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COMMON FIXED POINTS FOR SELF MAPS OF THE UNIT INTERVAL AND HAUSDORFF TOPOLOGICAL SPACES

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Abstract. Common fixed point theorems for continuous compatible self maps of the unit interval and general Hausdorff topological spaces are obtained. The self maps of the unit interval will typically be nonexpansive, whereas the self maps of the topological spaces will be continuous maps with proper orbits. We show, for example, that a commutative family of continuous self maps of a compact Hausdorff space has a common fixed point if all the maps have proper orbits.

1. Introduction

We first obtain common fixed point theorems for self maps of $I = [0, 1]$ by appealing to results of Jacek R. Jachymski [5] and Gerald Jungck [8], stated below. Next, common fixed point theorems for self maps of Hausdorff topological spaces are proved by using the concept of proper orbits. The use of proper orbits will be motivated by the above mentioned theorems for $[0, 1]$. Key to our approach is the use of compatible maps and Banach operator pairs. We begin with background regarding notation and definitions.

In [6], Jungck defined self maps f and g of a metric space (X, d) as being *compatible* if and only if $d(fg(x_n), gf(x_n)) \to 0$ when $f(x_n), g(x_n) \to c$ for some $c \in X$. In [7], it was shown that when (X, d) is compact and f and g are continuous, then f and g are compatible if and only if they commute on $C(f, g) = \{x \in X : f(x) = g(x)\}$, the set of *coincidence points* of f and g. And an example was provided of continuous self maps f and g of

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R (the reals) which were not compatible but satisfied $f(g(x)) = g(f(x))$ for $x \in C(f,g)$. This prompted the following definition stated here for topological spaces.

Definition 1.1 ([9]). Let X be a Hausdorff topological space and $f, g : X \to X$. f and g are *weakly compatible* if and only if $f(g(x)) = g(f(x))$ for $x \in C(f,g)$. If $C(f,g) \neq \phi$, f and g are said to be *nontrivially weakly compatible*.

We use N to denote the set of positive integers and $N_k = \{n \in N : n \leq k\}$ for $k \in N$. Let g be a continuous self map of a Hausdorff topological space X. Then K_g is the set of all continuous self maps f of X nontrivially weakly compatible with g. If $x \in X$, define $g^0(x) = x$ and $g^n(x) = g(g^{n-1}(x))$ for $n \in N$. We let $P(g) = \{x \in X : g^n(x) = x$ for some $n \in N\}$, the set of periodic points of g. $P_k(g) = \{x \in X : g^k(x) = x\}$ for $k \in N$, and $F(g) = P_1(g)$, the set of fixed points of g. Also, we let $C_1 = \{g : I \to I \mid g$ is continuous and $F(g)$ is a closed interval$\}$ and $C_2 = \{g : I \to I \mid g$ is continuous and $P(g) = F(g)\}$ [5].

Now consider the theorems below. Theorem 1.2 is Jungck's main result in paper [8]. Theorems 1.3 and 1.4 are excerpts from theorems of Jachymski [5].

Theorem 1.2. *A continuous map $g : I \to I$ has a common fixed point with each map $f \in K_g$ if and only if $P(g) = F(g)$.*

Theorem 1.3. *If g is a continuous self map of I, then the following are equivalent.*

(i) $g \in C_2$ $(P(g) = F(g))$.
(ii) *The sequence $\{g^n\}_{n=1}^{\infty}$ is point-wise convergent on I.*

Theorem 1.4. *If g is a continuous self map of I, then the following are equivalent.*

(i) $g \in C_1$ *($F(g)$ is a closed interval).*
(ii) *g has a common fixed point with every continuous map $f : I \to I$ that commutes with g on $F(g)$.*

Theorems 1.2 and 1.3 above show that when g is a continuous self map of I for which $P(g) = F(g)$, nice things happen. In the next section we ask what happens when $P(g) \neq F(g)$, mainly when g is *nonexpansive* ($|g(x) - g(y)| \leq |x - y|$).

2. Fixed Point Results for the Unit Interval $I = [0, 1]$

Lemma 2.1. *Let $f : I \to I$ be nonexpansive and suppose $P(f) \neq F(f)$. Then there exists $a \in I$ such that $P_2(f) = [a, f(a)]$ and $f(x) = f(a)+a-x$*

for $x \in [a, f(a)]$. Moreover, $x = (a + f(a))/2$ is the unique fixed point of f.

Proof. Since $P(f) \neq F(f)$, there exists $c \in [0, 1]$ such that $f^2(c) = c$ and $f(c) \neq c$ by [8, Theorem 3.4]. We assume without loss of generality that $f(c) > c$. Since f is continuous, there exists $a \in [0, 1]$ such that $a = \inf P_2(f) = f^2(a)$. Suppose that $f(a) \leq a$. Then $a \neq c$ and we have $f(a) \leq a < c < f(c)$; this is a contradiction since f is nonexpansive. Thus, $f(a) > a$.

Note that the points $(a, f(a))$ and $(f(a), f^2(a)) = (f(a), a)$ are on the line $y(x) = f(a) + a - x$. We show that $f(x) = y(x)$ for all $x \in [a, f(a)]$. For suppose there exists $x \in (a, f(a))$ such that $f(x) > y(x) = f(a)+a-x$. Observing that $f^2(a) = a$, we have $f(x) - f^2(a) > f(a) - x > 0$; this implies the contradiction $|f(x) - f(f(a))| > |x - f(a)|$. Similarly, $f(x) < y(x)$ for no $x \in [a, f(a)]$. Thus, $f(x) = f(a) + a - x$ on $[a, f(a)]$ and $d = (a + f(a))/2$ is a fixed point of f. In fact, d is the only fixed point of f since "f nonexpansive" implies that $f(x) \geq x + f(a) - a$ for $x \in [0, a]$ and $f(x) \leq x - f(a) + a$ on $[f(a), 1]$.

Moreover, $f^2(x) = f(a) + a - (f(a) + a - x) = x$ on $[a, f(a)]$; i.e., $[a, f(a)] \subset P_2(f)$. In fact, $P_2(f) = [a, f(a)]$. To verify this, first note that the designation of a implies $P_2(f) \subset [a, 1]$. Since f is nonexpansive, f^2 is also, and $P_2(f)$ must be a closed interval; therefore, $P_2(f) = [a, b]$ for some $b \geq f(a)$. Suppose $b > f(a)$. Remember that $z \in P_2(f)$ implies that $f(z) \in P_2(f)$. Now f is continuous, so we can choose z in the open interval $(f(a), b)$ and close to $f(a)$ such that $f^2(a) = a \leq f(z) < c < f(a)$. But $f(x) = f(a) + a - x$ on $[a, f(a)]$, so $z = f(f(z)) = f(a) + a - f(z) \leq f(a)$, contradicting $z \in (f(a), b)$. Thus, $b = f(a)$, and the proof is complete. □

The remark below follows easily by an inductive argument. Note that the given condition is *necessary* and *sufficient*.

Remark 2.2. Self maps f and g of a set X are *weakly compatible* if and only if $f^n(z) = g^n(z)$ for $n \in N$ when $f(z) = g(z)$.

Theorem 2.3. *Let $f : I \to I$ be nonexpansive and suppose $P(f) \neq F(f)$. Then there exists $c \in I$, which is the unique common fixed point of f, and all $g \in K_f \cap C_2$. Moreover, if $g \in K_f \cap C_2$ and $z \in C(f, g)$, then $f^n(z) = g^n(z)$ for $n \in N$ and $\{g^n(z)\}_{n \in N}$ converges to c.*

Proof. Let $g \in K_f \cap C_2$. Since $g \in K_f$, $f \in K_g$. But $g \in C_2$, so Theorem 1.2 implies that f and g have a common fixed point z. Lemma 2.1 implies that $z = c$, the unique fixed point of f. Thus, c is the unique common fixed point of f and all $g \in K_f \cap C_2$. Moreover, $g \in K_f \cap C_2$ implies that $g \in K_f$; i.e., f and g are weakly compatible. Consequently, if $z \in I$ such that $f(z) = g(z)$, then $f^n(z) = g^n(z)$ for $n \in N$ by Remark 2.2. But since

$g \in C_2$, Theorem 1.3 implies that $\{g^n(z)\}_{n\in N}$, and hence $\{f^n(z)\}_{n\in N}$, converges to some $x \in I$. Since f and g are continuous, $f(x) = g(x) = x$. Clearly, $x = c$ by the uniqueness of c. □

Example 2.4. To see that it is necessary to have $g \in K_f \cap C_2$ in Theorem 2.3, let $f(x) = 1 - x$ and $g(x) = 1 - x^2$ on I. These functions commute on their set of coincidence points {0,1}, and therefore $g \in K_f$. However, these points are also nontrivial periodic points of g so that $g \notin C_2$, and f and g have no common fixed point.

Corollary 2.5. *Let $f : I \to I$ be nonexpansive. If $P(f) = F(f)$, f has a common fixed point with each $g \in K_f$. If $P(f) \neq F(f)$, then f has a common fixed point with each $g \in K_f \cap C_2$.*

The following lemma leads us to a result guaranteeing a common fixed point for a family of self maps of I. The requirement that all maps have closed intervals as their fixed point set is prompted by a consideration of [8, Example 4.1]. The proof is comparable to that of [4, Theorem 4.3].

Lemma 2.6. *Let $\mathcal{F}$ be a family of self maps of I such that $\mathcal{F} \subset C_1$. If every pair $f, g \in \mathcal{F}$ has a common fixed point, then $\mathcal{F}$ has a common fixed point. In fact, $\cap\{F(f) : f \in \mathcal{F}\}$ is a nonempty closed interval.*

Proof. We first prove that any finite family in $\mathcal{F}$ has a common fixed point. To this end, let $P(n)$ denote "any n members of $\mathcal{F}$ have a common fixed point." $P(1)$ and $P(2)$ are true by hypothesis. Now let $n \in N$ such that $P(n)$ is true and suppose that $P(n+1)$ is false. (Note that by the above, $n + 1 \geq 3$.) Then there is a subset G of $\mathcal{F}$ having $n + 1$ functions f_i such that G has no common fixed point. However, since $P(n)$ is true, for each $i \in N_{n+1}$, we can let c_i be a common fixed point of the subset of G obtained by deleting f_i. Thus, $f_k(c_i) = c_i$ if and only if $i \neq k$. Clearly, all the c_i are distinct. For if there exist $i, k (i \neq k)$ such that $c_i = c_k$, we have the contradiction $f_k(c_k) = f_k(c_i) = c_i = c_k$. Consequently, since $n + 1 \geq 3$, we have distinct i, k, and j such that $c_i < c_k < c_j$, for which $f_k(c_i) = c_i$ and $f_k(c_j) = c_j$. But then the closed interval $[c_i, c_j]$ is a subset of $F(f_k)$ since by definition of C_1, $F(f_k)$ is an interval; therefore, we have the contradiction $f_k(c_k) = c_k$.

We have proved that the family $\{F(f) : f \in \mathcal{F}\}$ has the finite intersection property. Now I is compact and the sets $F(f)$ are closed since each function f is continuous. Therefore, the set $M = \cap\{F(f) : f \in \mathcal{F}\}$ is nonempty and compact. As such, M has a minimum element a and a maximum element b, so $M \subset [a, b]$. But if $f \in \mathcal{F}$, then $a, b \in F(f)$. Thus, $[a, b] \subset F(f)$ since $F(f)$ is an interval. We then have $[a, b] \subset M$ since f was an arbitrary element of $\mathcal{F}$, and we conclude that $M = [a, b]$. □

Theorem 2.7. *Let $\mathcal{F}$ be a family of self maps of I such that $\mathcal{F} \subset C_1$. Then $\cap\{F(f) : f \in \mathcal{F}\}$ is a nonempty closed interval provided one of the following is satisfied.*

(i) *$\mathcal{F} \subset C_2$, and each pair $f, g \in \mathcal{F}$ is nontrivially compatible.*
(ii) *Each pair $f, g \in \mathcal{F}$ commutes on at least one of $F(f)$ and $F(g)$.*

Proof. Suppose (i) holds and $f, g \in \mathcal{F}$. Since $f, g \in C_2$ and are nontrivially compatible, f and g have a common fixed point by Theorem 1.2. The conclusion follows by Lemma 2.6 since $\mathcal{F} \subset C_1$. On the other hand, assume that (ii) holds and $f, g \in \mathcal{F}$. Then $f, g \in C_1$ and are therefore continuous. By hypothesis we can assume that f commutes with g on $F(g)$. Then by Theorem 1.4 f and g have a common fixed point, and the desired conclusion again follows from Lemma 2.6. □

The following example shows it is unlikely that Theorem 2.7 can be generalized to higher dimensions.

Example 2.8. Let T denote the triangle in the plane (with the usual R^2 metric) having corners $A = (-1, 0), B = (1, 0)$, and $C = (0, 1)$. Define the functions f, g, $h : T \to T$ by $f(x, y) = (x - y, 0)$, $g(x, y) = (1 - y, y)$, and $h(x, y) = ((x+y-1)/2, (x+y+1)/2)$ for $x, y \in T$. Thus, $f(T)$ is the line segment $[A, B]$, $g(T) = [B, C]$ and $h(T) = [A, C]$. f actually sends the intersection of T with a line L parallel to $[A, C]$ into the intersection of L with $[A, B]$. g and h can be similarly described by lines parallel to $[A, B]$ and $[B, C]$, respectively. A check shows that $F(f) = [A, B]$, $F(g) = [B, C]$, and $F(h) = [C, A]$. Thus, the functions f, g, and h have " intervals" as fixed point sets and belong to C_1', a two-dimensional analogue of C_1. It is also clear that B is the only point of coincidence and the only common fixed point of f and g. And it is easy to show that any two functions commute at a common fixed point. Consequently, f and g are a nontrivially compatible pair. Likewise, g and h and f and h are nontrivially compatible pairs having C and A as their respective unique common fixed points. Moreover, none of the functions have nontrivial periodic points. However, $F(f) \cap F(g) \cap F(h) = \phi$. Consequently, a generalization of Theorem 2.7(i) is unlikely.

To see that a natural extension of Theorem 2.7(ii) is also unlikely, first note that f commutes with g on $F(f)$ if and only if $g(F(f)) \subset F(f)$. By the above, we see that $g(F(f)) = g([A, B]) = (1, 0) = B \subset F(f)$. Similarly, $h(F(g)) \subset F(g)$ and $f(F(h)) \subset F(h)$. Thus, (ii) is satisfied. (For future reference, note that even though, e.g., $h(F(g)) \subset F(g)$, it is not true that $g(F(h)) \subset F(h)$.)

If $g : I \to I$ is nonexpansive, then $g \in C_1$. Consequently, we have the following corollary to Theorem 2.7.

Corollary 2.9. *Let $\mathcal{F}$ be a family of nonexpansive self maps of I. Then $\cap\{F(f) : f \in \mathcal{F}\}$ is a nonempty closed interval provided one of the following holds.*

(i) *$\mathcal{F} \subset C_2$, and each pair $f, g \in \mathcal{F}$ is nontrivially compatible.*
(ii) *Each pair $f, g \in \mathcal{F}$ commutes on at least one of $F(f)$ and $F(g)$.*

Remark 2.10. Corollary 2.9(i) gives us [4, Theorem 4.3].

3. Proper Orbits

We now show how Theorem 1.3 identifies the concept of proper orbits as a possible means of obtaining generalizations/extensions of Theorem 1.2. Proper orbits were initially proposed in [10] as a topological generalization of diminishing orbital diameters, a metric space concept introduced by W. A. Kirk [12]. If g is a self map of a set X, the orbit $O(x)$ of g at x is defined by $O(x) = O_g(x) = \{g^n(x) : n \in N \cup \{0\}\}$; thus, $O(g^k(x)) = \{g^n(x) : n \geq k\}$ for $k \in N$. (In the following, we use $cl(M)$ to denote the closure of a set M.)

Definition 3.1 ([10])**.** Let g be a self-map of a topological space X and let $x \in X$. The orbit $O(x)$ of g at x is *proper* if and only if $O(x) = \{x\}$ or there exists $n = n_x \in N$ such that $cl(O(g^n(x))$ is a proper subset of $cl(O(x))$. If $O(x)$ is proper for each $x \in M \subset X$, we shall say that g has *proper orbits on M.* If $M = X$, we say g has *proper orbits.*

It is helpful to know that as a consequence of [11, Corollary 4.6], a continuous self map g of a Hausdorff topological space has proper orbits if and only if g has no nontrivial periodic points or recurrent points. (In [11] a point x is a *recurrent point* of g if and only if it is a limit point of $O(x)$.)

Example 3.2. Let $g : I \to I$ be continuous with the only other requirement being that $g(x) \neq x$ for $x \in (0, 1)$. g has proper orbits. (Verify)

Example 3.3. Let $C = \{(r, \theta) : \theta \in R\}$ be a circle in polar coordinates of radius $r > 0$ and let g be a rational rotation of C. Thus, $g(r, \theta) = (r, \theta + \phi)$ for some rational $\phi \in (0, 2\pi)$. It can be shown (see [10, Example 3.10]) that every point of C is a recurrent point of g, and therefore g does not have proper orbits. And if $\phi = \pi/2$, then g is periodic of period 4 and thus does not have proper orbits.

Theorem 1.3 tells us that if $g : I \to I$ is continuous, then $g \in C_2$ (i.e., $F(g) = P(g)$) if and only if $\{g^n\}_{n \in N}$ is point-wise convergent. However, by [11, Proposition 4.7], if g is a self map of a Hausdorff topological space and $\{g^n\}_{n \in N}$ is point-wise convergent, then g has proper orbits. On the other hand, we know that any continuous self map of a Hausdorff space with proper orbits has no nontrivial periodic points (i.e., $F(g) = P(g)$). Thus, if $g : I \to I$ is continuous, $g \in C_2$ if and only if g has proper orbits. Theorem 1.2 therefore reads, "If $g : I \to I$ is continuous, g has a common fixed point with each $f \in K_g$ if and only if g has proper orbits." From this we anticipate that the concept of proper orbits is a natural vehicle in our search for common fixed points in more general settings, particularly in the context of compatible maps. As evidence of this, consider the following theorem.

Theorem 3.4 ([10, [Corollary 3.7]). *Any continuous self map g of a compact Hausdorff topological space with proper orbits has a fixed point. Moreover, g has a common fixed point with each $f \in K_g$.*

The above theorem and the remaining results in this section suggest that the concept of proper orbits secures fixed point results with minimal restrictions on the underlying space.

Lemma 3.5. *Let g be a continuous self map of a Hausdorff topological space X having proper orbits. If M is a nonempty g-invariant compact subset of X, then $M \cap F(g) \neq \phi$ and compact.*

Proof. Since M is closed and g-invariant, $(g(M) \subset M))$, the restriction of g to M is a continuous self map of M having proper orbits. The conclusion follows from Theorem 3.4. □

In preparation for the first application of the above lemma, we remind the reader that if f and g are self maps of a set X, then $C(f, g)$ is the set of coincidence points of f and g. We shall also let $PC\,(f, g) = \{w = f(x) = g(x) : x \in C(f, g)\}$, called the points of coincidence of f and g. (See [1] and [4].) It is not difficult to verify that if f and g are weakly compatible, then

$$\text{(i)} \quad g(PC\,(f, g)) \subset PC(f, g) \subset C(f, g).$$

Theorem 3.6. *Let X be a Hausdorff topological space and $g : X \to X$ be continuous with proper orbits. If $f \in K_g$ and $cl\,(PC(f, g))$ is compact, then f and g have a common fixed point.*

Proof. First note that since g is continuous and $f \in K_g$, (i) implies that $g(cl(PC\,(f, g))) \subset cl(g(PC(f, g))) \subset cl(PC(f, g))$. Thus, $cl(PC(f, g))$ is g-invariant and nonempty ($f \in K_g$). Since $cl(PC(f, g))$ is compact by hypothesis, Lemma 3.5 implies there is a point $z \in cl(PC(f, g))$ such that $g(z) = z$. But since f and g are continuous, $C(f, g)$ is closed and (i) implies that $cl(PC(f, g)) \subset C(f, g)$. Therefore, $z \in C(f, g)$, and we have $f(z) = g(z) = z$, as promised. □

Let $g : [0, \infty) \to [0, \infty)$ be strictly increasing. Then the sequence $\{g^n(x)\}_{n \in N}$ is either constant or strictly monotonic for any x. It follows immediately that g has proper orbits.

Example 3.7. Let $g : [0, \infty) \to [0, \infty)$ be defined by $g(x) = x^p$ where p is a positive rational number greater than 1. Then g is strictly increasing and therefore has proper orbits. Let $f \in K_g$ and $x = a \in C(f,g)$. Then $f^n(a) = g^n(a)$ for $n \in N$ by Remark 2.2, and the orbit $O(g(a))$ is a subset of $PC(f, g)$. Suppose $cl(PC(f, g)) = cl(O(g(a))) \subset [0, 1]$. If $a \in \{0, 1\}$, then $cl(PC(f, g)) \subset F(g) = \{0, 1\}$. If $a \in (0, 1)$, $O(g(a))$ is infinite and the sequence $\{g^n(a)\}_{n \in N}$ converges to 0, a common fixed point of f and g. In this instance, $cl(PC(f, g)) = O(g(a)) \cup \{0\}$ and is compact. However, if $a > 1$, $cl(PC(f, g)) = cl(O(g(a))) \subset (1, \infty)$ and the sequence $\{g^n(a)\}_{n \in N}$ increases without bound; consequently, $cl(PC(f, g))$ is not compact, and the pair f and g has no common fixed point.

In the above example, let $g(x) = x^p$ be restricted to $[0, 1]$. When $a \in (0, 1)$, the points 0, 1, and $g^n(a)$ for $n \in N$ determine a unique *almost* piecewise linear function $l \in K_g$ as follows. Let $l(0) = 0$, $l(1) = 1$, and $l^n(a) = g^n(a)$ for $n \in N$. Since the sequence $\{g^n(a)\}_{n \in N}$ decreases to 0, we let the graph of l consist of $(0, 0)$ and the closed line segments connecting (right to left) the points $(1, 1), (a, g(a)), (g(a), g^2(a)), ..., (g^{n-1}(a), g^n(a)), ...$ Keeping in mind that the graph of g is concave up, Remark 2.2 assures us that g and the function l so constructed are weakly compatible. Thus, K_g actually contains uncountably many *almost* piecewise linear continuous functions l determined by the $a \in (0, 1)$.

We need the following definition for our next application of Lemma 3.5. The definition given in [4] is for metric spaces; we extend it to any set X.

Definition 3.8. Let f and g be self maps of a set X. The ordered pair (f, g) is a *Banach operator pair* (*B.O.P.*) if and only if $f(F(g)) \subset F(g)$. If (g, f) is also a B.O.P. so that $g(F(f)) \subset F(f)$, then f and g are called a *symmetric* B.O.P. (Note that (f, g) is a B.O.P. if and only if f and g commute on $F(g)$.)

Theorem 3.9. *Let $\mathcal{F}$ be a family of continuous self maps of a Hausdorff topological space X with proper orbits. If each pair in $\mathcal{F}$ is a symmetric B.O.P. and $cl(g(X))$ is compact for some $g \in \mathcal{F}$, then $\cap\{F(f) : f \in \mathcal{F}\}$ is nonempty and compact. Moreover, g has a common fixed point with each $f \in K_g$.*

Proof. We know that $cl(g(X))$ is compact by hypothesis and $cl(g(X))$ is g-invariant since g is continuous. It follows by Lemma 3.5 that $F(g) = F(g) \cap cl(g(X))$ is nonempty and compact. Let $\mathcal{F}^* = \{F(f) \cap F(g) : f \in \mathcal{F}\}$

and note that $\mathcal{F}^*$ is a collection of closed subsets of $F(g)$. We prove that $\mathcal{F}^*$ has the finite intersection property. To this end, let $f_1, f_2, ..., f_n \in \mathcal{F}$ and observe that $f_1(F(g)) \subset F(g)$ since f_1 and g are a symmetric B.O.P. Thus, $F(g)$ is a nonempty, compact, and f_1-invariant subset of X. Since f_1 has proper orbits by hypothesis, we know by Lemma 3.5 that $F(f_1) \cap F(g) \neq \phi$ and compact. Now suppose that $i \in N_{n-1}$ and that $F(g) \cap F(f_1) \cap ... \cap F(f_i)$ is nonempty and compact, and note that $f_{i+1}(F(g) \cap F(f_1) \cap ... \cap F(f_i)) \subset F(g) \cap F(f_1) \cap ... \cap F(f_i)$ since the pairs (f_{i+1}, g), $(f_{i+1}, f_1), ..., (f_{i+1}, f_i)$ are B.O.P.s. Thus, since f_{i+1} has proper orbits and $F(g) \cap F(f_1) \cap ... \cap F(f_i)$ is nonempty, compact, and f_{i+1}-noninvariant, Lemma 3.5 implies that $F(g) \cap F(f_1) \cap ... \cap F(f_i) \cap F(f_{i+1})$ is nonempty and compact. It follows inductively that $F(g) \cap \cap_{i=1}^{n} F(f_i) = \cap_{i=1}^{n}(F(f_i) \cap F(g))$ is nonempty. Since $f_1, f_2, ..., f_n$ were arbitrarily chosen, we conclude that $\mathcal{F}^*$ has the finite intersection property. Thus, $\cap \mathcal{F}^* = \cap \{F(f) : f \in \mathcal{F}\}$ is nonempty. Moreover, $\cap \{F(f) : f \in \mathcal{F}\}$ is compact since each member of $\mathcal{F}^*$ is compact.

To see that g has a common fixed point with each $f \in K_g$, observe that g has relatively compact proper orbits since $cl(g(X))$ is compact, and appeal to [10, Corollary 3.5]. □

Remark 3.10. Example 2.8 shows why it was necessary to require that the pairs of maps in the above theorem be symmetric Banach operator pairs. (The functions f, g, and h in Example 2.8 have proper orbits by a "pointwise convergence" argument.) Note also that a pair of self maps (f, g) is a symmetric B.O.P. if and only if they commute on $F(f) \cup F(g)$. Consequently, a pair of commuting maps ($fg = gf$) is certainly a symmetric B.O.P and we have the following corollary.

Corollary 3.11. *Let $\mathcal{F}$ be a commutative family of continuous self maps of a Hausdorff topological space having proper orbits. If $cl(g(X))$ is compact for some $g \in \mathcal{F}$, then $\cap\{F(f) : f \in \mathcal{F}\}$ is nonempty and compact. Moreover, g has a common fixed point with each $f \in K_g$.*

4. Retrospect

Theorem 3.4 above generalizes only the sufficiency portion of Theorem 1.2. In [3], M. Grinc and Ľ. Snoha obtain a complete generalization of Theorem 1.2 for triangular maps of I^n which yields Theorem 1.2 if $n = 1$. For other applications of weakly compatible maps in the spirit of Theorem 3.6, see [4]. And for an informative survey regarding coincidence values and common fixed points for commuting functions (and variants thereof), see Eric L. McDowell's article [13]. Theorems 1.2, 1.3, and 1.4 are discussed therein.

Finally, to more fully appreciate the scope/applicability of maps with proper orbits, check [11] and consider the following. First note that, by definition, a self map g of a first countable space X is *orbitally continuous* (o.c.) if and only if $g(g^{n_i}(x)) \to g(c)$ when $g^{n_i}(x) \to c$ as $i \to \infty$. $K_g(oc)$ is the set of all o.c. self maps of X which are nontrivially weakly compatible with g. And, if X has a metric d, X is *g-orbitally complete* if and only if any Cauchy sequence $\{g^{n_i}(x)\}_{i\in N}$ in $O(x)$ converges in X for all $x \in X$. We are now ready for the following theorems.

Theorem 4.1 ([11, Theorem 6.6])**.** *Suppose g is an orbitally continuous self map of a first countable Hausdorff topological space. If g has relatively compact proper orbits, then g has a common fixed point with each $f \in K_g(oc)$.*

Definition 4.2. Let $g : X \to X$ and let (X, d) be a metric space. g is called a *quasi-contraction* if and only if there exists $\gamma \in (0, 1)$ such that

$$d(g(x), g(y)) \le \gamma \max\{d(x, y), d(x, g(x)), d(y, g(y)), d(x, g(y)), d(y, g(x))\},$$
$$\text{for } x, y \in X.$$

Lj. B. Ćirić proved the following powerful result in his much cited paper [2].

Theorem 4.3. *If $g : X \to X$ is a quasi-contraction on a g-orbitally complete metric space, then*

(a) *g has a unique fixed point, say u;*
(b) $\lim_{n\to\infty} g^n(x) = u$, *for each $x \in X$;*
(c) $d(g^n(x), u) \le \gamma^n d(x, g(x))$.

Since by (a), $g(u) = u$, (b) implies that g is orbitally continuous. Moreover, (b) tells us that g has relatively compact proper orbits. But then, by Theorem 4.1, we have the following corollary.

Corollary 4.4. *If $g : X \to X$ is a quasi-contraction on a g-orbitally complete metric space, then*

(a) *g has a unique fixed point, say u;*
(b) $\lim_{n\to\infty} g^n(x) = u$, *for each $x \in X$;*
(c) $d(g^n(x), u) \le \gamma^n d(x, g(x))$;
(d) *u is the unique common fixed point of g and f for each $f \in K_g(oc)$.*

Moreover, since quasi-contractions are one of the most general metric space contractivity conditions for one function [14], (d) could be an addendum to a host of published "contractive" fixed point theorems for metric spaces.

We conclude with an example relating to Theorem 4.1.

Example 4.5. Let $g[0,\infty) \to [0,\infty)$ be defined by $g(x) = x + \sin(x)$. If $a \in [0,\infty)$, $a \in [2(k-1)\pi, 2k\pi] = I_k$ for some $k \in N$. If a is one of the endpoints of I_k, $g(a) = a$. Otherwise, as $n \to \infty$, $g^n(a) \to (2k-1)\pi$, the midpoint of I_k and a fixed point of g. Thus, g has relatively compact proper orbits and will have a common fixed point with each $f \in K_g$ by Theorem 4.1.

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