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ORDER IN SPACES OF PSEUDOQUOTIENTS

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Abstract. We investigate properties of pseudoquotient extensions of sets with relations to ordered spaces, in particular. In the case of ordered spaces, we compare the standard quotient topology in the space of pseudoquotients with the order topology.

1. Introduction

A *space of pseudoquotients*, denoted by $\mathcal{B}(X, S)$, is defined as the space of equivalence classes of pairs $x \times f$, where x is an element of a non-empty set X and f is an element of S, a commutative semigroup of injective maps from X to X, and $x \times f \sim y \times g$ if $gx = fy$. It is a generalization of the constructions of the field of quotients from an integral domain. If we take $X = \mathbb{Z}$ and $S = \mathbb{N}$ acting on $\mathbb{Z}$ by multiplication, then $\mathcal{B}(\mathbb{Z}, \mathbb{N}) = \mathbb{Q}$.

Let X be a topological space and S a commutative semigroup of continuous injections acting on X. We endow S with the discrete topology. Then $\mathcal{B}$ becomes a topological space with the topology defined as follows: we first define the product topology on $X \times S$ and then take the quotient topology in $\mathcal{B}$. This topology has natural properties (see, for example, [7] and [12]).

In the case of $\mathcal{B}(\mathbb{Z}, \mathbb{N})$, with the discrete topology in $\mathbb{Z}$, we do not obtain the natural topology in $\mathbb{Q}$. Instead we obtain the discrete topology. It is possible to obtain the natural topology in $\mathcal{B}(\mathbb{Z}, \mathbb{N})$ by considering order. In this note we consider a set X with a relation $\diamond$ and define an extension of $\diamond$ to $\mathcal{B}$. The defined extension has good properties; that is, it preserves various properties of $\diamond$ in X. Then we consider sets X with linear order and compare the order topology in $\mathcal{B}$ with its standard topology.

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The construction of pseudoquotients was introduced in [10] under the name of "generalized quotients." The motivation for the idea, early developments, and later modifications are described in [11]. The construction of pseudoquotients has desirable properties. For instance, it preserves the algebraic structure of X. There is growing evidence that pseudoquotients can be a useful tool (see, for example, [1], [2], or [3]).

2. Pseudoquotients

Let X be a nonempty set and let S be a commutative semigroup of injective maps from X into X. For $x \times f, y \times g \in X \times S$, we write $x \times f \sim y \times g$ if $gx = fy$. It is easy to check that this is an equivalence relation in $X \times S$. We define $\mathcal{B}(X,S) = (X \times S)/\sim$. If there is no ambiguity, we will write $\mathcal{B}$ instead of $\mathcal{B}(X,S)$. Elements of $\mathcal{B}$ will be called pseudoquotients. The equivalence class of $x \times f$ will be denoted by $\frac{x}{f}$. This is a slight abuse of notation, but we follow here the tradition of denoting rational numbers by $\frac{p}{q}$ even though the same formal problem is present there.

Elements of X can be identified with elements of $\mathcal{B}$ via the embedding $\iota : X \to B(X,S)$ defined by

$$\iota(x) = \frac{fx}{f},$$

where f is an arbitrary element of S. Action of S can be extended to $\mathcal{B}$ via

$$f\frac{x}{g} = \frac{fx}{g}.$$

If, for some $y \in X$, $f\frac{x}{g} = \iota(y)$, then we will write $f\frac{x}{g} \in X$ and $f\frac{x}{g} = y$, which is formally incorrect, but convenient and harmless. For instance, we have the natural equality $f\frac{x}{f} = x$.

Elements of S, when extended to maps on $\mathcal{B}$, become bijections. Indeed, for any $f \in S$ and $\frac{x}{g} \in \mathcal{B}$, we have

$$f\frac{x}{fg} = \frac{fx}{fg} = \frac{x}{g}.$$

The action of f^{-1} on $\mathcal{B}$ can be defined as

$$f^{-1}\frac{x}{g} = \frac{x}{fg}.$$

Consequently, S can be extended to a commutative group of bijections acting on $\mathcal{B}$.

3. Relations and Orders

Now we consider a nonempty set X with a relation $\diamond$ and a commutative semigroup S of injective maps from X into X such that every $f \in S$ is relation preserving, by which we mean that for every $x, y \in X$, we have

$$fx \diamond fy \quad \text{if and only if} \quad x \diamond y.$$

The relation $\diamond$ can be extended to $\mathcal{B}$ as follows:

$$\frac{x}{f} \diamond \frac{y}{g} \quad \text{if and only if} \quad gx \diamond fy.$$

Indeed, if $\frac{x}{f} \diamond \frac{y}{g}$, $x \times f \sim x' \times f'$, and $y \times g \sim y' \times g'$, then $gx \diamond fy$, and hence $g'gf'x \diamond f'fg'y$, which is equivalent to $g'gfx' \diamond f'fgy'$. Since f and g are relation preserving, we obtain $g'x' \diamond f'y'$, which means that $\frac{x'}{f'} \diamond \frac{y'}{g'}$. This shows that $\diamond$ is a well-defined relation in $\mathcal{B}$. Since $\frac{fx}{f} \diamond \frac{fy}{f}$ is equivalent to $f^2x \diamond f^2y$ and to $x \diamond y$, the relation defined in $\mathcal{B}$ is an extension of $\diamond$ in X. In other words, $x \diamond y$ if and only if $\iota(x) \diamond \iota(y)$.

If $h \in S$ and $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}$, then

$$\frac{x}{f} \diamond \frac{y}{g} \;\Leftrightarrow\; gx \diamond fy \;\Leftrightarrow\; hgx \diamond hfy \;\Leftrightarrow\; h\frac{x}{f} \diamond h\frac{y}{g}.$$

This shows that the extensions of functions in S to $\mathcal{B}$ are relation preserving.

Note that a finite number of pseudoquotients can always be written with the same denominator. In view of that, the extension of $\diamond$ to $\mathcal{B}$ can be defined as follows:

$$\frac{x}{f} \diamond \frac{y}{f} \quad \text{if and only if} \quad x \diamond y.$$

Theorem 3.1. *The following properties of $\diamond$ in X are inherited by the extension to $\mathcal{B}$:*

(1) *reflexivity,*
(2) *symmetry,*
(3) *antisymmetry,*
(4) *transitivity,*
(5) *if $x \diamond y$, then there exists $z \in X$ such that $x \diamond z \diamond y$.*

Proof. These are immediate consequences of the definitions. For example, to show that (5) is inherited by the extension to $\mathcal{B}$, it suffices to observe that if $\frac{x}{f} \diamond \frac{y}{f}$, then $\frac{x}{f} \diamond \frac{z}{f} \diamond \frac{y}{f}$ for any $z \in X$ such that $x \diamond z \diamond y$. □

Corollary 3.2. *If $(X, \leq)$ is a partially ordered set and S is a commutative semigroup of order-preserving injections in X, then the extension of $\leq$ to $\mathcal{B}$ is a partial order.*

Corollary 3.3. *If $(X, \leq)$ is a linearly ordered set and S is a commutative semigroup of order-preserving injections in X, then the extension of $\leq$ to $\mathcal{B}$ is a linear order.*

If $(X, \leq)$ is a partially ordered set, then by (a, b), we mean the set $\{x \in X : a < x < b\}$. Note that the interval $\left(\frac{a}{g}, \frac{b}{h}\right)$ can always be written in the form $\left(\frac{a'}{f}, \frac{b'}{f}\right)$, namely $\left(\frac{a}{g}, \frac{b}{h}\right) = \left(\frac{ha}{gh}, \frac{gb}{gh}\right)$. Moreover, if $\frac{x}{f} \in \left(\frac{a}{g}, \frac{b}{g}\right)$, then $\frac{gx}{fg} \in \left(\frac{fa}{fg}, \frac{fb}{fg}\right)$. In other words, if $\alpha, \beta, \gamma \in \mathcal{B}$ and $\gamma \in (\alpha, \beta)$, then α, β, and γ can be written with the same denominator. However, this does not mean that every element of the interval $\left(\frac{a}{f}, \frac{b}{f}\right)$ is of the form $\frac{x}{f}$.

Given a group X and a partial order $\leq$ on the set X, the pair $(X, \leq)$ is called an *ordered group* if, for all $w, x, y, z \in X$, $x \leq y$ implies $wxz \leq wyz$.

Theorem 3.4. *Let $(X, \leq)$ be an ordered group and let S be a commutative semigroup of injective order-preserving group homomorphisms from X to X. Then $\mathcal{B}$ is also an ordered group.*

Proof. The group operation in X can be extended to $\mathcal{B}$ as follows

$$\frac{x}{f}\frac{y}{g} = \frac{(gx)(fy)}{fg},$$

which simplifies to

$$\frac{x}{f}\frac{y}{f} = \frac{xy}{f}$$

when both elements are written with the same denominator. By Corollary 3.2, the extension to $\mathcal{B}$ of $\leq$ is a partial order. Now consider $\frac{w}{f}, \frac{x}{f}, \frac{y}{f}, \frac{z}{f} \in \mathcal{B}$. If $\frac{x}{f} \leq \frac{y}{f}$, then $x \leq y$, and thus $wxz \leq wyz$ since X is an ordered group. But this implies $\frac{w}{f}\frac{x}{f} \leq \frac{y}{f}\frac{z}{f}$, which completes the proof. □

Given a vector space X over the real numbers $\mathbb{R}$ and a partial order $\leq$ on the set X, the pair $(X, \leq)$ is called an ordered vector space if, for all $x, y, z \in X$ and $0 \leq \lambda \in \mathbb{R}$, $x \leq y$ implies $x + z \leq y + z$ and $\lambda y \leq \lambda x$.

Theorem 3.5. *Let $(X, \leq)$ be an ordered vector space and let S be a commutative semigroup of injective order preserving linear mappings from X to X. Then $\mathcal{B}$ is also an ordered vector space with the operations defined as $\frac{x}{f} + \frac{y}{g} = \frac{gx+fy}{fg}$ and $\lambda\frac{x}{f} = \frac{\lambda x}{f}$.*

Proof. We proceed as in Theorem 3.4 by writing all pseudoquotients with the same denominator and then the result follows trivially. □

Some properties of the order in X are not inherited by the extension to $\mathcal{B}$. For instance, if X has a largest element, $\mathcal{B}$ need not have a largest element. Indeed, consider $X = [0, 1]$ with its natural order. Let S be the semigroup consisting of functions of the form $f_\alpha(x) = \alpha x$ where $\alpha \in (0, 1)$. No pseudoquotient $\frac{x}{f_\alpha}$ can be the largest element in $\mathcal{B}$ since $\frac{x}{f_\alpha} < \frac{x}{f_\beta}$ for any $0 < \beta < \alpha$.

4. Order Topology and Quotient Topology in Spaces of Pseudoquotients

Now we assume that X is a topological space and S is a commutative semigroup of continuous injections acting on X. To define a topology in $\mathcal{B}$, we equip S with the discrete topology, then $X \times S$ with the product topology, and finally $\mathcal{B} = X \times S/\sim$ with the quotient topology. This topology has desirable properties. In particular, the embedding $\iota : X \to \mathcal{B}$ is continuous and S can be extended to a commutative group of homeomorphisms acting on $\mathcal{B}$. Various topological properties of pseudoquotients are investigated in [7], [5], [6], [9], and [12].

If $(X, \leq)$ is a linearly ordered set and S is a commutative semigroup of order-preserving injections in X, then we can use the extension of $\leq$ to $\mathcal{B}$ to define an order topology in $\mathcal{B}$. In this section we will compare the quotient topology and the order topology on $\mathcal{B}$. We will show that the quotient topology is always finer than the order topology. When the quotient topology is strictly finer than the order topology, we will analyze the quotient topology to determine when it is a GO topology.

By a *generalized ordered space* (GO space) (see, for example, [4]), we mean a triple $(X, \leq, \tau)$ where $\leq$ is a linear order of the set X and τ is a Hausdorff topology on X such that each point of X has a neighborhood base consisting of (possibly degenerate) intervals. It can be shown that GO spaces are topological subspaces of ordered spaces (see [8]).

Theorem 4.1. *The quotient topology on $\mathcal{B}$ is finer than the order topology on $\mathcal{B}$.*

Proof. Let $\frac{x}{f} \in \left(\frac{x_1}{h}, \frac{x_2}{h}\right)$. We will find a saturated set V open in $X \times S$ such that $x \times f \in V$ and $p(V) \subset (\frac{x_1}{h}, \frac{x_2}{h})$, where p is the quotient map from $X \times S$ to $\mathcal{B}$. Let

$$V = \bigcup_{g \in S} h^{-1}((gx_1, gx_2)) \times g. \tag{4.1}$$

Note that V is open in $X \times S$ because h is continuous. Also, $x \times f \in V$ because $hx \in (fx_1, fx_2)$. To show that V is saturated, let $y \times g \in V$ and let $y' \times g' \sim y \times g$. Since $\frac{x_1}{h} < \frac{y}{g} < \frac{x_2}{h}$, we have $\frac{x_1}{h} < \frac{y'}{g'} < \frac{x_2}{h}$. Therefore, $hy' \in (g'x_1, g'x_2)$, so that $y' \times g' \in V$. Thus, V is saturated, $p(V)$ is open, and $p(V)$ is clearly a subset of $\left(\frac{x_1}{h}, \frac{x_2}{h}\right)$. □

In general, the space of pseudoquotients $\mathcal{B}$ need not be Hausdorff even if X is Hausdorff (see [7]). Because the order topology is Hausdorff, from the above theorem we obtain the following result.

Corollary 4.2. *The quotient topology on $\mathcal{B}$ is Hausdorff.*

Throughout the rest of the section, $p : X \times S \to \mathcal{B}$ refers to the quotient map defined by $p(x \times f) = \frac{x}{f}$.

Theorem 4.3. *If X is connected, then the quotient topology and the order topology in $\mathcal{B}$ are the same.*

Proof. It suffices to show that the order topology is finer than the quotient topology. Let U be open in the quotient topology and let $\frac{x}{f} \in U$. Then $x \times f \in p^{-1}(U)$ and, since $p^{-1}(U)$ is open and the topology of S is discrete, there exists a basis element $(a, b) \times f$ of the topology of $X \times S$ such that $x \times f \in (a, b) \times f \subset p^{-1}(U)$. We show that $\frac{x}{f} \in (\frac{a}{f}, \frac{b}{f}) \subset p((a, b) \times f) \subset U$. Since $a < x < b$, clearly $\frac{a}{f} < \frac{x}{f} < \frac{b}{f}$. Let $\frac{c}{g} \in (\frac{a}{f}, \frac{b}{f})$. We show that there exists $d \in (a, b)$ such that $\frac{c}{g} \sim \frac{d}{f}$. We have that $g(a) < f(c) < g(b)$. Since X is connected and g is continuous, by the Intermediate Value Theorem, there exists $d \in (a, b)$ such that $f(c) = g(d)$, which implies that $\frac{c}{g} \sim \frac{d}{f}$ as desired. Thus, $(\frac{a}{f}, \frac{b}{f}) \subset p((a, b) \times f) \subset U$. □

From the above proof, we obtain the following result.

Theorem 4.4. *If X is connected, then any interval $\left(\frac{a}{f}, \frac{b}{f}\right) \subset \mathcal{B}$ is homeomorphic to an interval $(c, d) \subset X$.*

In [12] it is shown that $\mathcal{B}$ is connected if X is connected. From this result, we immediately obtain the following theorem.

Theorem 4.5. *If X is a linear continuum, then $\mathcal{B}$ is a linear continuum.*

If X is not connected, there are two possibilities: X is a union of open connected components or X has connected components that are not open. We will refer to connected components simply as components and to non-degenerate components as components with more than one point.

Theorem 4.6. *If all the connected components of X are open, then $\mathcal{B}$ with the quotient topology is a GO space.*

Proof. We first describe the GO topology τ that we claim to be equal to the quotient topology on $\mathcal{B}$. Let I be the set of $\frac{x}{f} \in \mathcal{B}$ such that if $y \times g \sim x \times f$, then y is an isolated point in X. Let L be the set of $\frac{x}{f} \in \mathcal{B} \setminus I$ such that if $y \times g \sim x \times f$, then y is either an isolated point or a left endpoint of a non-degenerate component in X, and let R be the set of $\frac{x}{f} \in \mathcal{B} \setminus I$ such that if $y \times g \sim x \times f$, then y is either an isolated point or a right endpoint of a non-degenerate component in X. The sets I, L, and R define a GO topology τ on $\mathcal{B}$: If $\frac{x}{f} \in I$, then $\{\frac{x}{f}\} \in \tau$; if $\frac{x}{f} \in L$, then $[\frac{x}{f}, \frac{y}{g}) \in \tau$ for all $\frac{y}{g} > \frac{x}{f}$; and if $\frac{x}{f} \in R$, then $(\frac{y}{g}, \frac{x}{f}] \in \tau$ for all $\frac{y}{g} < \frac{x}{f}$. Moreover, all sets in the standard open-interval topology are in τ.

Notice that if $\frac{x}{f}$ is not in I, L, or R, then it belongs to one of the following sets:

J is the set of $\frac{x}{f} \in \mathcal{B}$ for which there exists $y \times g \in X \times S$ such that $y \times g \sim x \times f$ and y is in the interior of a nondegenerate component.

K is the set of $\frac{x}{f} \in \mathcal{B}$ for which there exist $y \times g, z \times h \in X \times S$ such that $z \times h \sim y \times g \sim x \times f$, y is a left endpoint of a nondegenerate component, and z is a right endpoint of a nondegenerate component in X.

Consequently, $\mathcal{B} = I \cup L \cup R \cup J \cup K$.

We first show that τ is finer than the quotient topology. Let V be open in the quotient topology and let $\frac{x}{f} \in V$, so that $x \times f \in p^{-1}(V)$.

If $\frac{x}{f} \in I$, then $\frac{x}{f} \in \{\frac{x}{f}\} \subset V$.

If $\frac{x}{f} \in L$, then $x \times f \sim y \times g$, where y is a left endpoint of a nondegenerate component. $p^{-1}(V)$ is open, so $y \times g \in (a, b) \times g \subset p^{-1}(V)$ for some $a, b \in X$. Then $y \times g \in [y, b') \times g \subset (a, b) \times g \subset p^{-1}(V)$ for some b' such that $[y, b')$ is entirely inside a connected interval. Then, by an argument very similar to the one presented in Theorem 4.3, $\frac{x}{f} = \frac{y}{g} \in p([y, b') \times g) = [\frac{y}{g}, \frac{b'}{g}) \subset V$, as desired.

If $\frac{x}{f} \in R$, the proof is similar in view of the symmetry.

If $\frac{x}{f} \in J$, then there exists $y \times g$ such that $x \times f \sim y \times g$ and y is in the interior of a nondegenerate component, and we have $y \times g \in (a, b) \times g \subset p^{-1}(V)$ for some $a, b \in X$. We choose $a', b' \in X$ such that (a', b') is a proper subset of the component and $y \in (a', b') \subset (a, b)$. As in Theorem 4.3, we can say that $\frac{x}{f} = \frac{y}{g} \in p((a', b') \times g) = (\frac{a'}{g}, \frac{b'}{g}) \subset V$.

If $\frac{x}{f} \in K$, then there exist $y \times g$ and $z \times h$ such that $x \times f \sim y \times g \sim z \times h$, y is a left endpoint of a nondegenerate interval, and z is a right endpoint of a nondegenerate component. Then $\frac{x}{f} = \frac{y}{g} \in p([y, b') \times g) = [\frac{y}{g}, \frac{b'}{g}) \subset V$ and $\frac{x}{f} = \frac{z}{h} \in p((a', z] \times h) = (\frac{a'}{h}, \frac{z}{h}] \subset V$. Since $\frac{y}{g} = \frac{z}{h}$, we have $\frac{x}{f} \in (\frac{a'}{h}, \frac{b'}{g}) \subset V$, as desired.

Now we show that the quotient topology is finer than τ. By Theorem 4.1, open intervals are open in the quotient topology. It remains to show that the other sets in τ are open in the quotient topology.

If $\{\frac{x}{f}\} \in \tau$, then $\frac{x}{f} \in I$, and we have $p^{-1}(\{\frac{x}{f}\}) = \{y \times g : y \times g \sim x \times f\}$. Each y is an isolated point. Since the connected components of X are open, y must have an immediate predecessor a_y and successor b_y. Since

$\{y\} = (a_y, b_y)$, we have

$$p^{-1}\left(\left\{\frac{x}{f}\right\}\right) = \bigcup_{y\times g\sim x\times f} (a_y, b_y) \times g.$$

This set is open in $X \times S$, so $\{\frac{x}{f}\}$ is open in the quotient topology.

If $[\frac{x}{f}, \frac{y}{f}) \in \tau$, then $\frac{x}{f} \in L$. For $\frac{z}{g}$ in the interior of $[\frac{x}{f}, \frac{y}{f})$, we can find U open in the quotient topology such that $\frac{z}{g} \in U \subset (\frac{x}{f}, \frac{y}{f})$ as in Theorem 4.1. Otherwise, we find V open and saturated in $X \times S$ such that $\frac{x}{f} \in p(V) \subset [\frac{x}{f}, \frac{y}{f})$. Let

$$V = \bigcup_{g\in S} f^{-1}([gx, gy)) \times g.$$

By the same reasoning as in the proof of Theorem 4.1 (taking into account the left endpoint), V is saturated and $\frac{x}{f} \in p(V) \subset [\frac{x}{f}, \frac{y}{f})$. We must show that V is open. Note that $gx \times gf \sim x \times f$ implies that gx is either a left endpoint of an interval or an isolated point. In either case, since the connected components of X are open, $[gx, gy)$ must be an open set for all g. Since f is continuous, $f^{-1}([gx, gy))$ is open for all g. Thus, V is open and saturated, as desired. If $(\frac{x}{f}, \frac{y}{f}] \in \tau$, then $(\frac{x}{f}, \frac{y}{f}]$ is open in the quotient topology by a similar argument.

Thus, B with the quotient topology is a GO space. □

Corollary 4.7. *If X has finitely many connected components, then $\mathcal{B}$ with the quotient topology is a GO space.*

Corollary 4.8. *If the connected components of X are open and the order in X is dense, then the quotient topology equals the order topology in $\mathcal{B}$.*

Proof. It follows from the assumptions that there can be no isolated points in X. If the non-degenerate components had left or right endpoints, then they would be closed, for no neighborhood of an endpoint could be a subset of the component (since the endpoints cannot have immediate successors or predecessors). Thus, the non-degenerate components do not have endpoints. Therefore, the sets I, L, R, and K, as defined in the proof of Theorem 4.3, are empty, so τ is just the open-interval topology. □

Example 4.9. Let X be $(-\infty, -1] \cup [0, \infty)$ with the order inherited from $\mathbb{R}$. This space does not have dense order, but the connected components of X are open. The set $U = (-1, \infty) = [0, \infty)$ is open in the order topology on X. Let S be the set of functions $f : X \to X$ of the form $f(x) = px$ where $p \in [1, \infty)$. These functions are order preserving and injective, and they commute. The set $\mathcal{B}$ will consist of quotients of the form $\frac{x}{p}$ where $x \in X$ and $p \in [1, \infty)$; that is, $\mathcal{B} = \mathbb{R}$. Thus, basis elements in the order topology of $\mathcal{B}$ are of the form (a, b) where $a, b \in \mathbb{R}$. Consider the set $V = \bigcup_{f \in S} U \times f$. It is open because U is open. To prove V is saturated, suppose $x_1 \times p_1 \in V$ and $x_1 \times p_1 \sim x_2 \times p_2$. Then $x_2 = \frac{p_2 x_1}{p_1} \in U$, so that $x_2 \times p_2 \in V$. Thus, $p(V) = [0, \infty)$ is open in the quotient topology on $\mathcal{B}$ but not in the order topology on $\mathcal{B}$. Therefore, $\mathcal{B}$ with the quotient topology does not equal the order topology. However, $\mathcal{B}$ is a GO space, with $\mathcal{B} \setminus \{0\} = J$ and $I = \{0\}$.

Example 4.10. Let X be $(-\infty, -1] \cup \{0\} \cup [1, \infty)$. Then $\{0\}$ is an open set. Let S be defined as in the above example. Then $\mathcal{B}$ will again be $\mathbb{R}$. Consider the set $V = \bigcup_{f \in S} \{0\} \times f$. It is open in $X \times S$. It is also saturated since it consists of one equivalence class. Thus, $p(V) = \{0\}$ is open in $\mathcal{B} = \mathbb{R}$ in the quotient topology. Thus, the quotient topology does not equal the order topology. However, $\mathcal{B}$ is again a GO space, with $\mathcal{B} \setminus \{0\} = J$ and $L = \{0\}$.

If, on the other hand, X has connected components that are not open, then $\mathcal{B}$ need not be a GO space.

Example 4.11. Let $X = \mathbb{Q}_+$, the set of all positive rational numbers, and $S = \{f^n : n \in \mathbb{N}\}$ where $f : \mathbb{Q}_+ \to \mathbb{Q}_+$ is defined by $f(x) = x^2$. Notice that $\mathbb{Q}_+$ meets the above criteria; its connected components are points, which are not open. Also, S is a commutative semigroup of continuous, order-preserving injections. Then the space of pseudoquotients $\mathcal{B}$ will correspond to the subset of $\mathbb{R}$ of numbers which can be written in the form $q^{1/2^n}$ where q is rational. Fix rationals a and b such that $0 < a < 1 < b$. Consider the open set $V = \bigcup_{n \in \mathbb{N}} (a^{2^n}, b^{2^n}) \times f^n$ in $X \times S$. This set is saturated, for if $x \times f^n \in V$ and $x \times f^n \sim y \times f^m$, then clearly $a^{2^m} < y < b^{2^m}$. We will remove a set of equivalence classes from V to obtain a new saturated set V'. Let $a < q < 1$ be a positive rational such that q is not the square of any rational. We will remove all points corresponding to the equivalence classes $q^{1/2^n}$ for all $n \in \mathbb{N}$. Thus, if we remove $q^2 \times f$ from V, then we must also remove $q^{2^n} \times f^n$. In other words, from each open set $(a^{2^n}, b^{2^n}) \times f^n$, we remove the points $q^{2^i} \times f^n$ for $i = 1, 2, ..., n$. Note that V' will also be open because we have removed a finite number of points from each open set $(a^{2^n}, b^{2^n}) \times f^n$. Also, V' is saturated because we have removed whole equivalence classes. Thus, $p(V')$ is open in the quotient topology. Notice that $1 \in p(V')$. If $p(V')$ were open in a GO topology, then an interval that contains the point 1 and is a subset of $p(V')$ would exist. Note that any neighborhood of 1 in the order topology on $\mathcal{B}$ is not a subset of $p(V')$, for it will contain a number of the form $q^{1/2^n}$ for some n. Also, $\{1\}$ and $[1, x)$ are not open in the quotient topology because $p^{-1}(\{1\}) = \bigcup_{n \in \mathbb{N}} 1 \times f^n$ and $p^{-1}([1, x)) = \bigcup_{n \in \mathbb{N}} [1, x^{2^n}) \times f^n$, neither of which is open in $X \times S$. Therefore, no interval that is a subset of $p(V')$ containing the point 1 is open in the quotient topology. Thus, $\mathcal{B}$ with the quotient topology is not a GO space.

Example 4.12. Notice that in the last example, the connected components of X were points. The space $\mathbb{Q}_+ \times [0,1]$ with the dictionary order topology is an example of a space that has closed intervals as connected components. Let $S = \{f^n : n \in \mathbb{N}\}$ where $f : \mathbb{Q}_+ \to \mathbb{Q}_+$ is defined by $f(x,y) = (x^2, y)$. Note that S is a commutative semigroup of order-preserving injections. To show that the functions in S are continuous, note that$f^{-1}((x_1 \times y_1, x_2 \times y_2)) = \{x \times y : \sqrt{x}_1 < x < \sqrt{x}_2, 0 \leq y \leq 1\}$ if $\sqrt{x}_1$ and $\sqrt{x}_2$ are irrational, $\{x \times y : \sqrt{x}_1 < x < \sqrt{x}_2, 0 \leq y \leq 1\} \cup (\sqrt{x}_1 \times 0, \sqrt{x}_1 \times y)$ if $\sqrt{x}_1$ is rational and $\sqrt{x}_2$ is irrational, and likewise if $\sqrt{x}_2$ is rational or they are both rational. In each case, the set $f^{-1}((x_1 \times y_1, x_2 \times y_2))$ is open. If $\mathcal{B}'$ is the space of pseudoquotients obtained in the last example, then the space of pseudoquotients in this example will be $\mathcal{B} = \mathcal{B}' \times [0,1]$. Analogous to the above example, the set $V = (a \times 0, b \times 1) \setminus \bigcup_{n \in \mathbb{N}} (q^{1/2^n}, 0)$, for some $a, b \in \mathbb{Q}$ such that $0 < a < 1 < b$ and a q such that $a < q < 1$, will be open in $\mathcal{B}$. By a similar argument as in the previous example, a neighborhood of 1×0 that is open on the left would not be a subset of V, while a half-open interval of the form $[1 \times 0, x \times y)$ and the set $\{1 \times 0\}$ are not open in the quotient topology. Thus, there is no interval that contains the point 1×0, is a subset of V, and is open in the quotient topology. Thus, $\mathcal{B}$ with the quotient topology is not a GO space.

Here we summarize the results obtained in this section. If X is connected, the quotient and order topologies in $\mathcal{B}$ are equal. If X is not connected, we consider two cases. In the first case, X is a union of open connected components. Then the quotient topology is a GO topology, as described in Theorem 4.6. If, in addition, the order in X is dense, then the quotient topology equals the order topology. If the order in X is not dense, then the quotient topology need not equal the order topology (Example 4.9 and Example 4.10). In the second case, not all of the connected components of X are open. Then the quotient topology need not even be GO. We provided two examples, one in which the components of X are non-isolated points and one in which the components of X are non-open intervals. In both examples, $\mathcal{B}$ is not GO.

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