will fix a countable base $\mathcal{B}(\mathbb{R})$ for the usual topology in the set of real numbers $\mathbb{R}$.

For a subset A of a topological space X, $\mathrm{cl}_X(A)$ is the closure of A in X. If there is no possibility of confusion, we will simply write $\mathrm{cl}(A)$ instead of $\mathrm{cl}_X(A)$.

The *network weight* $\mathrm{nw}(X)$ of a space X is the minimal cardinality of a network in X. A space that has a countable network is called *cosmic*.

For any spaces X and Y the set $C(X, Y)$ consists of continuous functions from X to Y; if it has the topology induced from Y^X then the corresponding space is denoted by $C_p(X, Y)$. We write $C(X)$ instead of $C(X, \mathbb{R})$ and $C_p(X)$ instead of $C_p(X, \mathbb{R})$. Given a space X let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for all $n \in \omega$, i.e., $C_{p,n}(X)$ is the n-th iterated function space of X.

Let $f : X \to Y$ be a continuous function between the spaces X and Y. The *dual function* $f^* : C_p(Y) \to C_p(X)$ is defined as follows: if $g \in C_p(Y)$, then $f^*(g) = g \circ f$.

Let Y be a subspace of X. By π_Y we denote the function from $C_p(X)$ to $C_p(Y)$ which restricts each element in $C_p(X)$ to Y; that is, $\pi_Y(f) = f \restriction Y$.

If $X = \prod\{X_t : t \in T\}$ is a topological product, we denote by X_E the product $\prod\{X_t : t \in E\}$ for each $E \subset T$. Moreover, if $t \in T$ and $E \subset T$, then p_t and p_E denote the natural projections onto X_t and X_E, respectively.

A continuous map $f : X \to Y$ is called $\mathbb{R}$-*quotient* if, for any $g : Y \to \mathbb{R}$, the continuity of $g \circ f$ implies continuity of g.

Say that X is a *Sokolov* space if for any sequence $\{F_n : n \in \mathbb{N}\}$ where F_n is a closed subset of X^n for each $n \in \mathbb{N}$, there exists a continuous map $\varphi : X \to X$ such that $\varphi(X)$ is cosmic and $\varphi^n(F_n) \subset F_n$ for all $n \in \mathbb{N}$, the map $\varphi^n : X^n \to X^n$ is the n-th power of φ.

A function ϕ defined on a space X is a *neighborhood assignment* on X if, for any $x \in X$, the set $\phi(x)$ is an open neighborhood of the point x; let $\phi(A) = \bigcup\{\phi(x) : x \in A\}$ for every $A \subset X$. Call X a D-*space* if, for any neighborhood assignment ϕ on the space X, there exists a closed discrete subspace $D \subset X$ such that $\phi(D) = X$.

Let a^* be a point in $X = \prod\{X_t : t \in T\}$. For $x \in X$ let us set $\mathrm{supp}(x) = \{t \in T : x(t) \neq a^*(t)\}$. The subspace $\{x \in X : |\mathrm{supp}(x)| \leq \omega\}$ of X is called the *Σ-product* of the family $\{X_t : t \in T\}$ centered at the point a^*.

For those concepts and notations which appear in this article without a definition, consult [14] and [5].

3. Monotonically retractable spaces

Given a set A in a space X say that a family $\mathcal{N}$ of subsets of X is an *external network* of A in X if for any $x \in A$ and $U \in \tau(x, X)$ there exists $N \in \mathcal{N}$ such that $x \in N \subset U$.

Let $\mathcal{N}$ be a family of subsets of X and let f be a function from X onto Y. We say that $\mathcal{N}$ is a *network for Y modulo f* if for every $x \in X$ and each $U \in \tau(f(x), Y)$ there is $N \in \mathcal{N}$ such that $x \in N$ and $f(N) \subset U$.

Remark 3.1. Let f be a function from X onto Y and let $\mathcal{N}$ be a family of subsets of X which is a network for Y modulo f. Then:

(1) $f(\mathcal{N})$ is a network for Y.
(2) If g is a continuous function from Y onto Z, then $\mathcal{N}$ is a network for Z modulo $g \circ f$.
(3) If D is a subset of X with $D \cap N \neq \emptyset$ for any $N \in \mathcal{N}$, then $f(D)$ is a dense subset of Y.
(4) If $Z \subset X$ then $\{N \cap Z : N \in \mathcal{N}\}$ is a network for $f(Z)$ modulo $f \restriction Z$.

Given a family $\mathcal{A}$ of subsets of X, a family $\mathcal{B}$ of subsets of Y and $\psi : \mathcal{A} \to \mathcal{B}$; we say that ψ is *ω-monotone* if:

(1) $\psi(A)$ is countable whenever A is countable;
(2) if $A, B \in \mathcal{A}$ and $A \subset B$ then $\psi(A) \subset \psi(B)$;
(3) if $\{A_n : n \in \omega\} \subset \mathcal{A}$ and $A_n \subset A_{n+1}$ for any $n \in \omega$ then $\psi(\bigcup\{A_n : n \in \omega\}) = \bigcup\{\psi(A_n) : n \in \omega\}$.

Remark 3.2. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be families of subsets of X, Y and Z, respectively.

(1) If $\psi : \mathcal{A} \to \mathcal{B}$ and $\varphi : \mathcal{B} \to \mathcal{C}$ are ω-monotone, then $\varphi \circ \psi$ is ω-monotone.
(2) Suppose that $\mathcal{A}$ is closed under countable unions and that the assignment $\psi : \mathcal{A} \to \mathcal{A}$ is ω-monotone. If for each $A \in \mathcal{A}$ we choose $\varphi(A) = \bigcup\{\psi_n(A) : n \in \omega\}$, where $\psi_0(A) = A$ and $\psi_{n+1}(A) = \psi_n(A) \cup \psi(\psi_n(A))$ for each $n \in \omega$, then the assignment φ is ω-monotone.

For subsets $E_1, \ldots, E_n$ of a space X and subsets $U_1, \ldots, U_n$ of $\mathbb{R}$, we will use the symbol $[E_1, \ldots, E_n; U_1, \ldots, U_n]$ to denote the set $\{f \in C_p(X) : f(E_i) \subset U_i \text{ for } i = 1, \ldots, n\}$. If $\mathcal{E}$ is a family of subsets of X, then $\mathcal{W}(\mathcal{E})$ will be the family of all the sets of the form $[E_1, \ldots, E_n; B_1, \ldots, B_n]$ where $E_1, \ldots, E_n \in \mathcal{E}$, $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ and $n \in \omega$.

Remark 3.3. The assignment $\mathcal{E} \to \mathcal{W}(\mathcal{E})$ is ω-monotone.

Definition 3.4. Say that a space X is *monotonically retractable* if for any countable set $A \subset X$ we can assign $K(A)$, $r(A)$ and $\mathcal{N}(A)$ in such a form that: $A \subset K(A) \subset X$, $r(A) : X \to K(A)$ is a continuous retraction, $\mathcal{N}(A)$ is a family of subsets of X which is a network for $K(A)$ modulo $r(A)$, and the assignment $\mathcal{N}$ is ω-monotone.

Remark 3.5. By Remark 3.1 (1), the space $K(A)$ in the above definition has countable network weight; indeed $\{r(A)(N) : N \in \mathcal{N}(A)\}$ is a network for $K(A)$.

Proposition 3.6. *A space X is monotonically retractable if and only if for any countable set $A \subset X$ we can assign $L(A)$, $s(A)$ and $\mathcal{O}(A)$ in such a form that: $A \subset L(A) \subset X$, $s(A) : X \to L(A)$ is a continuous and onto function such that $s(A)(x) = x$ for any $x \in A$, $\mathcal{O}(A)$ is a family of subsets of X which is a network for $L(A)$ modulo $s(A)$, and the assignment $\mathcal{O}$ is ω-monotone.*

Proof. Clearly, a monotonically retractable space satisfies the conditions in the proposition. Suppose that for any countable set $A \subset X$ it is possible to assign $L(A)$, $s(A)$, and $\mathcal{O}(A)$ as in the proposition, we shall prove that X is monotonically retractable. Let $\mathcal{O}(X) = \bigcup\{\mathcal{O}(A) : A \text{ is a countable subset of } X\}$. We can suppose that all the elements of $\mathcal{O}(X)$ are non-empty. For any set $N \in \mathcal{O}(X)$ fix a point $x_N \in N$. If A is a countable subset of X let $\mathcal{E}(A) = \{x_N : N \in \mathcal{O}(A)\}$. Let us observe that the assignment $\mathcal{E}$ is ω-monotone.

Let A be a countable subset of X. Choose $D_0(A) = A$. If for some $n \in \omega$ we have defined $D_0(A), \ldots, D_n(A)$, let $D_{n+1}(A) = D_n(A) \cup \mathcal{E}(D_n(A))$. Let $\mathcal{D}(A) = \bigcup\{D_n(A) : n \in \omega\}$, $r(A) = s(\mathcal{D}(A))$, $K(A) = r(A)(X) = L(\mathcal{D}(A))$, and $\mathcal{N}(A) = \mathcal{O}(\mathcal{D}(A))$. First, let us observe that $A = r(A)(A) \subset K(A) \subset X$ and $\mathcal{N}(A)$ is a network for $K(A)$ modulo $r(A)$. Notice that $\mathcal{E}(\mathcal{D}(A)) = \mathcal{E}(\bigcup\{D_n(A) : n \in \omega\}) = \bigcup\{\mathcal{E}(D_n(A)) : n \in \omega\} \subset \bigcup\{D_{n+1}(A) : n \in \omega\} \subset \mathcal{D}(A)$. Since $\mathcal{E}(\mathcal{D}(A)) = \{x_N : N \in \mathcal{O}(\mathcal{D}(A))\} = \{x_N : N \in \mathcal{N}(A)\}$, we conclude that $\mathcal{D}(A) \cap N \neq \emptyset$ for any $N \in \mathcal{N}(\mathcal{A})$. It follows from Remark 3.1 (3) that $r(A)(\mathcal{D}(A))$ is dense in $K(A)$. Using this fact and the fact that $r(A)(x) = s(\mathcal{D}(A))(x) = x$ for any $x \in \mathcal{D}(A)$, we conclude that $r(A)$ is a continuous retraction.

It follows from Remark 3.2 (2) and the fact that the assignment $\mathcal{E}$ is ω-monotone, that the assignment $\mathcal{D}$ is ω-monotone. Since the assignment $\mathcal{O}$ is also ω-monotone, it follows from Remark 3.2 (1) that the assignment $\mathcal{N}$ is ω-monotone. This finishes the proof. □

Proposition 3.7. *If X is monotonically retractable, $Y \subset X$ and $t : X \to Y$ is a continuous retraction, then Y is monotonically retractable.*

Proof. Suppose that for any countable set $A \subset X$ we have assigned $K(A)$, $r(A)$ and $\mathcal{N}(A)$ as in Definition 3.4. By Proposition 3.6, it is enough to show that for any countable set $A \subset Y$ we can assign $L(A)$, $s(A)$ and $\mathcal{O}(A)$ in such a form that: $A \subset L(A) \subset Y$, $s(A) : Y \to L(A)$ is a continuous and onto function such that $s(A)(x) = x$ for any $x \in A$, $\mathcal{O}(A)$ is a family of subsets of Y which is a network for $L(A)$ modulo $s(A)$, and the assignment $\mathcal{O}$ is ω-monotone.

For each countable subset A of Y let $s(A) = t \circ r(A) \restriction Y$, $L(A) = s(A)(Y)$ and $\mathcal{O}(A) = \{N \cap Y : N \in \mathcal{N}(A)\}$. Let us observe that the assignment $\mathcal{O}$ is ω-monotone. Clearly, $s(A) : Y \to L(A)$ is a continuous and onto function. Moreover, if $x \in A$, then $x = t(x) = t(r(A)(x)) = s(A)(x) \in L(A)$. So, $s(A)(x) = x$ and $A \subset L(A)$. Also, since t is a continuous retraction onto Y, if $y \in Y$ then $s(A)(y) = t(r(A)(y)) \in Y$. Thus, $L(A) \subset Y$. Now, since $\mathcal{N}(A)$ is a network for $K(A) = r(A)(X)$ modulo $r(A)$, it follows from Remark 3.1 (4) that $\mathcal{O}(A)$ is a network for $r(A)(Y)$ modulo $r(A) \restriction Y$. Because of Remark 3.1 (2) and the continuity of t, we can see that $\mathcal{O}(A)$ is a network for $L(A) = s(A)(Y) = t \circ r(A)(Y)$ modulo $s(A) = t \circ r(A) \restriction Y$. □

Proposition 3.8. *Let $\{X_n : n \in \omega\}$ be a family of monotonically retractable spaces, then $X = \bigoplus\{X_n : n \in \omega\}$ also is monotonically retractable.*

Proof. Let $n \in \omega$, since X_n is monotonically retractable, for any countable set $A_n \subset X_n$ we can assign $K_n(A_n)$, $r_n(A_n)$ and $\mathcal{N}_n(A_n)$ as in Definition 3.4. For any countable set $A \subset X$, let $A_n = A \cap X_n$ for each $n \in \omega$. Let $K(A) = \bigcup\{K_n(A_n) : n \in \omega\}$, $r(A) = \bigoplus\{r_n(A_n) : n \in \omega\}$, and $\mathcal{N}(A) = \bigcup\{\mathcal{N}_n(A_n) : n \in \omega\}$. It is easy to verify that $K(A)$, $r(A)$ and $\mathcal{N}(A)$ satisfy the conditions in Definition 3.4; therefore, X is monotonically retractable. □

We will need the following notation.

Let $X = \prod\{X_t : t \in T\}$ be a product of monotonically retractable spaces and let Z be a Σ-product in X centered at some point $a^* \in X$.

Given a countable set $A \subset Z$ let $A^* = A \cup \{a^*\}$. For any $t \in T$ and any countable set $A_t \subset X_t$ choose $r_t(A_t)$, $K_t(A_t)$ and $\mathcal{N}_t(A_t)$ as in Definition 3.4. Given a countable set $A \subset Z$ and a countable set $E \subset T$, the retraction $r(A, E) : Z \to Z$ is given by:

$$r(A,E)(x)(t) = \begin{cases} r_t(p_t(A^*))(x(t)) & \text{if } t \in E; \\ a^*(t), & \text{if } t \in T \setminus E. \end{cases}$$

A subspace Y of Z is said to be *monotonically invariant* if for each countable subset A of Y we can assign a subset $\mathcal{E}(A)$ of T in such a way that; $\bigcup\{\text{supp}(x) : x \in A\} \subset \mathcal{E}(A)$, $r(A, \mathcal{E}(A))(Y) \subset Y$, and the assignment $\mathcal{E}$ is ω-monotone.

Remark 3.9. If Y is a monotonically invariant subspace of a Σ-product of monotonically retractable spaces and A is a countable subset of Y, then $r(A, \mathcal{E}(A)) \restriction Y : Y \to Y$ is a continuous retraction onto its image and $r(A, \mathcal{E}(A))(x) = x$ for any $x \in A$.

Remark 3.10. Let X be a product of monotonically retractable spaces and let Z be a Σ-product in X centered at some point $a^* \in X$. If we choose $\mathcal{E}(A) = \bigcup\{\text{supp}(x) : x \in A\}$ for any countable subset A of Z, then we can verify that the respective σ-product (Σ_s-product or Σ-product) in X centered at a^* is a monotonically invariant subspace of Z.

Theorem 3.11. *If Z is a Σ-product of monotonically retractable spaces and Y is a monotonically invariant subspace of Z, then Y is monotonically retractable.*

Proof. Let $\{X_t : t \in T\}$ be a family of monotonically retractable spaces. Suppose that for any $t \in T$ and any countable set $A_t \subset X_t$ we can assign $r_t(A_t)$, $K_t(A_t)$ and $\mathcal{N}_t(A_t)$ as in Definition 3.4. Let $X = \prod\{X_t : t \in T\}$ and let Z be a Σ-product in X centered at some point $a^* \in X$. Take a monotonically invariant subspace Y of Z. For each countable subset A of Y choose a subset $\mathcal{E}(A)$ of T in such a way that: $\bigcup\{\text{supp}(x) : x \in A\} \subset \mathcal{E}(A)$, $r(A, \mathcal{E}(A))(Y) \subset Y$, and the assignment $\mathcal{E}$ is ω-monotone. For each countable set $A \subset Y$ we shall construct $r(A)$, $K(A)$ and $\mathcal{N}(A)$ which witness that Y is monotonically retractable.

Given a set $A \subset Y$, let $r(A) = r(A, \mathcal{E}(A)) \restriction Y$ and let $K(A) = r(A)(Y)$. Moreover, let $\mathcal{N}(A)$ be the collection of all sets of the form $Y \cap \prod\{N_t : t \in T\}$, where $N_t \in \mathcal{N}_t(p_t(A^*))$ for any $t \in F$, $N_t = X_t$ for $t \in T \setminus F$, and F is a finite subset of $\mathcal{E}(A)$. Because of Remark 3.9, the map $r(A) : Y \to K(A)$ is a continuous retraction and $A \subset K(A) \subset Y$.

Claim 1. $\mathcal{N}(A)$ is a network for $K(A)$ modulo $r(A)$.

Proof of Claim 1. Let $y \in Y$ and $U \in \tau(r(A)(y), K(A))$. We can find a finite set $H \subset T$ and an open set $B = \prod\{B_t : t \in T\}$ with B_t open in X_t for $t \in H$ and $B_t = X_t$ for $t \in T \setminus H$, such that $r(A)(y) \in B \cap K(A) \subset U$.

Then $r_t(p_t(A^*))(y(t)) = r(A)(y)(t) \in B_t \cap K_t(p_t(A^*))$ for any $t \in \mathcal{E}(A)$. Let $F = H \cap \mathcal{E}(A)$. Since $\mathcal{N}_t(p_t(A^*))$ is a network for $K_t(p_t(A^*))$ modulo $r_t(p_t(A^*))$, for any $t \in F$ we can choose $N_t \in \mathcal{N}_t(p_t(A^*))$ such that $y(t) \in N_t$ and $r_t(p_t(A^*))(N_t) \subset B_t \cap K_t(p_t(A^*))$. For $t \in T \setminus F$ let $N_t = X_t$. Then we can see that $N = Y \cap \prod\{N_t : t \in T\} \in \mathcal{N}(A)$ and $y \in N$. We shall prove that $r(A)(N) \subset U$. Take $x \in N$. If $t \in F$ then $r(A)(x)(t) = r_t(p_t(A^*))(x(t)) \subset r_t(p_t(A^*))(N_t) \subset B_t$. If $t \in H \setminus F$ then $r(A)(x)(t) = a^*(t) = r(A)(y)(t) \in B_t$. Hence, $r(A)(x) \in B \cap K(A) \subset U$. Thus, $r(A)(N) \subset U$.

Claim 2. The assignment $\mathcal{N}$ is ω-monotone.

Proof of Claim 2. (1) Given a countable subset A of Y, since $\mathcal{E}(A)$ is countable and $\mathcal{N}_t(p_t(A^*))$ is countable for each $t \in T$, it is easy to see that $\mathcal{N}(A)$ is countable. (2) Suppose that $A \subset B \subset Y$, where B is a countable set. Let us observe that $\mathcal{E}(A) \subset \mathcal{E}(B)$. Moreover, $p_t(A^*) \subset p_t(B^*)$ and $\mathcal{N}_t(p_t(A^*)) \subset \mathcal{N}_t(p_t(B^*))$ for each $t \in T$. It follows from these facts that $\mathcal{N}(A) \subset \mathcal{N}(B)$. (3) Let $\{A_n : n \in \omega\}$ be a family of countable subsets of Y with $A_n \subset A_{n+1}$ for any $n \in \omega$, and let $A = \bigcup\{A_n : n \in \omega\}$. It follows from (2) that $\bigcup\{\mathcal{N}(A_n) : n \in \omega\} \subset \mathcal{N}(A)$. We shall prove the other contention. We know that $\mathcal{E}(A) = \bigcup\{\mathcal{E}(A_n) : n \in \omega\}$. Also, $\mathcal{N}_t(p_t(A^*)) = \mathcal{N}_t(p_t(\bigcup\{A_n^* : n \in \omega\})) = \bigcup\{\mathcal{N}_t(p_t(A_n^*)) : n \in \omega\}$ for each $t \in T$. Let $N \in \mathcal{N}(A)$, then there exists a finite set $F \subset \mathcal{E}(A)$ such that $N = Y \cap \prod\{N_t : t \in T\}$, where $N_t \in \mathcal{N}_t(p_t(A^*))$ for any $t \in F$ and $N_t = X_t$ for $t \in T \setminus F$. We can choose $n \in \omega$ such that $F \subset \mathcal{E}(A_n)$ and $N_t \in \mathcal{N}_t(p_t(A_n^*))$ for any $t \in F$. Then $N \in \mathcal{N}(A_n)$. Therefore $\mathcal{N}(A) \subset \bigcup\{\mathcal{N}(A_n) : n \in \omega\}$. $\square$

Corollary 3.12. *If Y is a σ-product of monotonically retractable spaces, then Y is monotonically retractable.*

Corollary 3.13. *If Y is a Σ_s-product of monotonically retractable spaces, then Y is monotonically retractable.*

Corollary 3.14. *If Y is a Σ-product of monotonically retractable spaces, then Y is monotonically retractable.*

Corollary 3.15. *Let $\{X_n : n \in \omega\}$ be a family of monotonically retractable spaces, then $X = \prod\{X_n : n \in \omega\}$ also is monotonically retractable.*

Proposition 3.16. [9] *Let $f : X \to Y$ be an onto and continuous function and $\mathcal{N}$ be a family of subsets of X which is a network for Y modulo f. Then $\mathcal{W}(\mathcal{N})$ is an external network of $f^*(C_p(Y))$ in $C_p(X)$.*

Proposition 3.17. [9] *Let $Y \subset X$. If $\mathcal{N}$ is an external network of Y in X, then the family $\mathcal{W}(\mathcal{N})$ of subsets of $C_p(X)$ is a network for $\pi_Y(C_p(X))$ modulo π_Y.*

Theorem 3.18. *Let X be a monotonically retractable space. Then $C_p(X)$ has the D-property and is a Lindelöf space.*

Proof. Since X is monotonically retractable, for any countable set $A \subset X$ we can assign $K(A)$, $r(A)$ and $\mathcal{N}(A)$ in such a form that: $A \subset K(A) \subset X$, $r(A) : X \to K(A)$ is a continuous retraction, $\mathcal{N}(A)$ is a family of subsets of X which is a network for $K(A)$ modulo $r(A)$, and the assignment $\mathcal{N}$ is ω-monotone. Let $\mathcal{O}(A) = \mathcal{W}(\mathcal{N}(A))$ for any countable subset A of X. It follows from Proposition 3.16 that $\mathcal{O}(A)$ is a family of subsets of $C_p(X)$ which is an external network for $r(A)^*(C_p(K(A)))$ in $C_p(X)$. Moreover, the assignment $\mathcal{O}$ is ω-monotone because of Remark 3.2 (1) and the fact that the assignments $\mathcal{W}$ and $\mathcal{N}$ are ω-monotone.

Let ϕ be a neighborhood assignment on $C_p(X)$. There is no loss of generality to assume that $\phi(f)$ is a standard open set in $C_p(X)$ for any $f \in C_p(X)$. Then for each $f \in C_p(X)$ we can choose a finite set $S(\phi(f)) \subset X$ such that $\phi(f) = \pi^{-1}_{S(\phi(f))}(\pi_{S(\phi(f))}(\phi(f)))$. We will construct in a recursive process a countable closed and discrete set $D \subset C_p(X)$ such that $\bigcup\{\phi(f) : f \in D\} = C_p(X)$. For any set $N \subset C_p(X)$ say that $f \in N$ is a *central point* of N if $N \subset \phi(f)$; denote by $Z(N)$ the set of all central points of N. It is easy to find a partition $\{\Omega_n : n \in \omega\}$ of ω in infinite subsets such that $\{0, \dots, n\} \subset \Omega_1 \cup, \dots, \cup \Omega_n$.

Step 0. Pick a function $f_0 \in C_p(X)$ arbitrarily. Let $A_0 = S(\phi(f_0))$ and let $\{N_k : k \in \Omega_0\}$ be an enumeration for the family $\mathcal{O}(A_0)$.

Proceeding inductively, assume that for $n \in \omega$ we have constructed countable sets $A_0, \dots, A_n \subset X$ and functions $f_0, \dots, f_n \in C_p(X)$ such that:

$a(n)$ $A_i \subset A_{i+1}$ for $i < n$;

$b(n)$ $S(\phi(f_i)) \subset A_i$ for $i \le n$;

$c(n)$ $\bigcup\{Z(N_j) : j < i\} \subset \bigcup\{\phi(f_j) : j \le i\}$ for $i \le n$;

$d(n)$ $f_{i+1} \in C_p(X) \setminus \bigcup\{\phi(f_j) : j \le i\}$ or $f_{i+1} = f_i$, for $i < n$;

$e(n)$ $\{N_k : k \in \Omega_i\}$ is an enumeration for the family $\mathcal{O}(A_i)$ for every $i \le n$.

Step $n+1$. Let $U_n = \bigcup\{\phi(f_i) : i \le n\}$. If for any $k \in \bigcup\{\Omega_i : i \le n\}$ we have $Z(N_k) \subset U_n$ then let $f_{n+1} = f_n$. In the other case, let $l(n)$ be the least element of $\bigcup\{\Omega_i : i \le n\}$ such that $Z(N_{l(n)}) \setminus U_n \neq \emptyset$ and choose $f_{n+1} \in Z(N_{l(n)}) \setminus U_n$. Let $A_{n+1} = A_n \cup S(\phi(f_{n+1}))$ and let $\{N_k : k \in \Omega_{n+1}\}$ be an enumeration of $\mathcal{O}(A_{n+1})$. This completes the inductive step in such a way that properties $a(n+1)$-$e(n+1)$ hold.

Therefore, we can construct the family $\mathcal{A} = \{A_i : i \in \omega\}$ of subsets of X together with a set $D = \{f_i : i \in \omega\} \subset C_p(X)$ such that the conditions $a(n)$-$e(n)$ are satisfied for all $n \in \omega$. Take $A = \bigcup\{A_i : i \in \omega\}$.

To show that $\bigcup\{\phi(f_i) : i \in \omega\} = C_p(X)$, fix an arbitrary point $f \in C_p(X)$. Let $g = (f \upharpoonright K(A)) \circ r(A) \in r(A)^*(C_p(K(A)))$. Since $\mathcal{O}(A)$ is an external network of $r(A)^*(C_p(K(A)))$ in $C_p(X)$, we can find a set $N \in \mathcal{O}(A)$ such that $g \in N \subset \phi(g)$, that is, $g \in Z(N)$. Since $\mathcal{O}$ is ω-monotone, we have $N \in \mathcal{O}(A) = \bigcup\{\mathcal{O}(A_i) : i \in \omega\}$. So, there exists $n \in \omega$ such that $N \in \mathcal{O}(A_n)$. By $e(n)$ we have $N = N_k$ for some $k \in \Omega_n \subset \omega$. By $c(k+1)$ we know that $g \in Z(N_k) \subset \bigcup\{\phi(f_i) : i \in \omega\}$. Choose $m \in \omega$ such that $g \in \phi(f_m)$. By $b(m)$ we know that $S(\phi(f_m)) \subset A_m \subset A \subset K(A)$. Since $r(A)(x) = x$ for any $x \in A$, we have $f \upharpoonright A = g \upharpoonright A$. In particular, $f \upharpoonright S(\phi(f_m)) = g \upharpoonright S(\phi(f_m))$. Thus, we conclude that $f \in \phi(f_m)$. The point $f \in C_p(X)$ was chosen arbitrarily, so $\{\phi(f_i) : i \in \omega\}$ is a cover of $C_p(X)$.

It follows from $d(n)$ for $n \in \omega$ and the fact that $\{\phi(f_i) : i \in \omega\}$ is a cover of $C_p(X)$ that D is closed and discrete in $C_p(X)$. Thus, $C_p(X)$ is a D-space.

Finally, to see that $C_p(X)$ is a Lindelöf space, let $\mathcal{U}$ be an open cover of $C_p(X)$. For any $f \in C_p(X)$ we can choose $\phi(f) \in \mathcal{U}$ such that $f \in \phi(f)$. Since ϕ is a neighborhood assignment, we can find a countable closed and discrete set $D \subset C_p(X)$ such that $C_p(X) = \bigcup\{\phi(f) : f \in D\}$. As a consequence, $\{\phi(f) : f \in D\}$ is a countable subcover of $\mathcal{U}$. □

Corollary 3.19. *Let X be a monotonically retractable space. Then the spaces $C_p(X)^\omega$ and $C_p(X^\omega)$ have the D-property and are Lindelöf spaces.*

Corollary 3.20. *For any monotonically retractable space X we have $t(X^\omega) = \omega$.*

Corollary 3.21. *Let X be a monotonically retractable space and suppose that $f : X \to Y$ is an $\mathbb{R}$-quotient map. Then $C_p(Y)^\omega$ has the D-property and is a Lindelöf space.*

Remark 3.22. Given a space X, let us note that if $K \subset X$, $f \in C_p(X)$ and $r : X \to K$ is a continuous retraction, then $f \upharpoonright K = ((f \upharpoonright K) \circ r) \upharpoonright K$.

Lemma 3.23. *If $K \subset X$ and $r : X \to K$ is a continuous retraction, then $r^* \circ \pi_K$ is a continuous retraction onto its image.*

Proof. Clearly $r^* \circ \pi_K$ is continuous. Let $g \in r^* \circ \pi_K(C_p(X))$, then $g = (f \upharpoonright K) \circ r$ for some $f \in C_p(X)$. Since r is a continuous retraction, by Remark 3.22, $r^* \circ \pi_K(g) = (g \upharpoonright K) \circ r = (((f \upharpoonright K) \circ r) \upharpoonright K) \circ r = (f \upharpoonright K) \circ r = g$. This shows that $r^* \circ \pi_K$ is a retraction onto its image. □

Lemma 3.24. [1] *Let Y be a dense subspace of the product $X = \prod\{X_t : t \in T\}$, where each space X_t is cosmic. Then, for every continuous real-valued function f on Y, there exists a countable set $A \subset T$ and a continuous real-valued function g on $p_A(Y)$ such that $f = g \circ p_A \restriction Y$.*

Theorem 3.25. *Let X be a monotonically retractable space. Then the space $C_pC_p(X)$ is also monotonically retractable.*

Proof. Suppose that for any countable set $A \subset X$ there exists a set $K(A) \subset X$, a continuous retraction $r(A) : X \to K(A)$, and a countable family $\mathcal{N}(A)$ of subsets of X as in Definition 3.4. For any countable set $E \subset C_pC_p(X)$ we shall construct a set $L(E) \subset C_pC_p(X)$, a retraction $s(E) : C_pC_p(X) \to L(E)$, and a countable family $\mathcal{O}(E)$ of subsets of $C_pC_p(X)$, which satisfy conditions in Definition 3.4. Take $A \subset X$ countable. Let $r(A,0) = r(A)$ and $K(A,0) = K(A)$. For each $n \in \omega$, let $r(A,n+1) = r(A,n)^* \circ \pi_{K(A,n)}$ and $K(A,n+1) = r(A,n+1)(C_{p,n+1}(X))$. Because of Lemma 3.23, $r(A,n) : C_{p,n}(X) \to K(A,n)$ is a retraction for any $n \in \omega$. Notice that if $f \in C_{p,n+1}(X)$ then $r(A,n+1)(f) = (f \restriction K(A,n)) \circ r(A,n)$.

Let $E \subset C_pC_p(X)$ be countable. By Lemma 3.24, for any $f \in C_pC_p(X)$ we can choose a countable set $A(f) \subset X$ and a continuous function $g(f) : \pi_{A(f)}(C_p(X)) \to \mathbb{R}$ such that $f = g(f) \circ \pi_{A(f)}$. Consider the countable set $\mathcal{A}(E) = \bigcup\{A(f) : f \in E\} \subset X$. Finally, let $s(E) = r(\mathcal{A}(E), 2)$, $L(E) = K(\mathcal{A}(E), 2) = s(E)(C_pC_p(X))$, and $\mathcal{O}(E) = \mathcal{W}(\mathcal{W}(\mathcal{N}(\mathcal{A}(E))))$. We shall prove that $s(E)$, $L(E)$, and $\mathcal{O}(E)$ satisfy conditions in Definition 3.4.

It is clear that $L(E) \subset C_pC_p(X)$, $s(E) : C_pC_p(X) \to L(E)$ is a continuous retraction, and $\mathcal{O}(E)$ is a countable family of subsets of $C_pC_p(X)$. It is easy to see that $\mathcal{A}$ is ω-monotone. Since $\mathcal{W}$ and $\mathcal{N}$ are also ω-monotone, by Remark 3.2 (1) the operator $\mathcal{O} = \mathcal{W} \circ \mathcal{W} \circ \mathcal{N} \circ \mathcal{A}$ is ω-monotone.

Claim 1. $E \subset L(E)$.

Proof of Claim 1. Take an arbitrary function $f \in E \subset C_pC_p(X)$. By construction we have $A(f) \subset \mathcal{A}(E) \subset K(\mathcal{A}(E))$, so there exists a continuous function $h(f) : \pi_{K(\mathcal{A}(E))}(C_p(X)) \to \mathbb{R}$ such that $f = h(f) \circ \pi_{K(\mathcal{A}(E))}$. For any $k \in C_p(X)$ since $r(\mathcal{A}(E))$ is a retraction onto $K(\mathcal{A}(E))$, by Remark 3.22 we have $\pi_{K(\mathcal{A}(E))}[(k \restriction K(\mathcal{A}(E))) \circ r(\mathcal{A}(E))] = [(k \restriction K(\mathcal{A}(E))) \circ r(\mathcal{A}(E))] \restriction K(\mathcal{A}(E)) = k \restriction K(\mathcal{A}(E)) = \pi_{K(\mathcal{A}(E))}(k)$. As a consequence $f(k) = h(f) \circ \pi_{K(\mathcal{A}(E))}(k) = h(f) \circ \pi_{K(\mathcal{A}(E))}[(k \restriction K(\mathcal{A}(E))) \circ r(\mathcal{A}(E))] = h(f) \circ \pi_{K(\mathcal{A}(E))}(r(\mathcal{A}(E),1)(k)) = f(r(\mathcal{A}(E),1)(k))$ for any $k \in C_p(X)$. Thus $f = (f \restriction K(\mathcal{A}(E),1)) \circ r(\mathcal{A}(E),1) = r(\mathcal{A}(E),2)(f) = s(E)(f) \in L(E)$. Therefore $E \subset L(E)$.

Claim 2. $\mathcal{O}(E)$ is a network for $L(E)$ modulo $s(E)$.

Proof of Claim 2. It follows from the election of $\mathcal{N}$ that the family $\mathcal{N}(\mathcal{A}(E))$ is a network for $K(\mathcal{A}(E))$ modulo $r(\mathcal{A}(E))$. We can apply Proposition 3.16 to see that the family $\mathcal{W}(\mathcal{N}(\mathcal{A}(E)))$ is an external network for $r(\mathcal{A}(E))^*(C_p(K(\mathcal{A}(E))))$. In particular, $\mathcal{W}(\mathcal{N}(\mathcal{A}(E)))$ is an external network for $r(\mathcal{A}(E))^*(\pi_{K(\mathcal{A}(E))}(C_p(X))) = r(\mathcal{A}(E),1)(C_p(X)) = K(\mathcal{A}(E),1)$. Now, we can apply Proposition 3.17 to see that the family $\mathcal{O}(E) = \mathcal{W}(\mathcal{W}(\mathcal{N}(\mathcal{A}(E))))$ is a network for $\pi_{K(\mathcal{A}(E),1)}(C_pC_p(X))$ modulo $\pi_{K(\mathcal{A}(E),1)}$. Since $r(\mathcal{A}(E),1)^*$ is an homeomorphism onto its image, by Remark 3.1 (2), we conclude that the family $\mathcal{O}(E)$ is a network for $r(\mathcal{A}(E),1)^*(\pi_{K(\mathcal{A}(E),1)}(C_pC_p(X)))$ modulo $r(\mathcal{A}(E),1)^* \circ \pi_{K(\mathcal{A}(E),1)}$. Finally, since $s(E) = r(\mathcal{A}(E),2) = r(\mathcal{A}(E),1)^* \circ \pi_{K(\mathcal{A}(E),1)}$ and $L(E) = s(C_pC_p(X)) = r(\mathcal{A}(E),1)^*(\pi_{K(\mathcal{A}(E),1)}(C_pC_p(X)))$, we conclude that the family $\mathcal{O}(E)$ is a network for $L(E)$ modulo $s(E)$. $\square$

Corollary 3.26. *Let X be a monotonically retractable space. Then, $C_{p,2n}(X)$ is monotonically retractable for every $n \in \omega$.*

Corollary 3.27. *Let X be a monotonically retractable space. Then, $C_{p,2n+1}(X)$ is Lindelöf and a D-space for every $n \in \omega$.*

Theorem 3.28. *Suppose that X is a first countable countably compact subspace of an ordinal, then X is monotonically retractable.*

Proof. We can suppose that $X = \{\alpha \in \mu : cf(\alpha) \leq \omega\}$ for some ordinal μ. There is no loss of generality to assume that X is infinite. Any interval is considered only for the points of X; in particular $[\alpha, \to) = \{\beta \in X : \alpha \leq \beta\}$ for each $\alpha \in X$ and $[\alpha, \beta) = \{z \in X : \alpha \leq z < \beta\}$ whenever $\alpha, \beta \in X$ and $\alpha < \beta$. For any $A \subset X$ let $\mathcal{I}(A) = \{\{\alpha\} : \alpha \in A\} \cup \{[\alpha, \to) : \alpha \in A\} \cup \{[\alpha, \beta) : \alpha, \beta \in A \text{ and } \alpha < \beta\}$. Let us observe that the assignment $A \to \mathcal{I}(A)$ is ω-monotone. Say that a set $A \subset X$ is *saturated* if $0 \in A$ and every isolated point of A is also isolated in X. For each $\alpha \in \mu$ non-isolated in μ, fix a countable strictly increasing sequence X_α of isolated ordinals converging to α.

Take a countable set $A \subset X$. Let us consider the set $S(A) = A \cup \{0\} \cup \{X_\alpha : \alpha \in A \text{ and } \alpha \text{ is non-isolated in } \mu\}$. The set $S(A)$ is countable as a countable union of countable sets. Notice that $S(A)$ and also $K(A) = \mathrm{cl}_X(S(A))$ are saturated sets. Since $S(A)$ is countable, it is standard to prove that $K(A)$ is compact and countable. Define a map $r(A) : X \to K(A)$ by the formula $r(A)(\alpha) = \max\{\beta \in K(A) : \beta \leq \alpha\}$ for each $\alpha \in X$. Finally, let $\mathcal{N}(A) = \mathcal{I}(S(A))$. It is clear that $\mathcal{N}(A)$ is countable. We shall prove that $r(A)$, $K(A)$ and $\mathcal{N}(A)$ satisfy conditions in Definition 3.4. Clearly $A \subset K(A) \subset X$. Since $K(A)$ is compact and saturated, by Lemma 2.4 in [15], the map $r(A) : X \to K(A)$ is a continuous retraction. Let us observe that the assignment $A \to S(A)$ is ω-monotone.

Since $\mathcal{I}$ is also ω-monotone, then $\mathcal{N}$ is ω-monotone. So, we only need to prove that $\mathcal{N}(A)$ is a network for $K(A)$ modulo $r(A)$.

Let $\xi \in X$ and $U \in \tau(r(A)(\xi), K(A))$. If $\xi \in S(A)$, then $\xi \in N$ and $r(A)(N) \subset U$ for $N = \{\xi\} \in \mathcal{N}(A)$. Otherwise, it follows from $r(A)(\xi) \in K(A) = \mathrm{cl}_X(S(A))$ that there exists $\alpha \in S(A)$ with $\alpha \le r(A)(\xi)$ and $[\alpha, r(A)(\xi)] \cap K(A) \subset U$. We have two cases to consider. (1) $\alpha < \xi$ for any $\alpha \in S(A)$. In this case $\xi \in N$ and $r(A)(N) \subset U$ for $N = [\alpha, \rightarrow) \in \mathcal{N}(A)$. (2) There exists $\gamma \in S(A)$ with $\xi < \gamma$. Let $\beta = \min\{\gamma \in S(A) : \xi < \gamma\}$. In this case $\xi \in N$ and $r(A)(N) \subset U$ for $N = [\alpha, \beta) \in \mathcal{N}(A)$. Thus, $\mathcal{N}(A)$ is a network for $K(A)$ modulo $r(A)$. □

Corollary 3.29. *Suppose that X is a first countable countably compact subspace of an ordinal. Then $C_{p,2n+1}(X)$ is Lindelöf and a D-space for every $n \in \omega$.*

Corollary 3.30. *Let κ be a cardinal number and let X be a Σ-product in $[0, \omega_1)^\kappa$, then $C_{p,2n+1}(X)$ is Lindelöf and a D-space for every $n \in \omega$.*

Theorem 3.31. *If Y is a closed subspace of a Σ-product of cosmic spaces, then Y is monotonically retractable.*

Proof. Let $\{X_t : t \in T\}$ be a family of cosmic spaces, and let $\mathcal{P}_t$ be a countable network for X_t for every $t \in T$. Let $X = \prod\{X_t : t \in T\}$, let Z be a Σ-product in X centered at some point $a^* \in X$, and let Y be a closed subspace of Z. For each countable set $A \subset Y$ we shall construct $r(A)$, $K(A)$ and $\mathcal{N}(A)$ which witness that Y is monotonically retractable.

For each set $D \subset Z$ let $\mathcal{S}(D) = \bigcup\{\mathrm{supp}(y) : y \in D\}$. Let us observe that $\mathcal{S}$ is ω-monotone. For each finite subset F of T we denote by $\mathcal{P}_F(Y)$ the collection of all non-empty sets of the form $Y \cap \prod\{P_t : t \in T\}$, where $P_t \in \mathcal{P}_t$ for any $t \in F$ and $P_t = X_t$ for $t \in T \setminus F$. Given a finite subset F of X and a set $N \in \mathcal{P}_F$, we fix a point $y_N \in N \cap Y$. Given a set $E \subset T$, let $\mathcal{P}(E) = \bigcup\{\mathcal{P}_F : F \in [E]^{<\omega}\}$ and $\mathcal{D}(E) = \{y_N : N \in \mathcal{P}(E)\}$. Let us observe that the assignments $E \to \mathcal{P}(E)$ and $E \to \mathcal{D}(E)$ are ω-monotone. Finally, for any set $E \subset T$ let $e_E : X_E \to X$ be the map given by $e_E(x)(t) = x(t)$ for $t \in E$ and $e_E(x)(t) = a^*(t)$ for $t \in T \setminus E$. It is easy to see that e_E is an embedding.

Take a countable set $A \subset Y$. Let $\mathcal{S}_0(A) = \mathcal{S}(A)$, and $\mathcal{S}_{n+1}(A) = \mathcal{S}_n(A) \cup \mathcal{S}(\mathcal{D}(\mathcal{S}_n(A)))$ for any $n \in \omega$. Take $\mathcal{E}(A) = \bigcup\{\mathcal{S}_n(A) : n \in \omega\} \subset T$. By Remark 3.2 (2), the assignment $\mathcal{S}(A) \to \mathcal{E}(A)$ is ω-monotone. Hence, the assignment $A \to \mathcal{E}(A)$ is ω-monotone. Finally, let $r(A) = e_{\mathcal{E}(A)} \circ p_{\mathcal{E}(A)} \restriction Y$, $K(A) = r(A)(Y)$, and $\mathcal{N}(A) = \mathcal{P}(\mathcal{E}(A))$. We shall prove that $r(A)$, $K(A)$ and $\mathcal{N}(A)$ satisfy the conditions in Definition 3.4. Since $\mathcal{E}$ and $\mathcal{P}$ are ω-monotone, by Remark 3.2 (1), the assignment $\mathcal{N}$ is ω-monotone.

Claim 1. $\mathcal{N}(A)$ is a network for $K(A)$ modulo $r(A)$.

Proof of Claim 1. Let $y \in Y$ and $U \in \tau(r(A)(y), K(A))$. We can find a finite set $H \subset T$ and a collection $\{B_t : t \in T\}$ with B_t open in X_t for each $t \in H$ and $B_t = X_t$ for $t \in T \setminus H$ such that, if $B = \prod\{B_t : t \in T\}$ then $r(A)(y) \in B \cap K(A) \subset U$. Let $F = H \cap \mathcal{E}(A)$. For any $t \in F$ we can find $P_t \in \mathcal{P}_t$ such that $y(t) = r(A)(y)(t) \in P_t \subset B_t$. For $t \in T \setminus F$ let $P_t = X_t$. Take $N = Y \cap \prod\{P_t : t \in T\}$, then $y \in N$ and $N \in \mathcal{P}_F \subset \mathcal{P}(\mathcal{E}(A)) = \mathcal{N}(A)$. We shall prove that $r(A)(N) \subset U$. Let $x \in N$. If $t \in F$ then $r(A)(x)(t) = x(t) \in P_t \subset B_t$. For $t \in H \setminus F \subset T \setminus \mathcal{E}(A)$, we have $r(A)(x)(t) = a^*(t) = r(A)(y)(t) \in B_t$. Thus, $r(A)(x) \in B$ for any $x \in N$ and so $r(A)(N) \subset B \cap K(A) \subset U$.

Claim 2. $A \subset K(A) \subset Y$ and $r(A) : Y \to K(A)$ is a continuous retraction.

Proof of Claim 2. Let us observe that if $y \in Y$ then $r(A)(y) = y$ if and only if $\mathrm{supp}(y) \subset \mathcal{E}(A)$. If $y \in A$ then $\mathrm{supp}(y) \subset \mathcal{S}_0(A) \subset \mathcal{E}(A)$, and so $y = r(A)(y) \in K(A)$. Hence, $A \subset K(A)$. If $y \in Y$ then $\mathrm{supp}(r(A)(y)) \subset \mathcal{E}(A)$ and so $r(A)(r(A)(y)) = r(A)(y)$. Thus, $r(A)$ is a continuous retraction. We only need to show that $K(A) \subset Y$.

Notice that, since $\mathcal{D}(\mathcal{E}(A)) = \{y_N : N \in \mathcal{P}(\mathcal{E}(A))\} = \{y_N : N \in \mathcal{N}(A)\}$, we can conclude that $\mathcal{D}(\mathcal{E}(A)) \cap N \neq \emptyset$ for each $N \in \mathcal{N}(A)$. Moreover, if $y \in \mathcal{D}(\mathcal{E}(A)) = \mathcal{D}(\bigcup\{\mathcal{S}_n(A) : n \in \omega\}) = \bigcup\{\mathcal{D}(\mathcal{S}_n(A)) : n \in \omega\}$, then $y \in \mathcal{D}(\mathcal{S}_n(A))$ for some $n \in \omega$, and so $\mathrm{supp}(y) \in \mathcal{S}(\mathcal{D}(\mathcal{S}_n(A))) = \mathcal{S}_{n+1}(A) \subset \mathcal{E}(A)$; that is, $y = r(A)(y)$. Therefore, because of Claim 1 and Remark 3.1 (3), $\mathcal{D}(\mathcal{E}(A)) = r(A)(\mathcal{D}(\mathcal{E}(A)))$ is dense in $K(A)$. Now, since $\mathcal{E}(A)$ is countable, it is clear that $e_{\mathcal{E}(A)} \circ p_{\mathcal{E}(A)}(X) \subset Z$. Since Y is closed in Z and by the continuity of $e_{\mathcal{E}(A)} \circ p_{\mathcal{E}(A)}$, we have $e_{\mathcal{E}(A)} \circ p_{\mathcal{E}(A)}(\mathrm{cl}_X(Y)) \subset \mathrm{cl}_Z(e_{\mathcal{E}(A)} \circ p_{\mathcal{E}(A)}(Y)) = \mathrm{cl}_Z(r(A)(Y)) = \mathrm{cl}_Z(K(A)) = \mathrm{cl}_Z(\mathcal{D}(\mathcal{E}(A))) \subset \mathrm{cl}_Z(Y) = Y$. In particular, $K(A) = r(A)(Y) = e_{\mathcal{E}(A)} \circ p_{\mathcal{E}(A)}(Y) \subset Y$. $\square$

Corollary 3.32. *Any Corson compact space is monotonically retractable.*

Corollary 3.33. *If Y is a Σ-product of a family of Corson compact spaces, then $C_p(Y)$ is Lindelöf and has the D-property.*

Corollary 3.34. *If Y is a closed subspace of a Σ-product of cosmic spaces, then $C_{p,2n+1}(Y)$ is a D-space for every $n \in \omega$.*

Example 3.35. *There exists a compact scattered Sokolov space which is not monotonically retractable.*

Proof. Given a countable limit ordinal α, a *ladder* on α is a set $S_\alpha = \{\alpha(n) : n \in \omega\}$ of isolated ordinals in α such that $\alpha(n) < \alpha(m)$ whenever $n < m$ and $\alpha = \sup\{\alpha(n) : n \in \omega\}$. Let L be a set of countable limit ordinals of ω_1 such that L and $\omega_1 \setminus L$ are stationary sets in ω_1 and fix a ladder S_α on α, for any $\alpha \in L$. We associate a compact space X_L in the following standard way. Let $Q = \omega_1$ and declare all points of $\omega_1 \setminus L$ to be isolated in Q. If $\alpha \in L$ then the local base of α in Q is the family $\{\{\alpha\} \cup (S_\alpha \setminus F) : F$ is a finite subset of $S_\alpha\}$. Finally, consider the one-point compactification X_L of the locally compact space Q with ω_1 being the point at infinity. It is easy to see that X_L is scattered. Since $\omega_1 \setminus L$ is a stationary set in ω_1, it follows from Proposition 4.2 in [10] that X_L is a Sokolov space and hence has countable tightness.

We will prove that X_L is not monotonically retractable. Suppose that for any countable set $A \subset X_L$, we have assigned $K(A)$, $r(A)$, and $\mathcal{N}(A)$ as in Definition 3.4. Let $T_\alpha = S_\alpha \cup \{\omega_1\}$ and $T_\alpha^n = \{\alpha(0), \dots, \alpha(n)\} \cup \{\omega_1\}$, for any $\alpha \in L$ and $n \in \omega$. It follows from $T_\alpha = \bigcup\{T_\alpha^n : n \in \omega\}$ that $\mathcal{N}(T_\alpha) = \bigcup\{\mathcal{N}(T_\alpha^n) : n \in \omega\}$. It follows from $r(T_\alpha)(\alpha(n)) = \alpha(n)$ for each $n \in \omega$ and the continuity of $r(T_\alpha)$ that $r(T_\alpha)(\alpha) = \alpha$. Since $\mathcal{N}(T_\alpha)$ is a network for $K(T_\alpha)$ modulo $r(T_\alpha)$, the family $\mathcal{N}(T_\alpha)$ contains an element N_α such that $\alpha \in N_\alpha$ and $\alpha \in r(T_\alpha)(N_\alpha) \subset \{\alpha\} \cup S_\alpha$. Also, there exists a natural number $n_\alpha \in \omega$ such that $N_\alpha \in \mathcal{N}(T_\alpha^{n_\alpha})$.

Since L is stationary, there exists a stationary set $L' \subset L$ such that, for some $m \in \omega$ we have $n_\alpha = m$ for every $\alpha \in L'$. The function $\alpha \to \alpha(0)$ is a regressive map on L'; so, there exists an ordinal $\mu(0) \in \omega \setminus L$ and a stationary set $L_0 \subset L'$ such that $\alpha(0) = \mu(0)$ for any $\alpha \in L_0$. The function $\alpha \to \alpha(1)$ is a regressive map on L_0; so, there exists an ordinal $\mu(1) \in \omega \setminus L$ and a stationary set $L_1 \subset L_0$ such that $\alpha(1) = \mu(1)$ for any $\alpha \in L_1$. Repeating this procedure $m+1$ times we will obtain a stationary set $L_m \subset L'$ and ordinals $\mu(0), \dots, \mu(m)$ such that $\alpha(i) = \mu(i)$ for any $\alpha \in L_m$ and $i \leq m$. Consequently $T_\alpha^m = M = \{\mu(0), \dots, \mu(m)\} \cup \{\omega_1\}$ for all $\alpha \in L_m$. Hence, $N_\alpha \in \mathcal{N}(T_\alpha^{n_\alpha}) = \mathcal{N}(T_\alpha^m) = \mathcal{N}(M)$ for any $\alpha \in L_m$. Since $\mathcal{N}(M)$ is countable, there exist $N \in \mathcal{N}(M)$ and a stationary set $L_N \subset L_m$ such that $N_\alpha = N$ for each $\alpha \in L_N$. Then $\alpha \in N_\alpha = N$ for each $\alpha \in L_N$, that is $L_N \subset N$. Finally, if we take any $\alpha \in L_N$ then $r(T_\alpha)(L_N) \subset r(T_\alpha)(N_\alpha) \subset \{\alpha\} \cup S_\alpha$. Being L_N uncountable and by the continuity of $r(T_\alpha)$ we have $r(T_\alpha)(\omega_1) \in \{\alpha\} \cup S_\alpha$. This is a contradiction because $r(T_\alpha)(\omega_1) = \omega_1$. □

Theorem 3.36. *Let $\mathcal{C}$ be the class of all spaces which are homeomorphic to a closed subspace in a Σ-product of cosmic spaces. Then $\mathcal{C}$ is closed under closed subspaces of Σ-products.*

Proof. Let $\{Y_\alpha : \alpha < \kappa\}$ be a family of spaces in $\mathcal{C}$, let Z' be a Σ-product in $X' = \prod\{Y_\alpha : \alpha < \kappa\}$ centered at some point $a^* \in X'$ and let Y be a closed subspace of Z'. We shall prove that Y is homeomorphic to a closed subspace of a Σ-product of cosmic spaces.

There is no loss of generality to assume that for each $\alpha < \kappa$ there exists a family of cosmic spaces $\{X_t : t \in T_\alpha\}$ such that Y_α is a closed subspace of the Σ-product Z_α in $X_\alpha = \prod\{X_t : t \in T_\alpha\}$ centered at some point $a^*_\alpha \in X_\alpha$. We can suppose that the family $\{T_\alpha : \alpha < \kappa\}$ is pairwise disjoint. Let $T = \bigcup\{T_\alpha : \alpha < \kappa\}$ and $X = \prod\{X_t : t \in T\}$. We canonically identify X with $\prod\{X_\alpha : \alpha < \kappa\}$. Since $a^*(\alpha) \in Y_\alpha \subset Z_\alpha$, then Z_α coincides with the Σ-product in X_α centered at $a^*(\alpha)$, so we can suppose that $a^*(\alpha) = a^*_\alpha$. Let Z be the Σ-product in X centered at a^*. Let us observe that Z and the Σ-product of $\{Z_\alpha : \alpha < \kappa\}$ centered at a^* are canonically identified. Also, observe that $Y \subset Z' \subset Z \subset X$, so in order to finish the proof it is enough to show that Z' is closed in Z.

We will show that $Z \setminus Z'$ is open in Z. Let $x \in Z \setminus Z'$. Notice that since $a^*(\alpha) = a^*_\alpha$ we have $x(\alpha) \in Z_\alpha$ for each $\alpha < \kappa$. Also, there exists a countable set $A \subset \kappa$ such that $x(\alpha) = a^*(\alpha) \in Y_\alpha$ for any $\alpha \in \kappa \setminus A$. Because of $x \notin Z'$, there exists $\xi \in A$ such that $x(\xi) \in Z_\xi \setminus Y_\xi$. Since Y_ξ is closed in Z_ξ we conclude that $U_\xi = Z_\xi \setminus Y_\xi$ is open in Z_ξ. For $\alpha \in \kappa \setminus \{\xi\}$ let $U_\alpha = Z_\alpha$. Then for the canonical open set $U = \prod\{U_\alpha : \alpha < \kappa\}$ in $\prod\{Z_\alpha : \alpha < \kappa\}$ we have $x \in U \cap Z \subset Z \setminus Z'$. This shows that $Z \setminus Z'$ is open in Z. □

Corollary 3.37. *Let κ be a cardinal number and let X be a closed subspace of a Σ-product in $[0, \omega_1)^\kappa$, then X is monotonically retractable.*

4. Open Questions

The following list of questions contains some interesting problems that the author could not solve while working on this paper.

Question 4.1. *Suppose that a compact space X is monotonically retractable. Must X be Corson compact?*

Question 4.2. *Suppose that a scattered compact space X is monotonically retractable. Must X be Corson compact?*

Question 4.3. *Suppose that X is a monotonically retractable compact space and $p(C_p(X)) = \omega$. Must X be metrizable?*

Question 4.4. *Suppose that X is a monotonically retractable realcompact space. Must X be Lindelöf?*

Question 4.5. *Suppose that a space X is monotonically retractable and ω_1 is a caliber of X. Must X have a countable network?*

Question 4.6. *Suppose that a space X is monotonically retractable and X has a small diagonal. Must X have a countable network?*

Question 4.7. *Suppose that X is a space such that $(X \times X) \setminus \Delta$ is monotonically retractable. Must X have a countable network? Here $\Delta = \{(x,x) : x \in X\}$ is the diagonal of the space X.*

Question 4.8. *Suppose that X is hereditarily monotonically retractable. Must X have a countable network?*

Question 4.9. *Suppose that X is a Lindelöf P-space. Must $C_p(X)$ be monotonically retractable?*

References

[1] A. V. Arhangel'skii and M. Tkachenko, *Topological groups and related structures*, Atlantis Press-World Scientific., 2008.

[2] R. Buzyakova, *In search for Lindelöf C_p's*, Comment. Math. Univ. Carolin. **45** (1) (2004), 145–151.

[3] E. K. van Douwen and W. F. Pfeffer, *Some properties of the Sorgenfrey line and related spaces*, Pacific J. Math. **81** (1979), 371–377.

[4] T. Eisworth, *On D-spaces, in: Open Problems in Topology II*, E. Pearl, ed., Elsevier, Amsterdam, 2007, 129–134.

[5] R. Engelking, *General Topology*, second edition, Sigma Series in Pure Mathematics, **6**, Heldermann, Berlin, 1989.

[6] G. Gruenhage, *A survey of D-spaces*, Contemporary Math. **553** (2011), 13–28.

[7] S. P. Gul'ko, *On the structure of spaces of continuous functions and their complete paracompactness*, Russian Math. Surveys **34:6** (1979), 36–44.

[8] L.-Xue Peng, *The D-property of some Lindelöf spaces and related conclusions*, Topology Appl. **154** (2007), 469–475.

[9] R. Rojas-Hernández, Á. Tamariz-Mascarúa, *D-property, monotone monolithicity and function spaces*, Topology Appl. **159** (2012), 3379–3391.

[10] G. A. Sokolov, *Lindelöf property and the iterated continuous function spaces*, Fund. Math. **143** (1993), 87–95.

[11] ———, *On some class of compact spaces lying in Σ-products*, Comment. Math. Univ. Carolin. **25:2** (1984), 219–231.

[12] D. T. Soukup, P. J. Szeptycki, *A counterexample in the theory of D-spaces*, Topology Appl. **159** (2012), 2669–2678.

[13] V. V. Tkachuk, *A nice class extracted from C_p-theory*, Comment. Math. Univ. Carolin. **46:3** (2005), 503–5013.

[14] ———, *A C_p-theory Problem Book. Topological and Function Spaces*, Springer, New York, 2011.

[15] ———, *Countably compact first countable subspaces of ordinals have the Sokolov property*, Quaestiones Mathematicae **34** (2011), 225–234.

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