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by

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SERGEY V. MEDVEDEV

Abstract. The classical theorems about closed embedding of the space of rationals, the space of irrationals, the Cantor set, and their products into analytic spaces are generalized from Polish spaces to non-separable complete metric spaces.

1. Introduction

All spaces under discussion are metric.

Let A be an analytic set in a compact metric space X and $B = X \setminus A$. In 1928, W. Hurewicz [8] proved that if A is not an F_σ in X, then there is a compact set $K \subseteq X$ such that $K \cap A \approx \mathcal{N}$ and $K \cap B \approx Q$, where $\mathcal{N}$ is the space of irrationals and Q is the space of rationals. In 1978, Jean Saint-Raymond [19] described conditions under which the intersection $K \cap B \approx Q \times \mathcal{N}$, and in 1984 Fons van Engelen with Jan van Mill [4] established when $K \cap B \approx Q \times \mathcal{C}$, where $\mathcal{C}$ is the Cantor set. Moreover, it was proved in the latter paper that the set K can be chosen homeomorphic to the Cantor set. In 1985, the author [13] announced a Hurewicz-type theorem (namely, Theorem 2.2) about embedding of the Baire space $B(\tau)$ into an A-set of a complete metric space for an arbitrary cardinal τ. Afterwards, A. S. Kechris, A. Louveau, and W. H. Woodin [10] and Louveau and Saint-Raymond [11] showed that the condition $B = X \setminus A$ may be sometimes weakened to the condition $A \cap B = \emptyset$ when X is a compact metric space. In 1995, Henryk Michalewski and Roman Pol [18] got the same theorem about embedding of the Cantor set for a non-separable complete metric space. Later, J. Chaber and Pol [2] obtained a similar result for the Baire space $B(\aleph_1)$. Now we continue such research.

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Take an A-set A in a complete metric space X and arbitrary $B \subset X$ with $A \cap B = \emptyset$. Suppose $\mathbf{\Gamma}$ is a family of subsets of X which contains F_σ-subsets. We shall show that if A is not $\mathbf{\Gamma}$-separated from B, then there exists a closed set $C \subseteq X$ such that $C \subseteq A \cup B$, $C \approx B^*(k)$, and $C \cap B$ is an F_σ-subset of X. Varying $\mathbf{\Gamma}$, we obtain that $C \cap B$ may be homeomorphic to Q, $Q(k)$, $Q(k) \times \mathcal{C}$, or $Q(k) \times B(\tau)$; the space $Q(k)$ is described before Theorem 2.1. For example, the set $C \cap B \approx Q$ for $\mathbf{\Gamma} = \{F : F \text{ is an } F_\sigma\text{-subset of } X\}$ (see Theorem 4.1).

For demonstration of the above results, game theory was used in [10] and [11]. In particular, compactness of the Cantor set is a necessary part of the proofs. But every non-separable metric space is not compact. Therefore, we modify the original technique from the Hurewicz paper [8] to the non-separable case by means of Hansell's theorem [6]. We also use some ideas due to Saint-Raymond [19] and to van Engelen with van Mill [4].

The paper is organized as follows. In section 2 we introduce notation and recall some statements. In section 3 the family $\mathbf{\Gamma}$ is described and the basic Lemma 3.4 is proved. In section 4 three Hurewicz-type theorems are obtained by varying the family $\mathbf{\Gamma}$. We first obtain the "light" Theorem 4.1 to clear the method. Later, we prove the "hard" Theorem 4.6 and give a sketch of the proof for the "middle" Theorem 4.9.

Remark 1.1. Note that Chaber, G. Gruenhage, and Pol [1, Theorem 4.1] proved Theorem 2.4 for the case $k = \aleph_1$. Theorem 4.1 was proved for a compact metric space X in [10] and for a Polish space in [9, Theorem 21.22]. Theorem 4.1 is equivalent to the result due to Michalewski and Pol [18] (see also [7, Remark 3.9]). Theorem 4.9 was obtained by Chaber and Pol (see [2, Proposition 6.1]) for the case $k = \aleph_1$.

2. Notation and Some Statements

For all undefined terms and notation, see [5] or [9].

$X \approx Y$ means that X and Y are homeomorphic spaces. Let $\mathcal{P}$ be a topological property. A space X is *nowhere* $\mathcal{P}$ if no nonempty open subset of X has the property $\mathcal{P}$. A *clopen* set is a set which is both closed and open. A space X is called *weight-homogeneous* if, for every nonempty open subset $U \subseteq X$, we have $w(U) = w(X)$. We say that X is *locally of weight* $<k$ if each point from X has a neighborhood of weight $<k$. A space X is said to be σ-locally of weight $<k$ (in symbols, $X \in \mathcal{L}_k$) provided $X = \cup\{X_i : i \in \omega\}$, where each X_i is locally of weight $<k$. In particular, $\emptyset \in \mathcal{L}_k$ for any k. Clearly, $X \in \mathcal{L}_\omega \Leftrightarrow X$ is σ-discrete.

Let $\mathcal{U}$ be a family of subsets of a metric space X. Put $\cup\mathcal{U} = \cup\{U : U \in \mathcal{U}\}$ and $\operatorname{mesh}\mathcal{U} = \sup\{\operatorname{diam}(U)\colon U \in \mathcal{U}\}$.

For an infinite cardinal k we denote by $B^*(k)$ the Baire space $B(k)$ of weight k if $k > \omega$ (see [5, Example 4.2.12]) or the Cantor set $\mathcal{C}$ if $k = \omega$. In [14] we proposed to consider the space

$$Q(k) = \{(x_0, x_1, \dots) \in k^\omega : \exists n \forall m (m \geq n \Rightarrow x_m = 0)\}$$

as a non-separable analogue of weight k for the space Q of rational numbers. Notice that $Q(\omega) \approx Q$. A topological description of the space $Q(k)$ is given by the following theorem (see [14]).

Theorem 2.1. *Let X be a σ-discrete weight-homogeneous metric space of weight k. Then X is homeomorphic to $Q(k)$.*

We identify cardinals with initial ordinals; in particular, $\omega = \{0, 1, 2, \dots\}$. Denote by $\emptyset$ the unique 0-tuple, i.e., $k^0 = \{\emptyset\}$. Let $k^1 = k$. If $t \in k^n$, then $\mathrm{lh}(t) = n$ is the length of t and $t\hat{\ } t_n = (t_0, \dots, t_{n-1}, t_n) \in k^{n+1}$ for $t_n \in k$. Let $k^{<\omega} = \cup\{k^i : i \in \omega\}$. For a point $x = (x_0, x_1, \dots) \in k^\omega$, we denote by $x{\restriction}n$ the initial n-tuple $(x_0, x_1, \dots, x_{n-1})$. For $t \in \omega^n$ put $\nu(t) = t_0 + \dots + t_{n-1} + n$; by definition, $\nu(\emptyset) = 0$.

For $n \in \omega$, we write $(\alpha, \beta) \in (\tau \times k)^n$ if $\alpha \in \tau^n$ and $\beta \in k^n$.

A set $Y \subset X$ is an *A-set* (or an *analytic set* [6], or a *Souslin set*) in a space X if, for some collection $\{F(t{\restriction}n) : n \in \omega, t \in \omega^\omega\}$ of closed subsets of X, we have $Y = \cup\{\bigcap\{F(t{\restriction}n) : n \in \omega\} : t \in \omega^\omega\}$. If $\mathcal{A}$ and $\mathcal{B}$ are two families of subsets of X, we say [6] that $\mathcal{B}$ is a *base* for $\mathcal{A}$ if each member of $\mathcal{A}$ is a union of members from $\mathcal{B}$. A *σ-discrete base* is a base which is also a σ-discrete family. A mapping $f : X \to Y$ is called *co-σ-discrete* if the image of each discrete family in X has a σ-discrete base in Y.

Let $\mathbf{\Gamma}$ be a family of subsets of a space X and let A and B be two subsets of X. We say that A is *$\mathbf{\Gamma}$-separated from* B if there exists a $C \in \mathbf{\Gamma}$ such that $A \subseteq C$ and $B \cap C = \emptyset$.

Theorem 2.2 was proved by the author in a difficult-to-access paper [12] and reproved in [16] (announced in [13]).

Theorem 2.2. *Let A be an A-set in a complete metric space X and τ be a cardinal with $\omega \leq \tau \leq w(X)$. Then the following conditions are equivalent:*

1. *A cannot be represented in the form $A = F \cup L$, where F is an F_σ-subset of X and $L \in \mathcal{L}_\tau$;*
2. *there exists a closed set $K \subseteq X$ such that $K \approx B^*(\tau)$, $K \cap A \approx B(\tau)$, and $K \setminus A \approx Q(\tau)$.*

A space X is called a *Baire space* if the intersection of countably many dense open subsets of X is again dense. We call a space *completely Baire* if every closed subset of it is Baire. It is well known that the space Q of rationals is not a Baire space. This implies that if X contains a closed subset homeomorphic to Q, then X is not completely Baire. From Theorem 2.2 with $\tau = \omega$ and $B = X \setminus A$ we immediately obtain the following corollary.

Corollary 2.3. *Let B be a CA-set in a complete metric space X. Then B is a G_δ-subset of X if and only if B is completely Baire.*

Theorem 2.4. *Let A be an A-set in a complete metric space X and $A \notin \mathcal{L}_\tau$ for a cardinal τ, where $\omega \leq \tau \leq w(X)$. Then there exists a closed set $B \subseteq X$ such that $B \approx B^*(\tau)$ and B is a nowhere dense subset of A.*

Proof. Since $B^*(\tau)$ contains a closed nowhere dense copy of itself, to prove the theorem it suffices to construct a closed subset B of X with $B \subset A$.

We distinguish two cases.

Case 1: There exist an F_σ-subset $F \subseteq X$ and $L \in \mathcal{L}_\tau$ such that $A = F \cup L$. Let $F = \{F_i : i \in \omega\}$, where each F_i is closed in X. Then $F_n \notin \mathcal{L}_\tau$ for some $n \in \omega$. The nowhere $\mathcal{L}_\tau$ kernel $F^* = F_n \setminus \bigcup\{U$ is open in $F_n : U \in \mathcal{L}_\tau\}$ is nonempty. Clearly, F^* is closed in X and nowhere of weight $<\tau$. By the Stone theorem [21], F^* contains a closed subset $B \approx B^*(\tau)$.

Case 2: Suppose A cannot be represented in the form $A = F \cup L$, where F is an F_σ-subset of X and $L \in \mathcal{L}_\tau$. For $\tau = \omega$ this means that A is not σ-discrete. According to the El'kin theorem [3], A contains a copy B of the Cantor set. As a compact space, B is closed in X. Now consider the case $\tau > \omega$. By Theorem 2.2, there exists a closed set $M \subseteq X$ such that $M \approx B(\tau)$, $M \cap A \approx B(\tau)$, and $M \setminus A \approx Q(\tau)$. The set $B_1 = \{(x_0, x_1, \ldots) \in \tau^\omega : \forall n (x_n \neq 0)\}$ is closed in $B(\tau)$ and misses the set $Q_1 = \{(x_0, x_1, \ldots) \in \tau^\omega : \exists n \forall m (m \geq n \Rightarrow x_m = 0)\}$. Clearly, $B_1 \approx B(\tau)$. By [15, Corollary 2], there exists a homeomorphism $f : B(\tau) \to M$ such that $f(Q_1) = M \setminus A$. Then $B = f(B_1)$ is as required. □

3. Basic Lemma

Definition 3.1. For a given space X, let $\mathbf{\Gamma}$ be a family of subsets of X satisfying the following conditions:

- (Γ1) if $E \in \mathbf{\Gamma}$ and F is a closed subset of E, then $F \in \mathbf{\Gamma}$;
- (Γ2) if every point $x \in E$ has a neighborhood U_x such that $U_x \cap E \in \mathbf{\Gamma}$, then $E \in \mathbf{\Gamma}$;
- (Γ3) if each $E_i \in \mathbf{\Gamma}$, then $\cup\{E_i : i \in \omega\} \in \mathbf{\Gamma}$;
- (Γ4) if E is an F_σ-subset of X, then $E \in \mathbf{\Gamma}$. In particular, $\emptyset \in \mathbf{\Gamma}$.

Remark 3.2. Note that, in general, the family $\mathbf{\Gamma}$ is not a σ-ideal.

Remark 3.3. The family $\mathbf{\Gamma}$ always contains any σ-discrete subset of X. So, if A and B are two disjoint subsets of a space X and A is σ-discrete, then A is $\mathbf{\Gamma}$-separated from B. Thus, below in Lemma 3.4 and Theorem 4.6, we actually deal with a non-σ-discrete subspace A.

Lemma 3.4. *Let $\mathbf{\Gamma}$ be a family of subsets of a complete metric space X satisfying* (Γ1)–(Γ4). *Let A be an A-set in X and B be an arbitrary subset of X with $A\cap B=\emptyset$. Suppose there is no $\mathbf{\Gamma}$-set separating A from B. Then there exist a complete metric space H with* $\mathrm{Ind}H=0$ *and a continuous mapping $\varphi : H \to A$ such that $\overline{\overline{\varphi(U)}\cap B}\setminus B$ is nowhere $\mathbf{\Gamma}$ for every nonempty open $U\subseteq H$, where the bar denotes the closure in X.*

Proof. Let $w(A)=p$. By virtue of Hansell's theorem ([6, Theorem 4.1]), there exists a continuous co-σ-discrete mapping $\varphi : B(p)\to A$ of the Baire space $B(p)$ onto A.

An open subset $U\subset B(p)$ is called *usual* if there exists an $E_U\in\mathbf{\Gamma}$ such that $\varphi(U)\subseteq E_U$ and $E_U\cap B=\emptyset$. Let us show that the set $U^*=\cup\{U : U$ is usual in $B(p)\}$ is usual in $B(p)$. Clearly, U^* is open in $B(p)$. Since an open subset of a usual set is usual, we can take a σ-discrete cover $\mathcal{U}$ of U^* by usual sets. Since φ is co-σ-discrete, there exists a σ-discrete base $\mathcal{V}$ in X for the family $\varphi(\mathcal{U})$. Then $\mathcal{V}=\{\mathcal{V}_n : n\in\omega\}$, where each family $\mathcal{V}_n$ is ε_n-metrically discrete for some $\varepsilon_n>0$. To each $V\in\mathcal{V}$ assign $U_V\in\mathcal{U}$ such that $V\subseteq\varphi(U_V)$. According to (Γ1), every $V^*=E_{U_V}\cap\overline{V}\in\mathbf{\Gamma}$. Hence, $\{V^* : V\in\mathcal{V}_n\}$ is an ε_n-metrically discrete family in X for each $n\in\omega$. From (Γ2) and (Γ3), it follows that the set $E=\cup\{V^* : V\in\mathcal{V}\}\in\mathbf{\Gamma}$. By construction, $\varphi(U^*)=\bigcup\mathcal{V}\subseteq E$. Clearly, $E\cap B=\emptyset$. Thus, U^* is the largest usual set in $B(p)$.

Let us show that $A\setminus E$ is nonempty. Assuming the converse, we have $A\subseteq E$. Hence, A is $\mathbf{\Gamma}$-separated from B, a contradiction.

The set $H=B(p)\setminus U^*$ is nonempty because $A\setminus E\neq\emptyset$. As a closed subset of $B(p)$, H is a complete metric space. Obviously, $\mathrm{Ind}H=0$.

Take a nonempty open (in H) set $U\subseteq H$. Let $U=U_1\cap H$, where U_1 is open in $B(p)$. Since $U_1\setminus U\subset U^*$, we have

$$\varphi(U_1)=\varphi(U)\cup\varphi(U_1\setminus U)\subseteq\overline{\varphi(U)}\cup E.$$

By (Γ4), $\overline{\varphi(U)}\setminus\overline{\overline{\varphi(U)}\cap B}\in\mathbf{\Gamma}$. If we assume that $\overline{\overline{\varphi(U)}\cap B}\setminus B\in\mathbf{\Gamma}$, we obtain that the set

$$E\cup\left(\overline{\varphi(U)}\setminus\overline{\overline{\varphi(U)}\cap B}\right)\cup\left(\overline{\overline{\varphi(U)}\cap B}\setminus B\right)=E\cup\left(\overline{\varphi(U)}\setminus B\right)$$

belongs to $\mathbf{\Gamma}$ and misses B. This implies that $U^*\cup U$ is a usual set. The last contradicts the maximality of U^*. Thus, $\overline{\overline{\varphi(U)}\cap B}\setminus B\notin\mathbf{\Gamma}$.

We claim that $\overline{\overline{\varphi(U)} \cap B} \setminus B$ is nowhere $\mathbf{\Gamma}$ for every nonempty open $U \subseteq H$. Assume the converse. Denote by $\widehat{U}$ the set $\overline{\overline{\varphi(U)} \cap B}$ if open $U \subseteq H$. Let $V \in \mathbf{\Gamma}$ be a nonempty open subset of some $\widehat{U} \setminus B$. Since $\varphi(U) \cap B = \emptyset$, the intersection $\overline{\varphi(U)} \cap B$ is a co-dense set in $\widehat{U}$. Then $\overline{\widehat{U} \setminus B} = \widehat{U}$. Take an open subset V_1 of $\widehat{U}$ such that $V_1 = \mathrm{Int}(\overline{V_1})$ with respect to $\widehat{U}$ and $\overline{V_1} \setminus B \subseteq V$. As a closed subset of V, the set $\overline{V_1} \setminus B \in \mathbf{\Gamma}$ by $\Gamma 1$). Since $\widehat{U}$ is closed in $\overline{\varphi(U)}$, we can find an open set V_2 in $\overline{\varphi(U)}$ such that $V_1 = V_2 \cap \widehat{U}$ and $\overline{V_1} = \overline{V_2} \cap \widehat{U}$. Then $V = V_2 \cap (\widehat{U} \setminus B)$. From continuity of φ it follows that the set $U_2 = U \cap \varphi^{-1}(V_2 \bigcap \varphi(U))$ is open in U, hence in H.

Since $\overline{\varphi(U_2)} = \overline{V_2 \cap \varphi(U)} = \overline{V_2}$, we get

$$\overline{\overline{\varphi(U_2)} \cap B} \setminus B = \left(\overline{\overline{V_2} \cap B} \cap \widehat{U}\right) \setminus B = \overline{V_1} \setminus B \in \mathbf{\Gamma}.$$

On the other hand, as above, $\overline{\overline{\varphi(U_2)} \cap B} \setminus B \notin \mathbf{\Gamma}$, a contradiction. Thus, $\overline{\overline{\varphi(U)} \cap B} \setminus B$ is nowhere $\mathbf{\Gamma}$. $\square$

4. Main Theorems

Theorem 4.1. *Let A be an A-set in a complete metric space X and B be an arbitrary subset of X with $A \cap B = \emptyset$. Then the following conditions are equivalent:*

(1) *A is not separated from B by an F_σ-subset of X;*
(2) *there exists a closed set $C \subseteq X$ such that $C \subset A \cup B$, $C \cap A$ is homeomorphic to the space of irrationals, $C \cap B$ is homeomorphic to the space of rationals, and C is homeomorphic to the Cantor set $\mathcal{C}$.*

Proof. We first verify $2 \Rightarrow 1$. Assume the converse. Take an F_σ-subset $F \subseteq X$ with $A \subseteq F$ and $F \cap B = \emptyset$. Then $F \cap C$ is F_σ in X. On the other hand, $F \cap C = A \cap C$ and $C \setminus F \approx Q$. Since the space of rationals is not a G_δ-subset of the Cantor set $\mathcal{C}$, we obtain that $F \cap C$ is not an F_σ-set in X, a contradiction.

Let us prove $1 \Rightarrow 2$. Put $\mathbf{\Gamma} = \{F : F \text{ is an } F_\sigma\text{-subset of } X\}$. According to Lemma 3.4, take a complete metric space H with $\mathrm{Ind} H = 0$ and a continuous mapping $\varphi : H \to A$ such that $\overline{\overline{\varphi(U)} \cap B} \setminus B$ is nowhere F_σ-subset of X for every nonempty open $U \subseteq H$, where the bar denotes the closure in X. Then $\overline{\varphi(U)} \cap B$ is nonempty and has no isolated points.

Let $H_\emptyset = H$ and $V_\emptyset = X$. Choose a point $z_\emptyset \in \overline{\varphi(H)} \cap B$ and a decreasing base $\{U_i : i \in \omega\}$ at $z_\emptyset$ such that each intersection $(U_i \setminus \overline{U_{i+1}}) \cap \varphi(H_\emptyset)$ is nonempty and $\mathrm{diam}(U_i) < (i+1)^{-1}$. Next, fix an

$i \in \omega$. Consider an open V_i subset of X such that $\overline{V_i} \subset U_i \setminus \overline{U_{i+1}}$ and $V_i \cap \varphi(H_\emptyset) \neq \emptyset$. From the continuity of φ, it follows that there is a nonempty clopen $H_i \subset H_\emptyset$ satisfying $\overline{\varphi(H_i)} \subset V_i$ and $\operatorname{diam}(H_i) < (i+1)^{-1}$. Choose a point $z_i \in \overline{\varphi(H_i)} \cap B$. Then the sequence $\{z_i : i \in \omega\}$ of points converges to $z_\emptyset$. Take a decreasing base $\{U_{i\hat{\ }j} : j \in \omega\}$ at z_i such that each intersection $(U_{i\hat{\ }j} \setminus \overline{U_{i\hat{\ }(j+1)}}) \cap \varphi(H_i)$ is nonempty, $U_{i\hat{\ }0} \subseteq V_i$, and $\operatorname{diam}(U_{i\hat{\ }j}) < (i+j+2)^{-1}$, and so on.

In such a way, by induction, for any $n \in \omega$ and $t \in \omega^n$ we shall construct a clopen set $H_t \subset H$, an open set $V_t \subset X$, a point $z_t \in B$, and a decreasing base $\{U_{t\hat{\ }j} : j \in \omega\}$ at the point z_t satisfying the following conditions:

(b1) $z_t \in \overline{\varphi(H_t)} \cap B$ and $\overline{\varphi(H_t)} \subset V_t$;
(b2) $\operatorname{diam}(H_t) < (n+1)^{-1}$ and $H_{t\hat{\ }j} \subseteq H_t$ for every $j \in \omega$;
(b3) $\operatorname{diam}(U_t) < \nu(t)^{-1}$ and $U_{t\hat{\ }0} \subseteq V_t$;
(b4) $V_{t\hat{\ }j} \subseteq \overline{V_{t\hat{\ }j}} \subseteq U_{t\hat{\ }j} \setminus \overline{U_{t\hat{\ }(j+1)}} \subset V_t$ for every $j \in \omega$;
(b5) $\overline{\cup\{V_{t\hat{\ }j} : j \in \omega\}} = \{z_t\} \cup \bigcup\{\overline{V_{t\hat{\ }j}} : j \in \omega\}$ and $\overline{V_{t\hat{\ }i}} \cap \overline{V_{t\hat{\ }j}} = \emptyset$ if $i \neq j$.

Clearly, the countable set $Z = \{z_t : t \in \omega^{<\omega}\}$ is homeomorphic to the space of rationals. From (b1) it follows that $Z \subset B$. One can check that for each $n \in \omega$ the family $\{U_{t\hat{\ }j} \cap Z : \nu(t\hat{\ }j) = n+1\}$ forms a finite cover of Z with measure less than $(n+1)^{-1}$. Hence, the set Z is totally bounded. Then the closure $C = \overline{Z}$ is a compact subset of X (see [5, Theorem 4.3.11 and Theorem 4.3.30]). Moreover, (b3)–(b5) imply that for each $n \in \omega$ the family $\{U_{t\hat{\ }j} \cap C : \nu(t\hat{\ }j) = n+1\}$ forms a finite discrete cover of C with measure less than $(n+1)^{-1}$. By [5, Theorem 7.3.1], we have $\dim C = 0$. According to the Brouwer theorem [5, Exercise 6.2.A], C is homeomorphic to the Cantor set. Then $C \setminus Z$ is homeomorphic to the space of irrationals.

It remains to show that $C \setminus Z \subset A$. For each $n \in \omega$, consider the sets $Z_n = \{z_t : t \in \omega^i, i \leq n\}$, $P_n = \cup\{\overline{\varphi(H_t)} : t \in \omega^n\}$, and $C_n = Z_n \cup P_{n+1}$. By induction, one can verify that each C_n is closed in X. By construction, $C_{n+1} \subset C_n$ and $C = \cap\{C_n : n \in \omega\}$. Take a point $x \in C \setminus Z$. Then $x \in P_n$ for each $n \in \omega$. By (b5), for each $n \in \omega$ there exists a unique $t(n) \in \omega^n$ with $x \in \overline{\varphi(H_{t(n)})}$. According to (b2), the intersection $\cap\{H_{t(n)} : n \in \omega\}$ in a complete metric space H consists of one point ξ. From continuity of φ, it follows that $\varphi(\xi) \in \cap\{\overline{\varphi(H_{t(n)})} : n \in \omega\}$. Using (b1) and (b3), we conclude $x = \varphi(\xi) \in A$. □

Definition 4.2. For a subset B of a space X and infinite cardinals k and τ with $\max\{k, \tau\} \leq w(X)$, let us define the family $\mathbf{F}_{k,\tau}(B) = \{E \subseteq X : E = (F \cup L) \setminus D$, where F is an F_σ-subset of X, $L \in \mathcal{L}_k$ with $L \cap B = \emptyset$, and $D \in \mathcal{L}_\tau$ with $D \subseteq B \cap F\}$.

Lemma 4.3. *For a given $B \subset X$, the family $\mathbf{F}_{k,\tau}(B)$ satisfies* (Γ1)–(Γ4).

Proof. One easily verifies that the family $\mathbf{F}_{k,\tau}(B)$ satisfies (Γ1) and (Γ4).

For each $i \in \omega$, take a set $E_i = (F_i \cup L_i) \setminus D_i \in \mathbf{F}_{k,\tau}(B)$, where sets F_i, L_i, and D_i are as in Definition 4.2. Clearly, $F = \cup\{F_i : i \in \omega\}$ is an F_σ-subset of X, the set $L = \cup\{L_i : i \in \omega\} \in \mathcal{L}_k$, and the set $D = \cup\{D_i : i \in \omega\} \in \mathcal{L}_\tau$. By construction, $L \cap B = \emptyset$ and $D \subseteq B \cap F$. Put

$$D^* = \cup\left\{D_i \setminus \bigcup\{F_j \setminus D_j : j \in \omega,\, j \neq i\} : i \in \omega\right\}.$$

From $D^* \subseteq D$ it follows that $D^* \in \mathcal{L}_\tau$. Then

$$\cup\{E_i : i \in \omega\} = (F \cup L) \setminus D^* \in \mathbf{F}_{k,\tau}(B).$$

Property (Γ3) is checked.

Let us verify property (Γ2). For every point $x \in E$ choose a neighborhood U_x with $U_x \cap E \in \mathbf{F}_{k,\tau}(B)$. By the Stone theorem [5, Theorem 4.4.1], the cover $\{U_x : x \in E\}$ of E has a σ-discrete open refinement $\mathcal{V} = \{\mathcal{V}_i : i \in \omega\}$ such that each family $\mathcal{V}_i$ is discrete in X and $V \cap E \in \mathbf{F}_{k,\tau}(B)$ for every $V \in \mathcal{V}$. One can check that each $E_i = \cup\{V \cap E : V \in \mathcal{V}_i\} \in \mathbf{F}_{k,\tau}(B)$. As above, we conclude $E = \cup\{E_i : i \in \omega\} \in \mathbf{F}_{k,\tau}(B)$. □

To prove the main Theorem 4.6, we shall use the following two statements. Lemma 4.4 is well known. Theorem 4.5 was announced in [13] and proved in [17, Theorem 3.5].

Lemma 4.4. *Let a subset A of a metric space X be nowhere of local weight $<k$, where the cardinal $k > \omega$. Then there exists a closed discrete set $F \subset X$ of cardinality k such that $F \subset A$.*

Theorem 4.5. *Let X be a weight-homogeneous metric space of weight k such that* $\mathrm{Ind} X = 0$ *and every nonempty clopen subset of X contains a closed copy of the space $B^*(\tau)$, where $\omega \leq \tau \leq k$. Suppose that $X = \cup\{X_n : n \in \omega\}$, where each X_n is a closed nowhere dense subset of X and each X_n is locally homeomorphic to $B^*(\tau)$. Then X is homeomorphic to $Q(k) \times B^*(\tau)$.*

Theorem 4.6. *Let A and B be disjoint A-sets in a complete metric space X and let τ and k be cardinals with* $k^* = \max\{k, \tau\} \leq w(X)$*. Then the following conditions are equivalent:*

1. *A is not $\mathbf{F}_{k,\tau}(B)$-separated from B;*
2. *there exists a closed set $C \subseteq X$ such that $C \subseteq A \cup B$, $A \cap C$ is homeomorphic to the Baire space $B(k^*)$, $B \cap C$ is homeomorphic to the space $Q(k^*) \times B^*(\tau)$, and $C \approx B^*(k^*)$.*

Proof. We first verify $2 \Rightarrow 1$. Assume the converse. Take a set $E \in \mathbf{F}_{k,\tau}(B)$ with $A \subseteq E$ and $E \cap B = \emptyset$. Without loss of generality, $B \cap C$ is a dense subset of C. According to Definition 4.2, $E = (F \cup L) \setminus D$, where F is an F_σ-subset of X, $L \in \mathcal{L}_k$ with $L \cap B = \emptyset$, and $D \subseteq B \cap F$. Replacing L by $L \setminus F$ if it is necessary, we may assume that $F \cap L = \emptyset$. Since $L \in \mathcal{L}_k$, we obtain that $C \cap L$ is of first category in C. Clearly, $B \cap C \approx Q(k^*) \times B^*(\tau)$ is of first category in itself. Then $C \setminus F = (C \bigcap L) \cup (C \bigcap B)$ is of first category in C. This contradicts the Baire category theorem [5, Theorem 3.9.3] because $C \setminus F$ is metrizable by a complete metric as a G_δ-subset of C.

Let us prove the implication $1 \Rightarrow 2$. By Lemma 4.3, the family $\mathbf{F}_{k,\tau}(B)$ satisfies the conditions (Γ1)–(Γ4). According to Lemma 3.4, take a complete metric space H with $\mathrm{Ind} H = 0$ and a continuous mapping $\varphi : H \to A$ such that $\overline{\overline{\varphi(U)} \cap B \setminus B}$ is nowhere $\mathbf{F}_{k,\tau}(B)$ for every nonempty open $U \subseteq H$, where the bar denotes the closure in X.

We first consider the case $\tau > \omega$. Then $k^* > \omega$.

By induction, for any $n \in \omega$ and $(t, \gamma, \alpha) \in (\omega \times \tau \times k^*)^n$, we shall construct a clopen set $H(t, \gamma, \alpha) \subseteq H$, an open set $V(t, \gamma, \alpha) \subseteq X$, a sequence $\mathcal{U}(t\hat{}i, \gamma, \alpha) = \{U(t\hat{}i, \gamma\hat{}\delta, \alpha) : \delta \in \tau\}$ of open subsets of X, where $i \in \omega$, and a closed nowhere dense set $Z(t, \gamma, \alpha) \subset X$ satisfying the following conditions:

(a1) $\overline{\varphi(H(t, \gamma, \alpha))} \subset V(t, \gamma, \alpha)$;
(a2) $\mathrm{diam}(H(t, \gamma, \alpha)) < (n + 1)^{-1}$ and, for fixed $t \in \omega^n$, the family $\{H(t, \gamma, \alpha) : (\gamma, \alpha) \in (\tau \times k^*)^n\}$ is discrete in H;
(a3) $\mathrm{diam}(V(t, \gamma, \alpha)) < \nu(t)^{-1}$ and, for fixed $t \in \omega^n$, the family $\{\overline{V(t, \gamma, \alpha)} : (\gamma, \alpha) \in (\tau \times k^*)^n\}$ is discrete in X;
(a4) $H(t\hat{}i, \gamma\hat{}\delta, \alpha\hat{}\beta) \subset H(t, \gamma, \alpha)$ and $\overline{V(t\hat{}i, \gamma\hat{}\delta, \alpha\hat{}\beta)} \subset V(t, \gamma, \alpha)$ for every $(i, \delta, \beta) \in \omega \times \tau \times k^*$;
(a5) $Z(t, \gamma, \alpha) = \cap\{\bigcup \mathcal{U}(t\hat{}i, \gamma, \alpha) : i \in \omega\}$, $Z(t, \gamma, \alpha) \approx B(\tau)$, and $Z(t, \gamma, \alpha)$ is a closed nowhere dense subset of $\overline{\varphi(H(t, \gamma, \alpha))} \cap B$;
(a6) $\overline{\cup\{V : V \in \mathcal{V}(t, \gamma, \alpha)\}} = Z(t, \gamma, \alpha) \cup \bigcup\{\overline{V} : V \in \mathcal{V}(t, \gamma, \alpha)\}$ and $Z(t, \gamma, \alpha) \cap \overline{V} = \emptyset$ for every $V \in \mathcal{V}(t, \gamma, \alpha)$, where $\mathcal{V}(t, \gamma, \alpha) = \{V(t\hat{}i, \gamma\hat{}\delta, \alpha\hat{}\beta) : (i, \delta, \beta) \in \omega \times \tau \times k^*\}$;
(a7) $\cup\mathcal{U}(t\hat{}0, \gamma, \alpha) \subset V(t, \gamma, \alpha)$ and $\cup\mathcal{U}(t\hat{}(i + 1), \gamma, \alpha) \subset \cup\mathcal{U}(t\hat{}i, \gamma, \alpha)$ for each $i \in \omega$;
(a8) $\overline{V(t\hat{}i, \gamma\hat{}\delta, \alpha\hat{}\beta)} \subset U(t\hat{}i, \gamma\hat{}\delta, \alpha) \setminus \overline{U(t\hat{}(i + 1), \gamma\hat{}\delta, \alpha)}$ for every $(i, \delta, \beta) \in \omega \times \tau \times k^*$.

Put $V(\emptyset, \emptyset, \emptyset) = X$ and $H(\emptyset, \emptyset, \emptyset) = H$. By Lemma 3.4, the set $\overline{\overline{\varphi(H)} \cap B \setminus B} \notin \mathbf{F}_{k,\tau}(B)$. Then the A-set $\overline{\varphi(H)} \cap B \notin L_\tau$. By Theorem 2.4, the last set contains a nowhere dense subset $Z(\emptyset, \emptyset, \emptyset)$ which is closed in X and homeomorphic to $B(\tau)$.

Now suppose that the sets $H(t, \gamma, \alpha)$, $V(t, \gamma, \alpha)$, and $Z(t, \gamma, \alpha)$ have been defined for every $(t, \gamma, \alpha) \in (\omega \times \tau \times k^*)^m$ with $m \leq n$.

Fix $(t, \gamma, \alpha) \in (\omega \times \tau \times k^*)^n$. We shall work within the nonempty set $V(t, \gamma, \alpha)$. Since $\dim Z(t, \gamma, \alpha) = 0$, there exists a sequence $\mathcal{U}_i^* = \{U_{i,\delta}^* : \delta \in \tau\}$ of clopen (in $Z(t, \gamma, \alpha)$) covers of $Z(t, \gamma, \alpha)$ such that $\mathcal{U}_{i+1}^*$ refines $\mathcal{U}_i^*$ and $\mathrm{mesh}(\mathcal{U}_i^*) \to 0$ as $i \to \infty$. Then every $\mathcal{U}_i^*$ is a discrete closed family in X. Since X is a metric space, every $\mathcal{U}_i^*$ has a discrete swelling $\mathcal{U}_i = \{U_{i,\delta} : \delta \in \tau\}$ in X, where each $U_{i,\delta}$ is open in X and $U_{i,\delta}^* \subset U_{i,\delta}$. Then the boundary of each $U_{i,\delta}$ misses $Z(t, \gamma, \alpha)$. Without loss of generality, $\cup \mathcal{U}_0 \subseteq V(t, \gamma, \alpha)$, $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$, and $\mathrm{mesh}(\mathcal{U}_i) \to 0$ as $i \to \infty$. Then $\cap\{\bigcup \mathcal{U}_i : i \in \omega\} = Z(t, \gamma, \alpha)$. Moreover, we may assume that every $\overline{U_{i+1,\delta}} \subset U_{i,\delta}$ and the intersection $W_{i,\delta} = (U_{i,\delta} \setminus \overline{U_{i+1,\delta}}) \cap \overline{\varphi(H(t, \gamma, \alpha))}$ is non empty because $Z(t, \gamma, \alpha)$ is a closed nowhere dense subset of $V(t, \gamma, \alpha) \cap \overline{\varphi(H(t, \gamma, \alpha))}$

We claim that each $\overline{W_{i,\delta}} \cap B$ is nowhere of weight $<k^*$, where $k^* = \max\{k, \tau\}$. Indeed, according to Lemma 3.4, $\overline{\overline{W_{i,\delta}} \cap B} \setminus B$ is nowhere $\mathbf{F}_{k,\tau}(B)$. In particular, the last set is nowhere $\mathcal{L}_k$; hence, it is nowhere of weight $<k$. Then $\overline{W_{i,\delta}} \cap B$ is nowhere of weight $<k$. Next, if there exists a nonempty open (in $\overline{\overline{W_{i,\delta}} \cap B}$) subset $O \subseteq \overline{\overline{W_{i,\delta}} \cap B}$ with $O \cap B \in \mathcal{L}_\tau$, then $O \setminus B \in \mathbf{F}_{k,\tau}(B)$, a contradiction. Hence, $\overline{\overline{W_{i,\delta}} \cap B}$ is nowhere $\mathcal{L}_\tau$. Then $\overline{W_{i,\delta}} \cap B$ is nowhere of weight $<\tau$. The claim is verified.

By Lemma 4.4 we can choose a closed discrete set $D_{i,\delta} = \{d_{i,\delta}(\beta) : \beta \in k^*\}$ of cardinality k^* with $D_{i,\delta} \subset \overline{W_{i,\delta}} \cap B$. Then there exists a discrete family $\{V_{i,\delta}(\beta) : \beta \in k^*\}$ such that $\overline{V_{i,\delta}(\beta)} \subset U_{i,\delta} \setminus \overline{U_{i+1,\delta}}$ and every $V_{i,\delta}(\beta)$ is a neighborhood of $d_{i,\delta}(\beta)$ with diameter less than $\nu(t\hat{\ }i)^{-1}$. From continuity of φ it follows that the set $H(t, \gamma, \alpha) \cap \varphi^{-1}(V_{i,\delta}(\beta))$ contains a nonempty clopen subset $H_{i,\delta}(\beta)$ such that $\mathrm{diam}(H_{i,\delta}(\beta)) < (n+2)^{-1}$ and $\overline{\varphi(H_{i,\delta}(\beta))} \subset V_{i,\delta}(\beta)$. By Lemma 3.4, the set $\overline{\overline{\varphi(H_{i,\delta}(\beta))} \cap B} \setminus B \notin \mathbf{F}_{k,\tau}(B)$. Then the A-set $\overline{\varphi(H_{i,\delta}(\beta))} \cap B \notin L_\tau$. By Theorem 2.4, the last set contains a nowhere dense subset $Z_{i,\delta}(\beta)$ which is closed in X and homeomorphic to $B(\tau)$.

Put $Z(t\hat{\ }i, \gamma\hat{\ }\delta, \alpha\hat{\ }\beta) = Z_{i,\delta}(\beta)$, $V(t\hat{\ }i, \gamma\hat{\ }\delta, \alpha\hat{\ }\beta) = V_{i,\delta}(\beta)$, $U(t\hat{\ }i, \gamma\hat{\ }\delta, \alpha) = U_{i,\delta}$, and $H(t\hat{\ }i, \gamma\hat{\ }\delta, \alpha\hat{\ }\beta) = H_{i,\delta}(\beta)$ for every $(i, \delta, \beta) \in \omega \times \tau \times k^*$.

One can check that conditions (a1)–(a8) are satisfied. This completes the induction.

For each $n \in \omega$, consider the set

$$Z_n = \cup\{Z(t, \gamma, \alpha) : (t, \gamma, \alpha) \in (\omega \times \tau \times k^*)^l,\ l \leq n\}.$$

Put $Z = \cup\{Z_n : n \in \omega\}$. From (a5) it follows that $Z \subset B$. By construction, each $Z_n \subset Z_{n+1}$. Conditions (a6), (a5), and (a1) imply that Z_n is a closed nowhere dense subset of Z_{n+1} and of X. Then Z is of first category in itself. Moreover, Z is an F_σ-subset of X. It follows from (a3) and (a5) that each $Z_{n+1} \setminus Z_n$ is locally homeomorphic to $B(\tau)$. From (a8), (a6), and (a2), it follows that Z is a weight-homogeneous space of weight k^*. Applying Theorem 4.5, we conclude $Z \approx Q(k^*) \times B(\tau)$.

For each $n \in \omega$, consider the set

$$P_n = \cup \left\{ \overline{\varphi(H(t,\gamma,\alpha))} : (t,\gamma,\alpha) \in (\omega \times \tau \times k^*)^n \right\}.$$

Put $C_n = Z_n \cup P_{n+1}$. From (a4) and (a1), it follows that $C_{n+1} \subset C_n$.

Let us show by induction that each C_n is closed in X. From (a1) and (a6), it follows that the set

$$C_0 = Z(\emptyset,\emptyset,\emptyset) \cup \bigcup \left\{ \overline{\varphi(H(t,\gamma,\alpha))} : (t,\gamma,\alpha) \in \omega \times \tau \times k^* \right\}$$

is closed in X. Suppose C_{n-1} is closed. Since $C_n \subset C_{n-1}$, we have $\overline{C_n} \subset C_{n-1}$. Take a point $x \in \overline{C_n}$. Clearly, if $x \in Z_n$, then $x \in Z_n \subset C_n$. Let $x \in \overline{C_n} \setminus Z_n$. Then $x \in \overline{P_{n+1}}$. On the other hand,

$$x \in \overline{\varphi(H(t,\gamma,\alpha))} \subset P_n \subset C_{n-1}$$

for some $(t,\gamma,\alpha) \in (\omega \times \tau \times k^*)^n$. From (a6) and (a2), it follows that

$$\overline{\varphi(H(t,\gamma,\alpha))} \cap \overline{P_{n+1}} =$$

$$Z(t,\gamma,\alpha) \cup \bigcup \left\{ \overline{\varphi(H(t\hat{}i, \gamma\hat{}\delta, \alpha\hat{}\beta))} : (i,\delta,\beta) \in \omega \times \tau \times k^* \right\}.$$

The last family is disjoint according to (a8) and (a3). Hence, for some $(i,\delta,\beta) \in \omega \times \tau \times k^*$, we obtain $x \in \overline{\varphi(H((t\hat{}i, \gamma\hat{}\delta, \alpha\hat{}\beta))} \subset C_n$. So $\overline{C_n} = C_n$.

The last statement implies that $C = \cap\{C_n : n \in \omega\}$ is a closed subset of X. By construction, each $Z_{n+1} \subset B \cap C_n$. Then $Z \subset B \cap C$. From (a5) and (a3), it follows that Z is dense in C. Hence, $C = \overline{Z}$ is a weight-homogeneous space of weight k^*.

Let us verify that $\dim C = 0$.

Define the family $\mathcal{U}_n = \{U : U \in \mathcal{U}(t\hat{}i,\gamma,\alpha)$ with $\nu(t\hat{}i) = n+1\}$. We start with $\mathcal{U}_0 = \{U(0,\emptyset,\emptyset)\} = \{X\}$.

We claim that $C_n \subset \cup\mathcal{U}_n$ for each $n \in \omega$. Consider a point $x \in C_n$. If $x \in Z_n$, then $x \in Z(t,\gamma,\alpha)$ for some $(t,\gamma,\alpha) \in (\omega \times \tau \times k^*)^j$ with $j \leq n$. If $\nu(t) \leq n$, then $\nu(t\hat{}(n-\nu(t))) = n+1$; hence, $x \in U(t\hat{}(n-\nu(t)),\gamma,\alpha) \subset \cup\mathcal{U}_n$. In the case $\nu(t) > n$, by construction, the family $\mathcal{U}_{\nu(t)}$ refines $\mathcal{U}_n$. Then $x \in \cup\mathcal{U}_{\nu(t)} \subset \cup\mathcal{U}_n$. Now, let $x \in P_{n+1}$. By (a1), we have $x \in \overline{\varphi(H(t,\gamma,\alpha))} \subset V(t,\gamma,\alpha)$ for some $(t,\gamma,\alpha) \in (\omega \times \tau \times k^*)^{n+1}$. By (a7), $\cup\mathcal{U}(t\hat{}0,\gamma,\alpha) \subset V(t,\gamma,\alpha)$. Since $\nu(t\hat{}0) > n+1$, the family $\mathcal{U}_{\nu(t)}$ refines the family $\mathcal{U}_n$. Hence, $x \in \cup\mathcal{U}_n$. Thus, $C_n \subset \cup\mathcal{U}_n$.

From (a3), (a7), and (a8), it follows that each family $\mathcal{U}_n$ is discrete in X. This implies that for each $n \in \omega$, the family $\{U \cap C : U \in \mathcal{U}_n\}$ forms a discrete cover of C with measure less than $(n+1)^{-1}$. By [5, Theorem 7.3.1], $\dim C = 0$. Applying the Stone theorem [20], we have $C \approx B(k^*)$.

To prove $Z = B \cap C$ it suffices to show that $C \setminus Z \subset A$. Take a point $x \in C \setminus Z$. Then $x \in P_n$ for each $n \in \omega$. From the conditions (a1) and (a4), it follows that for each $n \in \omega$, there exists a unique $(t, \gamma, \alpha)_n \in (\omega \times \tau \times k^*)^n$ such that $x \in \overline{\varphi(H(t, \gamma, \alpha)_n)}$. According to (a2) and (a4), the intersection $\cap\{H(t, \gamma, \alpha)_n : n \in \omega\}$ in a complete metric space H consists of one point ξ. The mapping φ is continuous. Therefore, $\varphi(\xi) \in \bigcap \left\{\overline{\varphi(H(t, \gamma, \alpha)_n)} : n \in \omega\right\}$. Since $\operatorname{diam}(\varphi(H(t, \gamma, \alpha)_n) \to 0$ as $n \to \infty$, we conclude $x = \varphi(\xi) \in A$. Thus, $C \setminus Z \subset A$. Since Z is a meager F_σ-subset of C, we obtain $C \setminus Z \approx B(k^*)$.

This completes the proof of the theorem in the case $\tau > \omega$.

Let us point out the changes in the proof for $\tau = \omega$ omitting like details.

In the case $\tau = \omega$ and $k = k^* > \omega$, every $Z(t, \gamma, \alpha)$ is homeomorphic to the Cantor set $\mathcal{C}$. This implies that the cover $\mathcal{U}_i^*$ of $Z(t, \gamma, \alpha)$ is finite for every i, i.e., $\mathcal{U}_i^* = \{U_{i,\delta} : \delta \in \Delta_i\}$ for some $\Delta_i < \omega$. Using Theorem 4.5, we obtain $Z \approx Q(k^*) \times \mathcal{C}$. As above, one can verify that $C = \overline{Z} \approx B(k^*)$, $C \setminus Z \approx B(k^*)$, and $C \setminus Z \subset A$.

Finally, in the case $\tau = k = k^* = \omega$, we shall construct sets $H(t, \gamma)$, $V(t, \gamma)$, and $Z(t, \gamma) \approx \mathcal{C}$ and families $\mathcal{U}_i(t, \gamma)$, where $i \in \omega$, satisfying conditions (a1)–(a8) from which the third index α is omitted. Formally, we may assume that each $\alpha = 0$. One can verify that in this case each family $\{U \cap C : U \in \mathcal{U}_n\}$ forms a finite discrete cover of C with measure less than $(n + 1)^{-1}$. Then C is totally bounded in the complete metric space X. Hence, C is compact. By Brouwer's theorem [5, Exercise 6.2.A], C is homeomorphic to the Cantor set. Theorem 4.6 is proved. □

The following corollary for a compact metric space X was proved by van Engelen and van Mill [4].

Corollary 4.7. *Let A and B be A-sets in a complete metric space X such that $B = X \setminus A$. Then B contains a relatively closed subset homeomorphic to $Q \times \mathcal{C}$ if and only if B cannot be represented as $B = G \cup D$, where G is a G_δ-subset of X and D is a σ-discrete subset of X.*

Proof. The corollary follows from Theorem 4.6 when $k = \tau = \omega$.

Suppose there exists a relatively closed subset Z of B such that $Z \approx Q \times \mathcal{C}$. Using [4], we may assume that the closure $C = \overline{Z}$ is homeomorphic to $\mathcal{C}$. By Theorem 4.6, A is not $\mathbf{F}_{\omega,\omega}(B)$-separated from B. If we assume that $B = G \cup D$, where G is a G_δ-subset of X and D is σ-discrete, then the set $(X \setminus G) \setminus D \in \mathbf{F}_{\omega,\omega}(B)$ separates A from B, a contradiction.

The converse statement is checked in a similar way. □

Definition 4.8. For a space X and a cardinal k with $k \leq w(X)$, let us define the family $\mathbf{F}_k(X) = \{F \subseteq X : F = F_1 \cup F_2$, where F_1 is an F_σ-subset of X and $F_2 \in \mathcal{L}_k\}$.

Clearly, the family $\mathbf{F}_k(X)$ satisfies the conditions (Γ1)–(Γ4).

The following is a generalization of Theorem 2.2 and Theorem 4.1.

Theorem 4.9. *Let A be an A-set in a complete metric space X and B be an arbitrary subset of X with $A \cap B = \emptyset$. For a cardinal k satisfying $\omega \leq k \leq w(X)$, the following conditions are equivalent:*

(1) *A is not $\mathbf{F}_k(X)$-separated from B;*
(2) *there exists a closed set $C \subseteq X$ such that $C \subseteq A \cup B$, $A \cap C$ is homeomorphic to the Baire space $B(k)$, $B \cap C$ is homeomorphic to the space $Q(k)$, and $C \approx B^*(k)$.*

Proof. We first verify $2 \Rightarrow 1$. Assume the converse. Take a set $F \in \mathbf{F}_k(X)$ with $A \subseteq F$ and $F \cap B = \emptyset$. Then $F \cap C \in \mathbf{F}_k(X)$ by (Γ1). On the other hand, since $F \cap C = A \cap C \approx B(k)$ and $C \setminus F \approx Q(k)$, Theorem 2.2 implies that $F \cap C \notin \mathbf{F}_k(X)$, a contradiction.

The proof of the implication $1 \Rightarrow 2$ is similar to the proof of Theorem 4.6 and simpler. We only point out the changes in the proof and omit like details.

According to Lemma 3.4, take a complete metric space H with $\mathrm{Ind}H = 0$ and a continuous mapping $\varphi : H \to A$ such that $\overline{\overline{\varphi(U)} \cap B}\backslash B$ is nowhere $\mathbf{F}_k(X)$ for every nonempty open $U \subseteq H$, where the bar denotes the closure in X. In fact, we shall use that $\overline{\varphi(U)} \cap B$ is nonempty and nowhere of weight $<k$. So, analyticity of B is not necessary. Of course, $k^* = k$.

In the case $k > \omega$, by induction, for any $n \in \omega$ and $(t, \alpha) \in (\omega \times k)^n$, we shall construct a clopen set $H(t, \alpha) \subseteq H$, an open set $V(t, \alpha) \subseteq X$, and a base $\{U(t\hat{}i, \alpha) : i \in \omega\}$ at a point $z(t, \alpha) \in B$ satisfying conditions (a1)–(a8) from which the second index γ is deleted. In other words, we replace the set $Z(t, \gamma, \alpha)$ by a point $z(t, \alpha)$ and the family $\mathcal{U}_i(t, \gamma, \alpha)$ by a base $\{U(t\hat{}i, \alpha) : i \in \omega\}$ at the point $z(t, \alpha) \in \overline{\varphi(H(t, \alpha))} \cap B$.

For each $n \in \omega$, put $Z_n = \{z(t, \alpha) : (t, \alpha) \in (\omega \times k)^l, l \leq n\}$. The set $Z = \cup\{Z_n : n \in \omega\}$ is a σ-discrete weight-homogeneous space of weight k. By Theorem 2.1, we have $Z \approx Q(k)$. By construction, $Z \subset B$.

For each $n \in \omega$, consider sets $P_n = \bigcup\{\overline{\varphi(H(t, \alpha))} : (t, \alpha) \in (\omega \times k)^n\}$ and $C_n = Z_n \cup P_{n+1}$. As above, each C_n is closed in X. Then $C = \cap\{C_n : n \in \omega\}$ is the same. The set Z is dense in C and $C \setminus Z \subset A$. For each $n \in \omega$, the family $\{U \cap C : U \in \mathcal{U}_n\}$ forms a discrete cover of C with measure less than $(n+1)^{-1}$, where $\mathcal{U}_n = \{U(t\hat{}i, \alpha) : \nu(t\hat{}i) = n+1\}$. Then $\dim C = 0$. By the Stone theorem [20], $C \approx B(k)$. Moreover, $C \setminus Z \approx B(k)$. This completes the proof in the case $k > \omega$.

If $k = \omega$, the family $\mathbf{F}_\omega(X)$ coincides with the family of F_σ-subsets of X. Now the conclusion of the theorem follows from Theorem 4.1. Theorem 4.9 is proved. □

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