TOPOLOGY PROCEEDINGS
Volume 43, 2014
Pages 319–330

http://topology.auburn.edu/tp/

Continuous characters of weakened group topologies for $\mathbb{R}^n$

by

T. Christine Stevens

Electronically published on October 13, 2013

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 43 (2014)
Pages 319-330

E-Published on October 13, 2013

CONTINUOUS CHARACTERS OF WEAKENED GROUP TOPOLOGIES FOR $\mathbb{R}^n$

T. CHRISTINE STEVENS

Abstract. We study the continuous characters of a collection of metrizable group topologies for $\mathbb{R}^n$ that are weaker than the usual topology. These topologies are defined by choosing a sequence $\{v_j\}$ of elements of $\mathbb{R}^n$ and specifying the approximate rate $\{p_j\}$ at which it converges to zero. If $\{v_j\}$ goes to infinity sufficiently fast in the usual topology, then such a group topology always exists. Since neither the resulting groups nor their completions are locally compact, classical duality theory does not apply to them. We investigate the group of their continuous characters, proving, for example, that there are infinitely many non-trivial characters, which are arranged in a "fractal-like" fashion. These group topologies for $\mathbb{R}^n$ are not reflexive, in the sense of Pontryagin-van Kampen duality.

1. Introduction

In this paper we investigate the continuous characters of a collection of metrizable group topologies for $\mathbb{R}^n$ that are created by choosing a sequence $\{v_j\}$ of elements of $\mathbb{R}^n$ and specifying the approximate rate $\{p_j\}$ at which it will converge to zero. The topologies in question, which are always weaker than the usual topology, are defined in [11].

The current paper is part of a larger project, part of which has been conducted in collaboration with Jon W. Short, that explores the local and global properties of these topological groups. Previous papers ([9], [10], [12]) have studied the effects on the topology of changing either the "converging sequence" $\{v_j\}$ or the "rate sequence" $\{p_j\}$. We now turn our attention to the continuous characters of these topological groups.

2010 *Mathematics Subject Classification.* Primary 22A05, 43A40; Secondary 54H11, 46A20.

Key words and phrases. Character, dual group, reflexive, metrizable group, Pontryagin duality, abelian topological group.

The author thanks the anonymous referee for many helpful suggestions.

Since their completions are not locally compact, the Pontryagin-van Kampen duality theorem does not apply to them. Moreover, it is not obvious *a priori* that they will have *any* non-trivial continuous characters, for Nienhuys [8] gave an example of a group topology for $\mathbb{R}$ that is metrizable and weaker than the usual topology, but for which the only continuous character is the trivial one.

Our main theorem settles this question by proving that the topological groups under consideration always have infinitely many non-trivial continuous characters. In fact, the group of continuous characters is algebraically isomorphic to an uncountable subgroup of $\mathbb{R}^n$ that is dense in $\mathbb{R}^n$ in the usual topology, and the complement of that subgroup is also uncountable and dense.

Our work is related to recent results on the duality theory of abelian topological groups that are not locally compact. These results were thoroughly surveyed by M.J. Chasco, D. Dikranjan, and E. Martín-Peinador in [4]. Indeed, it was a question from Dikran Dikranjan that prompted the author to undertake a careful study of the continuous characters of the groups in this paper. As we will see, our main theorem implies that these groups are not Pontryagin-reflexive.

After providing the necessary background and terminology in Section 2, we state the main theorem in Section 3 and prove it in Section 4. Section 5 proves a theorem involving Pontryagin duality and lays out possible avenues for future research.

2. Notation and Terminology

$\mathbb{R}$ will denote the set of real numbers and $\mathbb{R}^n$ the (set-theoretic) product of n copies of $\mathbb{R}$; the group operation on these sets will always be addition. We will often write an element x of $\mathbb{R}^n$ as $x = (x_1, \ldots, x_n)$, where each $x_i \in \mathbb{R}$. If $x, y \in \mathbb{R}^n$, then $\|x\|$ will denote the usual Euclidean norm of x, and $d(x, y) = \|x - y\|$ is the usual Euclidean distance from x to y. Since we will be examining many different group topologies for $\mathbb{R}^n$, topological statements will always mention the specific topology under consideration. The usual topology for $\mathbb{R}^n$ will be denoted by $\mathcal{U}$.

$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ will denote, respectively, the natural numbers, the integers, and the rational numbers. Unless stated otherwise, all sums will be assumed to have only finitely many terms.

$\mathbb{T}$ will denote the multiplicative group of complex numbers of modulus 1, and it is always assumed to have the usual topology that it inherits from the complex numbers. We define an invariant metric ρ on $\mathbb{T}$ by letting $\rho(e^{ix}, e^{iy}) = \min\{|x - y - 2\pi n| : n \in \mathbb{Z}\}$, and we note that ρ generates the usual topology for $\mathbb{T}$.

If $(H,\mathcal{T})$ is an abelian topological group, then $(H,\mathcal{T})^\wedge$ will denote the set of all $\mathcal{T}$-continuous homomorphisms from $(H,\mathcal{T})$ to $\mathbb{T}$. The compact-open topology makes $(H,\mathcal{T})^\wedge$ a topological group, known as the dual of $(H,\mathcal{T})$, and $(H,\mathcal{T})^\wedge$ is complete whenever $(H,\mathcal{T})$ is metrizable [4, Fact A, p. 2294].

Definition 2.1. An abelian topological group $(H,\mathcal{T})$ is *Pontryagin-reflexive* if the evaluation map from $(H,\mathcal{T})$ to $(H,\mathcal{T})^{\wedge\wedge}$ is a topological isomorphism, where $(H,\mathcal{T})^{\wedge\wedge}$ is the dual of $(H,\mathcal{T})^\wedge$ and has the compact-open topology.

Our strategy for constructing group topologies on $\mathbb{R}^n$ relies on the notion of a *groupnorm* (or simply a *norm*).

Definition 2.2. A *groupnorm* on an abelian group G is a function $\nu : G \to \mathbb{R}$ satisfying, for all $x, y \in G$,

(i) $\nu(x) \geq 0$;
(ii) $\nu(x) = 0$ if and only if $x = 0$;
(iii) $\nu(x + y) \leq \nu(x) + \nu(y)$;
(iv) $\nu(x) = v(-x)$.

If ν is a groupnorm on G, then the function $r(x,y) = \nu(x-y)$ defines a translation-invariant metric on G, and the corresponding metric topology makes G a topological group. Blurring the distinction between the norm ν, the metric r, and the topology it induces on G, we will denote by (G,ν) the group G with the topology induced by r.

In [11] the author introduced the following method for constructing metrizable group topologies on $\mathbb{R}^n$ that are weaker than the usual topology. Their continuous characters are the subject of this paper.

Proposition 2.3. [11, Proposition 4.1] *Let $\{p_j : j \in \mathbb{N}\}$ be a non-increasing sequence of positive real numbers which converges to zero in the usual topology on $\mathbb{R}$, and let $\{v_j : j \in \mathbb{N}\}$ be a sequence of non-zero elements of $\mathbb{R}^n$ such that $\{\|v_j\|\}$ is non-decreasing and the sequence $\{p_{j+1}\|v_{j+1}\|/\|v_j\|\}$ has a positive lower bound. Then the function $\nu : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\nu(x) = \inf\left\{\sum |c_j| p_j + \left\| x - \sum c_j v_j \right\| : c_j \in \mathbb{Z}\right\}$$

is a groupnorm on $\mathbb{R}^n$ such that $\nu(x) \leq \|x\|$ for all $x \in \mathbb{R}^n$ and $\nu(v_j) \leq p_j$. ν gives rise to a metrizable group topology on $\mathbb{R}^n$, weaker than the standard topology, in which $v_j \to 0$.

Another way to describe ν is that it is the largest norm on $\mathbb{R}^n$ such that $\nu(v_j) \leq p_j$ for all $j \in \mathbb{N}$ and $\nu(x) \leq \|x\|$ for all $x \in \mathbb{R}^n$. Roughly speaking, we can say that an element x of $\mathbb{R}^n$ is near the origin in the ν-topology if there is an element of the subgroup generated by the sequence $\{v_j\}$ that is near x in the usual topology and also near the origin in the ν-topology.

Definition 2.4. If the sequences $\{v_j\}$ and $\{p_j\}$ satisfy the hypothesis of Proposition 2.3, then $(\{v_j\}, \{p_j\})$ will be called a *sequential-norming pair* (*SNP*) for $\mathbb{R}^n$. If the groupnorm they induce is ν, then $(\{v_j\}, \{p_j\}, \nu)$ will be called a *sequential-norming triple* (*SNT*) for $\mathbb{R}^n$.

For example, $(\{(j! + \sqrt{2}, j\pi)\}, \{1/j\})$ is an SNP on $\mathbb{R}^2$. If ν is the corresponding norm, then $\nu(j! + \sqrt{2}, j\pi) \leq 1/j$, and thus $\{(j! + \sqrt{2}, j\pi)\}$ converges to zero in $(\mathbb{R}^2, \nu)$ at least as fast as $1/j$ converges to zero in the usual topology for $\mathbb{R}$.

It is important to note that, if $(\{v_j\}, \{p_j\}, \nu)$ is an SNT on $\mathbb{R}^n$, then the ν-topology might not be a product topology, in the sense that the projections of $(\mathbb{R}^n, \nu)$ onto the coordinate axes (with the subspace topology) need not be continuous.

3. Continuous characters

Let $(\{v_j\}, \{p_j\}, \nu)$ be an SNT for $\mathbb{R}^n$, let $f : \mathbb{R}^n \to \mathbb{T}$ be a ν-continuous homomorphism, and let $x = (x_1, \ldots, x_n)$ be in $\mathbb{R}^n$. Since the ν-topology is weaker than the usual topology $\mathcal{U}$ for $\mathbb{R}^n$, it follows that f is also $\mathcal{U}$-continuous, and thus f must have the form

$$f(x) = \exp(i(\theta_1 x_1 + \cdots + \theta_n x_n)), \tag{3.1}$$

where $\theta_k \in \mathbb{R}$, $1 \leq k \leq n$. For any $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, we will let f_θ denote the homomorphism defined by (3.1). Given an SNT, we want to study the set of all $\theta \in \mathbb{R}^n$ such that f_θ is ν-continuous. It is easy to see that this set, which we will denote by G, is a subgroup of $\mathbb{R}^n$. Our main theorem is the following:

Theorem 3.1. *Let $G = \{\theta \in \mathbb{R}^n : f_\theta$ is ν-continuous$\}$. G is an uncountable $\mathcal{U}$-dense subgroup of $\mathbb{R}^n$ whose complement is also uncountable and $\mathcal{U}$-dense in $\mathbb{R}^n$.*

The proof of Theorem 3.1 is contained in Section 4. Before embarking on that proof, we give some examples of continuous characters and note a situation in which, without Theorem 3.1, the existence of continuous characters might not be readily apparent.

We begin with a few examples where $n = 1$, that is, with SNT's for the real numbers $\mathbb{R}$. If $(\{v_j\}, \{p_j\}, \nu)$ is an SNT for $\mathbb{R}$, let V be the subgroup of $\mathbb{R}$ that is generated by the converging sequence $\{v_j\}$. If V is discrete in the usual topology for $\mathbb{R}$, then it is easy to find non-trivial continuous characters of $(\mathbb{R}, \nu)$, since any $\mathcal{U}$-continuous homomorphism from $\mathbb{R}$ to $\mathbb{T}$ whose kernel contains V will be ν-continuous.

For the SNT $(\{j!\}, \{1/j\}, \nu_1)$, for instance, we have $V = \mathbb{Z}$. It is clear that $2\pi\mathbb{Z} \subseteq G$, because any character of the form $f_{2\pi m}$, where $m \in \mathbb{Z}$, will map all the integers to 1 and thus will be ν_1-continuous. We claim, moreover, that $2\pi\mathbb{Q} \subseteq G$. Let $p, q \in \mathbb{Z}$, with $q \neq 0$. Then q divides $j!$ whenever $j \geq q$, and thus $f_{2\pi(p/q)}(j!) = 1$, for all sufficiently large j. Therefore $2\pi\mathbb{Q} \subseteq G$, and G is dense in $\mathbb{R}$, as Theorem 3.1 predicts.

If we look instead at the SNT $(\{j!+1\}, \{1/j\}, \nu_2)$, then $2\pi\mathbb{Z}$ is still a subset of G, for the same reason as before. But $\pi \notin G$, because $f_\pi(v_j) = \exp(i\pi(j!+1)) = -1$ for all $j \geq 2$, and thus G does not contain $2\pi\mathbb{Q}$. In this example, it is not as easy to find the closure of G, without the help of Theorem 3.1. From this example we also see that G might not be closed under scalar multiplication by rational numbers, and thus it is not necessarily a vector space.

Finally, consider the SNT $(\{j! + \sqrt{2}\}, \{1/j\}, \nu_3)$. Then V is a dense subgroup of $\mathbb{R}$ in the usual topology, and we cannot construct non-trivial ν_3-continuous characters simply by mapping every element of V to the identity. Indeed, without Theorem 3.1, it is not obvious that there are *any* non-trivial ν_3-continuous characters.

Turning our attention to SNT's on $\mathbb{R}^n$, where $n > 1$, we find that the situation remains complicated. For the SNT $(\{(j!, (-1)^j)\}, \{1/j\}, \nu_4)$ on $\mathbb{R}^2$, for example, it is easy to see that $2\pi\mathbb{Q} \times 2\pi\mathbb{Z} \subseteq G$, but it is difficult to find the closure of G without using Theorem 3.1. When $n > 1$, we must also deal with situations where the subgroup generated by the sequence $\{v_j\}$ is neither discrete nor dense in $(\mathbb{R}^n, \mathcal{U})$.

4. Proof of Main Theorem

Throughout this section, $(\{v_j\}, \{p_j\}, \nu)$ will be an SNT on $\mathbb{R}^n$, and G and f_θ will be as in Theorem 3.1. The proof of Theorem 3.1 is accomplished by means of a lemma and four propositions.

The following preparatory lemma provides a "Lipschitz-like" condition that is sufficient to guarantee that a homomorphism $f_\theta : \mathbb{R}^n \to \mathbb{T}$ is ν-continuous. It says that f_θ will be ν-continuous if the image under f_θ of the "converging sequence" $\{v_j\}$ converges to the identity at roughly the rate $\{p_j\}$.

Lemma 4.1. *Let $f : \mathbb{R}^n \to \mathbb{T}$ be a $\mathcal{U}$-continuous homomorphism. If there is a positive real number α such that $\rho(f(v_j), 1) \leq \alpha p_j$ for all sufficiently large j, then f is ν-continuous.*

Proof. Let m be a natural number such that $\rho(f(v_j), 1) \leq \alpha p_j$ for all $j \geq m$, and let $\epsilon > 0$ be given. Without loss of generality, we may assume that $\epsilon < p_m$. Because f is $\mathcal{U}$-continuous, there is a $\delta > 0$ such that $\rho(f(x), 1) < \epsilon/2$ whenever $\|x\| < \delta$. We may assume that $\delta < \min(\epsilon, \epsilon/(2\alpha))$. If $\nu(x) < \delta$, then the definition of ν implies that x can be written in the form $x = y + \sum c_j v_j$, where $c_j \in \mathbb{Z}$, $y \in \mathbb{R}^n$, the sum is finite, and $\|y\| + \sum |c_j| p_j < \delta$. Note that every non-zero c_j in this expression must have $p_j \le |c_j|p_j < \delta < \epsilon < p_m$, and thus all such c_j have index $j > m$, so that $\rho(f(v_j), 1) \le \alpha p_j$. Since $\|y\| < \delta$, we know that $\rho(f(y), 1) < \epsilon/2$. Since f is a homomorphism and ρ is invariant, we know that, for all $a, b \in \mathbb{R}^n$, $\rho(f(a+b), 1) = \rho(f(a)f(b), 1) = \rho(f(a), f(b)^{-1}) \le \rho(f(a), 1) + \rho(1, f(b)^{-1}) = \rho(f(a), 1) + \rho(f(b), 1)$. It follows that

$$\begin{aligned}\rho(f(x), 1) = \rho(f(y + \sum c_j v_j), 1) &\le \rho(f(y), 1) + \sum |c_j| \rho(f(v_j), 1) \\ &< \epsilon/2 + \alpha \sum |c_j| p_j < \epsilon/2 + \delta\alpha < \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Hence f is ν-continuous. □

The next four propositions prove Theorem 3.1 by showing, successively, that G is infinite, $\mathcal{U}$- dense in $\mathbb{R}^n$, and uncountable, and that the complement of G in $\mathbb{R}^n$ is also dense and uncountable. We begin by noting that, since $(\{v_j\}, \{p_j\}, \nu)$ is an SNT, we can choose a positive constant k such that

$$0 < k < \frac{p_{j+1}\|v_{j+1}\|}{\|v_j\|} \tag{4.1}$$

for all $j \in \mathbb{N}$. Our first proposition states that G is infinite.

Proposition 4.2. *The group G is infinite.*

Proof. We will prove that G is non-trivial, from which it will immediately follow that G is infinite. We start by establishing some notation. Choose $\alpha > 0$ such that

$$\pi/\alpha < k. \tag{4.2}$$

For each $j \in N$, we define a collection of parallel hyperplanes in $\mathbb{R}^n$. If $j \in \mathbb{N}$ and $m \in \mathbb{Z}$, we write v_j as $v_j = (v_{j_1}, \dots v_{j_n})$ and let

$$H_{j,m} = \{x \in \mathbb{R}^n : v_{j_1}x_1 + \cdots + v_{j_n}x_n = 2\pi m\}.$$

Then H_j, m is a hyperplane in $\mathbb{R}^n$. If we fix $j \in \mathbb{N}$ and take the union of $\{H_{j,m} : m \in \mathbb{Z}\}$, then we get the set of all $\theta \in \mathbb{R}^n$ such that $f_\theta(v_j) = 1$.

For future reference, we compute the distance (in the usual metric) between parallel hyperplanes. If $r, s \in \mathbb{R}$ and $H_r = \{x \in \mathbb{R}^n : v_{j_1}x_1 + \cdots + v_{j_n}x_n = r\}$ and $H_s = \{x \in \mathbb{R}^n : v_{j_1}x_1 + \cdots + v_{j_n}x_n = s\}$, it is an exercise in multi-variable calculus to show that the distance between H_r and H_s is $|r - s|/\|v_j\|$. (See, for example, Exercise 60 on p. 88 of [7].) In particular, the distance between $H_{j,m}$ and $H_{j,m+1}$ is $2\pi/\|v_j\|$.

We will now describe a procedure for finding a non-zero $\theta \in \mathbb{R}^n$ such that $\rho(f_\theta(v_j), 1) \le \alpha p_j$, for all sufficiently large j. According to Lemma 4.1, this will enable us to conclude that f_θ is ν-continuous.

Since k and α were chosen to satisfy (4.2) and since $p_j \to 0$ in the usual metric, we can choose $s \in \mathbb{N}$ such that, for all $j \geq s$,

$$p_j + \frac{\pi}{\alpha} < k \tag{4.3}$$

and

$$p_j \alpha < \pi. \tag{4.4}$$

For $j \geq s$, we will inductively choose integers $m_j \in \mathbb{Z}$ and points P_{j,m_j} on the hyperplane H_{j,m_j} such that, if A_{j,m_j} is the closed ball around P_{j,m_j} of radius $\alpha p_j/\|v_j\|$ (in the usual metric), then $A_{j+1,m_{j+1}} \subseteq A_{j,m_j}$. We begin by choosing an arbitrary $m_s \in \mathbb{Z}$ and then choose as P_{s,m_s} to be any point on the hyperplane H_{s,m_s}.

Assuming that P_{j,m_j} has been chosen, we show how to choose $P_{j+1,m_{j+1}}$ so that $A_{j+1,m_{j+1}} \subseteq A_{j,m_j}$. Consider the family $\{H_{j+1,z} : z \in \mathbb{Z}\}$ of hyperplanes. Since consecutive members of this family are $2\pi/\|v_{j+1}\|$ apart, every element of $\mathbb{R}^n$ is at most $\pi/\|v_{j+1}\|$ away from one (or two) of these hyperplanes. In particular, we can choose $m_{j+1} \in \mathbb{Z}$ so that P_{j,m_j} is at most $\pi/\|v_{j+1}\|$ away from $H_{j+1,m_{j+1}}$. Let $P_{j+1,m_{j+1}}$ be a point on $H_{j+1,m_{j+1}}$ such that $d(P_{j,m_j}, P_{j+1,m_{j+1}}) \leq \pi/\|v_{j+1}\|$, and let $A_{j+1,m_{j+1}}$ be the closed ball around $P_{j+1,m_{j+1}}$ of radius $\alpha p_{j+1}/\|v_{j+1}\|$. To show that $A_{j+1,m_{j+1}} \subseteq A_{j,m_j}$, let $Q \in A_{j+1,m_{j+1}}$. Then

$$\begin{aligned} d(P_{j,m_j}, Q) &\leq d(P_{j,m_j}, P_{j+1,m_{j+1}}) + d(P_{j+1,m_{j+1}}, Q) \\ &\leq \frac{\pi}{\|v_{j+1}\|} + \frac{\alpha p_{j+1}}{\|v_{j+1}\|} = \frac{\pi + \alpha p_{j+1}}{\|v_{j+1}\|}. \end{aligned}$$

It follows from (4.3) that $\alpha p_{j+1} + \pi < k\alpha$, and thus

$$d(P_{j,m_j}, Q) < k\alpha/\|v_{j+1}\|.$$

Then (4.1) and the fact that $\{p_j\}$ is non-increasing imply that $d(P_{j,m_j}, Q) < p_{j+1}\alpha/\|v_j\| \leq p_j\alpha/\|v_j\|$, and thus $Q \in A_{j,m_j}$.

Proceeding in this way, we get a decreasing sequence

$$A_{s,m_s} \supseteq A_{s+1,m_{s+1}} \supseteq \dots \tag{4.5}$$

of non-empty $\mathcal{U}$-compact subsets of $\mathbb{R}^n$, whose intersection is therefore non-empty. In fact, since the diameter of the closed ball A_{j,m_j} converges to zero as $j \to \infty$, the intersection consists of a single point θ. We claim that f_θ is ν-continuous.

To establish this claim, we first show that, for each $j \geq s$, A_{j,m_j} is a subset of the closed region bounded by the hyperplanes that are defined by the equations

$$v_{j_1} x_1 + \cdots + v_{j_n} x_n = 2\pi m_j \pm \alpha p_j.$$

This follows from the fact that these hyperplanes are $2\alpha p_j/\|v_j\|$ apart and the fact that P_{j,m_j}, which is on H_{j,m_j}, is halfway between them. Since any point in A_{j,m_j} is at most $\alpha p_j/\|v_j\|$ from P_{j,m_j}, it must lie in the desired closed region.

Thus θ must satisfy the inequality $|v_{j_1}\theta_1 + \dots v_{j_n}\theta_n - 2\pi m_j| \leq \alpha p_j$ for all $j \geq s$. It follows from the definition of the invariant metric ρ that $\rho(f_\theta(v_j), 1) \leq \alpha p_j$, and we conclude from Lemma 4.1 that f_θ is ν-continuous. By choosing the initial integer m_s and the initial point P_{s,m_s} in such a way that P_{s,m_s} is more than $\alpha p_s/\|v_s\|$ away from the origin (in the usual metric), we can guarantee that $0 \notin A_{s,m_s}$. Thus $\theta \neq 0$, and f_θ is not the trivial homomorphism. It follows that G, as a non-trivial subgroup of $\mathbb{R}^n$, is infinite. (Another way to show that G is infinite is to note that there are infinitely many choices for the initial point P_{s,m_s} that make the corresponding closed balls A_{s,m_s} pairwise disjoint.) $\square$

We now prove that G is $\mathcal{U}$-dense in $\mathbb{R}^n$.

Proposition 4.3. *G is dense in* $(\mathbb{R}^n, \mathcal{U})$.

Proof. To prove this proposition, we refine the procedure that was given in the proof of Proposition 4.2 for choosing the sequence (4.5). Let $Q \in \mathbb{R}^n$ and $\epsilon > 0$ be given, and recall that there is a natural number s such that inequalities (4.3) and (4.4) hold for all $j \geq s$. Since $p_j \to 0$ and $\|v_j\| \to \infty$ in the usual metric, we may assume, without loss of generality, that

$$\alpha p_s/\|v_s\| \leq \epsilon/2 \tag{4.6}$$

and

$$2\pi/\|v_s\| \leq \epsilon.$$

Let E_1 and E_2 be, respectively, the closed balls around Q of radius ϵ and $\epsilon/2$ (in the usual metric). Since E_2 has diameter ϵ and consecutive hyperplanes in the family $\{H_{s,z} : z \in \mathbb{Z}\}$ are $2\pi/\|v_s\| \leq \epsilon$ apart, we can choose $m_s \in \mathbb{Z}$ such that H_{s,m_s} intersects E_2. If we choose the initial point P_{s,m_s} to be any point in that intersection, then the triangle inequality and (4.6) imply that, for every $X \in A_{s,m_s}$,

$$d(Q, X) \leq d(Q, P_{s,m_s}) + d(P_{s,m_s}, X) \leq \epsilon/2 + \alpha p_s/\|v_s\| \leq \epsilon,$$

and thus $A_{s,m_s} \subseteq E_1$. If we proceed as in the proof of Proposition 4.2 to construct the sequence (4.5), then the corresponding θ will be in E_1. Therefore G is $\mathcal{U}$-dense in $\mathbb{R}^n$. $\square$

To prove that G is uncountable, we will modify the procedure that was described in the proof of Proposition 4.2 for choosing the sequence $\{P_{j,m_j}\}$. Our strategy is first to choose a fixed natural number $t > 1$, and then to choose the point $P_{j+1,m_{j+1}}$ in such a way that, at each step, there are at least t acceptable choices for $P_{j+1,m_{j+1}}$. The proof of the next proposition contains the details.

Proposition 4.4. *G is uncountable.*

Proof. After choosing $t \in \mathbb{N}$ with $t > 1$, we adopt a new criterion for selecting α and s. Instead of making (4.2) true, as we did in the proof of Proposition 4.2, we choose $\alpha > 0$ to satisfy the stronger inequality $t\pi/\alpha < k$, and we then choose $s \in \mathbb{N}$ so that the inequalities (4.4) and

$$p_j + \frac{t\pi}{\alpha} < k \tag{4.7}$$

hold for all $j \geq s$. We choose the initial point P_{s,m_s} and the corresponding closed ball A_{s,m_s} as in the proof of Proposition 4.2.

Assuming that P_{j,m_j} has been chosen, we consider the closed ball F around it of radius $\frac{\alpha p_j}{\|v_j\|} - \frac{\alpha p_{j+1}}{\|v_{j+1}\|}$ (in the usual metric). Clearly $F \subseteq A_{j,m_j}$, and we note that the radius of F must be strictly positive, for the inequalities (4.7) and (4.1), combined with the fact that the sequence $\{p_j\}$ is non-increasing, imply that

$$p_{j+1} < k < \frac{p_{j+1}\|v_{j+1}\|}{\|v_j\|} \leq \frac{p_j\|v_{j+1}\|}{\|v_j\|},$$

so that $\frac{p_{j+1}}{\|v_{j+1}\|} < \frac{p_j}{\|v_j\|}$. Our goal is to choose $m_{j+1} \in \mathbb{Z}$ and a point $P_{j+1,m_{j+1}}$ on the hyperplane $H_{j+1,m_{j+1}}$ in such a way that $A_{j+1,m_{j+1}} \subseteq A_{j,m}$, and we want to estimate the number of distinct integers m_{j+1} that make this possible. We claim that F intersects at least t distinct hyperplanes in the family $\{H_{j+1,z} : z \in \mathbb{Z}\}$. To prove this, first divide the diameter of F by the distance $2\pi/\|v_{j+1}\|$ between consecutive hyperplanes in this family, obtaining the ratio

$$2\left(\frac{\alpha p_j}{\|v_j\|} - \frac{\alpha p_{j+1}}{\|v_{j+1}\|}\right)\frac{\|v_{j+1}\|}{2\pi} = \frac{\alpha}{\pi}\left(\frac{\|v_{j+1}\|p_j}{\|v_j\|} - p_{j+1}\right).$$

Using the fact that $p_j \geq p_{j+1}$ and then applying (4.1) and (4.7), we find that

$$\frac{\alpha}{\pi}\left(\frac{\|v_{j+1}\|p_j}{\|v_j\|} - p_{j+1}\right) \geq \frac{\alpha}{\pi}\left(k - p_{j+1}\right) > t.$$

Thus there are at least t hyperplanes in the family $\{H_{j+1,z} : z \in \mathbb{Z}\}$ that intersect F, and we choose $P_{j+1,m_{j+1}}$ to be any point in F that is on one of them. We again let let $A_{j+1,m_{j+1}}$ be the closed ball around $P_{j+1,m_{j+1}}$ of radius $\alpha p_{j+1}/\|v_{j+1}\|$. Then the triangle inequality guarantees that the distance from any point in $A_{j+1,m_{j+1}}$ to P_{j,m_j} is no more than

$$\frac{\alpha p_{j+1}}{\|v_{j+1}\|} + d(P_{j+1,m_{j+1}}, P_{j,m_j}) \leq \frac{\alpha p_{j+1}}{\|v_{j+1}\|} + \frac{\alpha p_j}{\|v_j\|} - \frac{\alpha p_{j+1}}{\|v_{j+1}\|} = \frac{\alpha p_j}{\|v_j\|},$$

so that $A_{j+1,m_{j+1}} \subseteq A_{j,m_j}$. As before, we obtain a decreasing sequence (4.5) of non-empty compact sets that converges to a $\theta \in \mathbb{R}^n$ that makes f_θ ν-continuous.

Finally, we observe that two choices for $P_{j+1,m_{j+1}}$ that lie on different hyperplanes in the family $\{H_{j+1,z} : z \in \mathbb{Z}\}$ will yield closed balls $A_{j+1,m_{j+1}}$ that are disjoint, for (4.4) implies that the radius $\frac{\alpha p_{j+1}}{\|v_{j+1}\|}$ of the closed ball $A_{j+1,m_{j+1}}$ is less than half the distance between any two distinct hyperplanes in the family $\{H_{j+1,z} : z \in \mathbb{Z}\}$, which is at least $\frac{2\pi}{\|v_{j+1}\|}$. Thus choosing $P_{j+1,m_{j+1}}$ to be on a different hyperplane will produce a different value of θ. Since there are at least t distinct hyperplanes to choose from at each step in the construction of the sequence (4.5), and since these choices yield distinct continuous characters, we conclude that G is uncountable. Moreover, the elements of G are arranged in a "fractal-like" fashion. □

We now turn our attention to the complement of G in $\mathbb{R}^n$.

Proposition 4.5. *The complement of G is uncountable and dense in $(\mathbb{R}^n, \mathcal{U})$.*

Proof. To show that the complement of G is uncountable, we use an argument similar to the proof of Proposition 4.4 (which in turn refers back to the proof of Proposition 4.2). In those proofs, we now replace the hyperplane $H_{j,m}$ by

$$H'_{j,m} = \{x \in \mathbb{R}^n : v_{j_1}x_1 + \cdots + v_{j_n}x_n = (2m+1)\pi\}.$$

We again obtain a decreasing sequence (4.5) of non-empty compact sets that converges to some $\theta \in \mathbb{R}^n$.

In the modified construction, however, the closed ball A_{j,m_j} is contained within the closed region bounded by the hyperplanes

$$v_{j_1}x_1 + \cdots + v_{j_n}x_n = (2m_j+1)\pi \pm \alpha p_j,$$

and thus $|v_{j_1}\theta_1 + \ldots v_{j_n}\theta_n - 2\pi m_j - \pi| \leq \alpha p_j$. It follows that

$$\rho(\exp(i(v_{j_1}\theta_1 + \cdots + v_{j_n}\theta_n - \pi)), 1) \leq \alpha p_j.$$

Since ρ is invariant, this implies that

$$\rho(f_\theta(v_j), -1) = \rho(-f_\theta(v_j), 1) = \rho(-\exp(i(v_{j_1}\theta_1 + \cdots + v_{j_n}\theta_n)), 1)$$

$$= \rho(\exp(i(v_{j_1}\theta_1 + \cdots + v_{j_n}\theta_n - \pi)), 1) \leq \alpha p_j.$$

Hence $f_\theta(v_j) \to -1$ as $j \to \infty$, and thus f_θ is not ν-continuous. Therefore the complement of G is non-empty. Moreover, as in the proof of Proposition 4.4, we can arrange things so that, at each step, the point $P_{j+1,m_{j+1}}$ can be chosen to be on any one of at least $t > 1$ distinct hyperplanes, each of which will lead to a different value of θ that makes f_θ not ν-continuous. It follows that the complement of G is uncountable.

Finally, we prove that the complement of G is $\mathcal{U}$-dense in $\mathbb{R}^n$. This is easily accomplished by taking the proof of Proposition 4.3 and replacing each hyperplane $H_{j,m}$ by the hyperplane $H'_{j,m}$, as defined above. The argument in the proof of Proposition 4.3 then shows, *mutatis mutandis*, that every $Q \in \mathbb{R}^n$ is arbitrarily close to a θ such that f_θ is not ν-continuous. $\square$

To obtain a proof of Theorem 3.1, we simply combine Propositions 4.2 through 4.5.

5. Duality and some open questions

We conclude with some remarks about duality and some questions for future investigation. If $(\{v_j\}, \{p_j\}, \nu)$ is an SNT for $\mathbb{R}^n$, then Theorem 3.1 describes the group G as a subgroup of $(\mathbb{R}^n, \mathcal{U})$. It says nothing, however, about the compact-open topology $\mathcal{T}_{co}$ that G acquires as the dual of $(\mathbb{R}^n, \nu)$. It would be useful to determine the topological structure of G in the topology $\mathcal{T}_{co}$. Using Theorem 3.1, we can prove the following result:

Theorem 5.1. *The topology $\mathcal{U}_G$ that G inherits from $\mathcal{U}$ is properly contained in $\mathcal{T}_{co}$, and $(\mathbb{R}^n, \nu)$ is not Pontryagin-reflexive.*

Proof. Since the dual of $(\mathbb{R}^n,\mathcal{U})$ is $(\mathbb{R}^n,\mathcal{U})$, we can think of $\mathcal{U}_G$ as the compact-open topology that G acquires when regarded as a group of characters of $(\mathbb{R}^n,\mathcal{U})$. Because ν is weaker than $\mathcal{U}$, it follows that $\mathcal{U}_G \subseteq \mathcal{T}_{co}$. As we noted in Section 2, G is complete in the uniform structure associated with the compact-open topology $\mathcal{T}_{co}$. If the two topologies were equal, then G would also be complete in the uniform structure generated by $\mathcal{U}_G$, contradicting the fact that G is not closed in $(\mathbb{R}^n,\mathcal{U})$.

To prove that $(\mathbb{R}^n, \nu)$ is not Pontryagin-reflexive, we first note that it cannot be complete, by [5, p. 99, Cor. 6], and then invoke the fact that, for metrizable groups, completeness is a necessary condition for Pontryagin reflexivity [3, Cor. 2] $\square$

We close by indicating some possible directions for future research. As noted in [2, p. 84], the dual group of an abelian topological group is always locally quasi-convex in its compact-open topology, and thus a group that is Pontryagin-reflexive must be locally quasi-convex. Although topological groups that are generated by SNT's are not Pontryagin-reflexive, it would be interesting to determine whether they are nevertheless locally quasi-convex.

One can also invert the question posed in this paper. Instead of finding the continuous characters of a given SNT, one could ask what subgroups of $\mathbb{R}^n$ can occur as the character groups of SNT's. Given an uncountable, $\mathcal{U}$-dense subgroup G of $\mathbb{R}^n$, can one find an SNT $(\{v_j\}, \{p_j\}, \nu)$ such that G is the group of ν-continuous characters?

Finally, the strategies used here to construct topologies and continuous characters on $\mathbb{R}^n$ might be adapted to other abelian groups, thus shedding light on the long-standing problem (mentioned in [6], p. 649) of identifying those abelian Hausdorff topological groups that are Pontryagin-reflexive. According to Theorem 3 in [3], the completion of $(\mathbb{R}^n, \nu)$ will be Pontryagin-reflexive if and only if it is reflexive with respect to the convergence structure described in [1].

The author hopes to investigate some of these questions in a future paper.

References

[1] M. M. Bruguera and E. Martín-Peinador,*Open subgroups, compact subgroups and Binz-Butzmann reflexivity*, Topology Appl. **72** (1996), 101–111.

[2] M. Bruguera, M. J. Chasco, E. Martín-Peinador, and V. Tarieladze, *Completeness properties of locally quasi-convex groups*, Topology Appl. **111** (2001), 81–93.

[3] M. J. Chasco, *Pontryagin duality for metrizable groups.*, Arch. Math. (Basel) 70 (1998), 22-28.

[4] M. J. Chasco, D. Dikranjan, and E. Martín-Peinador, *A survey on reflexivity of abelian topological groups*, Topology Appl. **159** (2012), 2290–2309.

[5] Taqdir Husain, *Introduction to Topological Groups*, W. B. Saunders Company, 1966.

[6] Samuel Kaplan, *Extensions of the Pontrjagin Duality I: Infinite Products*, Duke Math. J. **15** (1948), 549–658.

[7] Jerrold E. Marsden, Anthony J. Tromba, and Alan Weinstein, *Basic Multivariable Calculus*, Springer Verlag/W. H. Freeman, 1993.

[8] J. W. Nienhuys, *A solenoidal and monothetic minimally almost periodic group* Fund. Math **73** 1971/72, 167–169.

[9] Jon W. Short and T. Christine Stevens, *Weakened Lie groups and their locally isometric completions*, Topology Appl. **135** (2004), 47–61.

[10] ———, *Comparing locally isometric topologies for* $\mathbb{R}^n$, Topology Proc. 37 (2011), pp. 349–366.

[11] T. Christine Stevens, *Decisive subgroups of analytic groups*, Trans. Amer. Math. Soc. **274** (1982), 101–108.

[12] ———, *Rates of convergence in weakened group topologies for* $\mathbb{R}^n$, Topology Appl. **159** (2012), 2528–2534.

Department of Mathematics and Computer Science; Saint Louis University; Saint Louis, Missouri 63103

E-mail address: stevensc@slu.edu