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FINITE GRAPHS HAVE UNIQUE HYPERSPACE $HS_n(X)$

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Abstract. For a metric continuum X and a positive integer n, we consider the hyperspaces $C_n(X)$, ($F_n(X)$, respectively) of all nonempty closed subsets of X with at most n components (n points, respectively). Let $HS_n(X)$ be the quotient space $C_n(X)/F_n(X)$ which is obtained from $C_n(X)$ by identifying $F_n(X)$ into a one-point set. In this paper we prove that if X is a finite graph and Y is a continuum such that $HS_n(X)$ is homeomorphic to $HS_n(Y)$, then X is homeomorphic to Y. This answers a question by Sergio Macías and Sam B. Nadler, Jr.

1. Introduction

A *continuum* is a compact connected metric space with more than one point. The set of positive integers is denoted by $\mathbb{N}$. Given a continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X:

$2^X = \{A \subset X : A \text{ is a nonempty closed subset of } X\}$,
$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$,
$C(X) = C_1(X)$, and
$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}$.

All these hyperspaces are considered with the Hausdorff metric H. The hyperspace $F_n(X)$ is called the *n-th symmetric product of* X. We extend the definition of $C_n(X)$ by defining $C_0(X) = \emptyset$.

The *n-fold hyperspace suspension* $HS_n(X)$ is defined as the quotient space $C_n(X)/F_n(X)$ which is obtained from $C_n(X)$ by identifying $F_n(X)$

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into a one-point set [14]. The symbol q_X^n denotes the natural projection $q_X^n : C_n(X) \to HS_n(X)$, and F_X^n denotes the element $q_X^n(F_n(X))$. Notice that $q_X^n|_{C_n(X)-F_n(X)} : C_n(X) - F_n(X) \to HS_n(X) - \{F_X^n\}$ is a homeomorphism.

A *finite graph* is a continuum that can be written as the union of a finite number of arcs such that each two of them intersect in a finite set. Given a finite graph X, let $E(X)$ denote the set of end points of X and let $R(X)$ denote the set of ramification points of X. Notice that $R(X) = \emptyset$ if and only if X is either an arc or a simple closed curve. If X is a finite graph and $p \in X$, the *order* of p in X is denoted by $\text{ord}_X(p)$. The unit circle in the Euclidean plane $\mathbb{R}^2$ is denoted by S^1. Given a subset A of a space Z, the interior of A in Z is denoted by $\text{int}_Z(A)$ or A° when there is no possibility of confusion.

A continuum X is said to have *unique hyperspace* $HS_n(X)$ provided that the following implication holds: If Y is a continuum and $HS_n(X)$ is homeomorphic to $HS_n(Y)$, then X is homeomorphic to Y. Our main result answers [15, Question 3.3] by proving that finite graphs have unique hyperspace $HS_n(X)$ for each $n \in \mathbb{N}$. A general exposition on the topic of uniqueness of hyperspaces can be found in [12].

A continuum Z is said to be *an absolute n-fold hyperspace suspension* [15] provided that for each pair of different points p and q in Z, there exists a continuum Y (that depends on p and q) and there exists a homeomorphism $g : Z \to HS_n(Y)$ such that $g(p) = q_Y^n(Y)$ and $g(q) = F_Y^n$. In [15], it is shown that the 2-sphere is the only finite-dimensional absolute 1-fold hyperspace suspension; it is also shown, by [15, Theorem 4.2], that $HS_n([0,1])$ and $HS_n(S^1)$ are the only two possible finite-dimensional absolute n-fold hyperspace suspensions for each $n \geq 3$ and there are no finite-dimensional absolute 2-fold hyperspace suspensions.

Answering [15, Question 4.11], we also show that for each $n \geq 3$, there are no finite-dimensional absolute n-fold hyperspace suspensions. Thus, combining our result and the results of [15], we obtain that (a) the 2-sphere is the only finite-dimensional absolute 1-fold hyperspace suspension and (b) for each $n \geq 2$, there are no finite-dimensional absolute n-fold hyperspace suspensions.

Here we use the most recent and advanced techniques in the area of uniqueness of hyperspaces. We widely use the precise and important formula developed by Verónica Martínez-de-la-Vega [16] to compute the dimension in $C_n(X)$ for a finite graph X. That is, if X is a finite graph and $A \in C_n(X)$, then

$$\text{(MV)} \qquad \dim_A[C_n(X)] = 2n + \sum_{p \in R(X) \cap A} (\text{ord}_X(p) - 2).$$

2. Auxiliary Results

In this section we develop the necessary tools to obtain the main results of this paper. We use, adapt, and generalize results that have been used in the area of uniqueness of hyperspaces; they were introduced in [9] and [10] and have been used and improved in various papers, such as [1], [2], [3], [4], [5], [6], [7], and [11]. The general ideas of these techniques are as follows. Suppose that $h : \mathcal{K}(X) \to \mathcal{K}(Y)$ is a homeomorphism where $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are certain hyperspaces of a finite graph X and a continuum Y, respectively. Using arguments of dimension and local connectedness, conclude that Y is also a finite graph. Define families of subsets $\mathcal{R}_1(X), \mathcal{R}_2(X), \ldots$ and $\mathcal{R}_1(Y), \mathcal{R}_2(Y), \ldots$ in the hyperspaces $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ by using topological properties that are preserved under homeomorphisms and obtain that $h(\mathcal{R}_i(X)) = \mathcal{R}_i(Y)$ for each i. Describe the structure of the graphs X and Y in terms of the sets $\mathcal{R}_1(X), \mathcal{R}_2(X), \ldots$ and $\mathcal{R}_1(Y), \mathcal{R}_2(Y), \ldots$ and conclude that the topological structure of X is the same as that of Y, so X and Y are homeomorphic.

Given a finite graph X, a *cycle* in X is a simple closed curve S in X such that S has at most one ramification point of X. A *free arc* in X is an arc α with end points a and b such that $\alpha - \{a, b\}$ is open in X. A *maximal free arc* is a free arc that is maximal with respect to the inclusion. Let

$$\mathfrak{A}_S(X) = \{J \subset X : J \text{ is a maximal free arc or } J \text{ is a cycle in } X\},$$
$$\mathfrak{A}_R(X) = \{J \in \mathfrak{A}_S(X) : J \text{ is a cycle}\}; \text{ and}$$
$$\mathfrak{A}_E(X) = \{J \in \mathfrak{A}_S(X) : J \text{ is an arc and } |J \cap R(X)| = 1\}.$$

Notice that the elements of $\mathfrak{A}_S(X)$ are the edges of the graph X. This implies that if $J, K \in \mathfrak{A}_S(X)$ and $J \neq K$, then $J^o \cap K = \emptyset$.

We do not use the notation $\mathcal{R}_i(X)$; instead, we use more suggestive notations.

Given a finite graph X and $n \in \mathbb{N}$, let

$$\mathcal{M}_n(X) = \{A \in C_n(X) : A \notin C_{n-1}(X) \text{ and } A \cap R(X) = \emptyset\},$$
$$\mathcal{L}_n(X) = \{A \in C_n(X) : A \text{ has a neighborhood } \mathcal{M} \text{ in } C_n(X) \text{ such that } \mathcal{M} \text{ is a } 2n\text{-cell}\},$$
$$\partial\mathcal{L}_n(X) = \{A \in C_n(X) : A \text{ has a neighborhood } \mathcal{M} \text{ in } C_n(X) \text{ such that } \mathcal{M} \text{ is a } 2n\text{-cell and } A \text{ is in the manifold boundary of } \mathcal{M}\},$$
$$\mathcal{HL}_n(X) = \{A \in HS_n(X) : A \text{ has a neighborhood } \mathcal{M} \text{ in } HS_n(X) \text{ such that } \mathcal{M} \text{ is a } 2n\text{-cell}\},$$
$$\partial\mathcal{HL}_n(X) = \{A \in HS_n(X) : A \text{ has a neighborhood } \mathcal{M} \text{ in } HS_n(X) \text{ such that } \mathcal{M} \text{ is a } 2n\text{-cell and } A \text{ is in the manifold boundary of } \mathcal{M}\},$$

$\mathcal{D}_n(X) = \{A \in C_n(X) : A \notin \mathcal{L}_n(X)$ and A has a basis $\mathcal{B}$ of neighborhoods in $C_n(X)$ such that for each $\mathcal{U} \in \mathcal{B}$, $\dim \mathcal{U} \leq 2n$, and $\mathcal{U} \cap \mathcal{L}_n(X)$ is arcwise connected$\}$, and

$\mathcal{HD}_n(X) = \{A \in HS_n(X) : A \notin \mathcal{HL}_n(X)$ and A has a basis $\mathcal{B}$ of neighborhoods in $HS_n(X)$ such that for each $\mathcal{U} \in \mathcal{B}$, $\dim \mathcal{U} \leq 2n$, and $\mathcal{U} \cap \mathcal{HL}_n(X)$ is arcwise connected$\}$, and

$\mathcal{HE}_n(X) = \{A \in HS_n(X) : \dim_A[HS_n(X)] = 2n\}$.

Given $J, K \in \mathfrak{A}_S(X)$, let

$\mathcal{D}(J,K) = \mathrm{cl}_{C_2(X)}(\partial \mathcal{L}_2(X) \cap \langle J^{\circ}, K^{\circ} \rangle_2) \cap \mathrm{cl}_{C_2(X)}(\partial \mathcal{L}_2(X) - \langle J^{\circ}, K^{\circ} \rangle_2)$,

$\mathcal{HF}_2(J) = \{F_X^2\} \cup \mathrm{cl}_{HS_2(X)}(\partial \mathcal{HL}_2(X) \cap q_X^2(\langle J^{\circ} \rangle_2)) - q_X^2(\langle J^{\circ} \rangle_2)$, and

$\mathcal{HD}(J,K) = \mathrm{cl}_{HS_2(X)}(\partial \mathcal{HL}_2(X) \cap q_X^2(\langle J^{\circ}, K^{\circ} \rangle_2)) \cap \mathrm{cl}_{HS_2(X)}(\partial \mathcal{HL}_2(X) - q_X^2(\langle J^{\circ}, K^{\circ} \rangle_2))$.

Let X be a finite graph such that $R(X) \neq \emptyset$. Given $J \in \mathfrak{A}_S(X)$, in the case that J is an arc, let p_J and q_J be its end points; we suppose that $q_J \in \mathrm{Fr}_X(J)$ (then, if $J \notin \mathfrak{A}_E(X)$, then $p_J \in \mathrm{Fr}_X(J)$, and if $J \in \mathfrak{A}_E(X)$, then $p_J \notin \mathrm{Fr}_X(J)$). If J is a cycle, let q_J be the unique point in J such that $J - \{q_J\}$ is open. Since X is not a simple closed curve, $q_J \notin J^{\circ}$. Given $J \in \mathfrak{A}_S(X)$, define $\mathcal{E}(J)$ in the following way: If J is an arc, let $\mathcal{E}(J) = C(J)$. In the case that J is a simple closed curve, let $\mathcal{E}(J) = \{A \in C(J) : A = J$ or $A = \{p\}$ for some $p \in J$ or A is a subarc of J such that $q_J \notin A$ or A is a subarc of J such that q_J is one of its end points$\}$. Note that, in both cases, $\mathcal{E}(J) = \mathrm{cl}_{C(X)}(\langle J^{\circ} \rangle_n \cap C(X))$.

For the reader's convenience, we mention some consequences for finite graphs of results of [2] that we will use in this paper.

Lemma 2.1 (See [2, Lemma 32]). *Let X be a finite graph. Then* $\partial \mathcal{L}_2(X) = \{A \in \mathcal{L}_2(X) :$ *A is connected or A has a degenerate component or A contains an end point of X*$\}$.

Lemma 2.2 (See [2, Lemma 33]). *Let X be a finite graph. Let $J, K \in \mathfrak{A}_S(X)$. Then* $\mathcal{D}(J,K) = \{\{p\} \cup A : p \in Fr_X(J)$ *and* $A \in \mathcal{E}(K)$ *or* $p \in Fr_X(K)$ *and* $A \in \mathcal{E}(J)\}$.

Given subsets $U_1, \ldots, U_m$ of X, let

$$\langle U_1, \ldots, U_m \rangle_n = \{A \in C_n(X) : A \subset U_1 \cup \cdots \cup U_m \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\}\}.$$

It is known [13, Theorem 3.1] that the family of all sets of the form $\langle U_1, \ldots, U_m \rangle_n$, where $m \in \mathbb{N}$ and each U_i is open in X, is a basis for the topology in $C_n(X)$.

Lemma 2.3. *Let X be a finite graph, $n \in \mathbb{N}$, and $A \in C_n(X) - F_n(X)$. If $A \cap R(X) \neq \emptyset$, then* $\dim_{q_X^n(A)}[HS_n(X)] \geq 2n+1$.

Proof. Since $q_X^n|_{C_n(X)-F_n(X)} : C_n(X) - F_n(X) \to HS_n(X) - \{F_X^n\}$ is a homeomorphism and $C_n(X) - F_n(X)$ is open in $C_n(X)$, it is enough to show that $\dim_A[C_n(X)] \geq 2n+1$. This is immediate from (MV). □

Lemma 2.4. *Let X be a finite graph such that $R(X) \neq \emptyset$ and let $n \in \mathbb{N}$. Then, for each neighborhood $\mathcal{U}$ of F_X^n in $HS_n(X)$, we have* $\dim[\mathcal{U}] \geq 2n+1$.

Proof. Let $\mathcal{U}$ be an open neighborhood of F_X^n in $HS_n(X)$ and $\mathcal{V} = (q_X^n)^{-1}(\mathcal{U})$. Then $\mathcal{V}$ is open in $C_n(X)$. Fix a point $p \in R(X)$. Since $\{p\} \in \mathcal{V}$, we can choose an arc A in X such that $p \in A$ and $A \in \mathcal{V}$. Thus, $q_X^n(A) \in \mathcal{U}$. Notice that $\dim_{q_X^n(A)}[HS_n(X)] \leq \dim[\mathcal{U}]$, and, by Lemma 2.3, $2n+1 \leq \dim_{q_X^n(A)}[HS_n(X)] \leq \dim[\mathcal{U}]$. □

The next two lemmas are the equivalent versions of results proved in [10], in which the uniqueness of the hyperspace $C_n(X)$ for a finite graph X is studied. For the reader's convenience, we include those results here.

Lemma 2.5 (See [10, Lemma 3.2])**.** *Let X be a finite graph and $n \in \mathbb{N}$. Then $\mathcal{M}_n(X) \subset \mathcal{L}_n(X)$.*

Lemma 2.6 (See [10, Lemma 3.5])**.** *Let X be a finite graph and $n \geq 3$. Then $\mathcal{M}_n(X) = \mathcal{L}_n(X)$.*

Lemma 2.7. *Let X be a finite graph and $n \in \mathbb{N}$. Suppose that either $n \geq 3$ or $R(X) \neq \emptyset$. Then $q_X^n(\mathcal{L}_n(X) - F_n(X)) = \mathcal{HL}_n(X)$.*

Proof. Since $q_X^n|_{C_n(X)-F_n(X)} : C_n(X) - F_n(X) \to HS_n(X) - \{F_X^n\}$ is a homeomorphism and $C_n(X) - F_n(X)$ and $HS_n(X) - \{F_X^n\}$ are open in the respective continua $C_n(X)$ and $HS_n(X)$, $q_X^n(\mathcal{L}_n(X) - F_n(X)) = \mathcal{HL}_n(X) - \{F_X^n\}$. In the case that $R(X) \neq \emptyset$, by Lemma 2.4, $\mathcal{HL}_n(X) - \{F_X^n\} = \mathcal{HL}_n(X)$. Thus, $q_X^n(\mathcal{L}_n(X) - F_n(X)) = \mathcal{HL}_n(X)$. Now, suppose that $R(X) = \emptyset$ and $n \geq 3$. We claim that $F_X^n \notin \mathcal{HL}_n(X)$. Suppose to the contrary that $F_X^n \in \mathcal{HL}_n(X)$. Let $\mathcal{M}$ be a neighborhood of F_X^n in $HS_n(X)$ such that $\mathcal{M}$ is a $2n$-cell. Fix a point $p \in X$. Then $(q_X^n)^{-1}(\mathcal{M})$ is a neighborhood of $\{p\}$ in $C_n(X)$. Choose an element $A \in C_n(X)$ such that A has exactly $n-1$ nondegenerate components and $q_X^n(A) \in \mathrm{int}_{HS_n(X)}(\mathcal{M})$. Then there exists a $2n$-cell $\mathcal{N}$ such that $q_X^n(A) \in \mathrm{int}_{HS_n(X)}(\mathcal{N})$ and $F_X^n \notin \mathcal{N}$. Then $(q_X^n)^{-1}(\mathcal{N})$ is a $2n$-cell that contains A in its interior in $C_n(X)$. This contradicts Lemma 2.6 and proves that $F_X^n \notin \mathcal{HL}_n(X)$. Therefore, $\mathcal{HL}_n(X) - \{F_X^n\} = \mathcal{HL}_n(X)$ and $q_X^n(\mathcal{L}_n(X) - F_n(X)) = \mathcal{HL}_n(X)$. □

Lemma 2.8 (See [10, Lemma 3.6])**.** *Let X be a finite graph and $n \geq 3$. Then $\mathcal{D}_n(X) = \{A \in C_n(X) : A$ is connected and $A \cap R(X) = \emptyset\}$.*

Lemma 2.9. *Let X be a finite graph. Then*

(a) *if $n \geq 3$, then $\mathcal{HD}_n(X) - \{F_X^n\} = \{q_X^n(A) \in HS_n(X) : A \in C(X) - F_n(X)$ and $A \cap R(X) = \emptyset\}$,*

(b) *if $n \geq 3$ and $R(X) \neq \emptyset$, then $\mathcal{HD}_n(X) = \{q_X^n(A) \in HS_n(X) : A \in C(X) - F_n(X)$ and $A \cap R(X) = \emptyset\}$,*

(c) *$\mathcal{HL}_2(X) \subset \{B \in HS_2(X)$: for each $A \in (q_X^2)^{-1}(B)$ and $A \cap R(X) = \emptyset\}$,*

(d) *if X is not a simple closed curve, then $\mathcal{HL}_2(X) = \{B \in HS_2(X) :$ for each $A \in (q_X^2)^{-1}(B)$ and $A \cap R(X) = \emptyset\}$.*

Proof. (a) Let $B = q_X^n(A) \in \mathcal{HD}_n(X) - \{F_X^n\}$. Then $A \in C_n(X) - F_n(X)$ and $\dim_A[C_n(X)] = \dim_B[HS_n(X)] \leq 2n$. By Lemma 2.3, $A \cap R(X) = \emptyset$. Since $B \notin \mathcal{HL}_n(X)$, $A \notin \mathcal{L}_n(X)$. Let $\mathcal{B}$ be a basis of neighborhoods of B in $HS_n(X)$ as in the definition of $\mathcal{HD}_n(X)$. Since $B \neq F_X^n$, we may assume that $F_X^n \notin \mathcal{U}$ for each $\mathcal{U} \in \mathcal{B}$. By Lemma 2.7, for each $\mathcal{U} \in \mathcal{B}$, $(q_X^n)^{-1}(\mathcal{U} \cap \mathcal{HL}_n(X)) = (q_X^n)^{-1}(\mathcal{U}) \cap (q_X^n)^{-1}(\mathcal{HL}_n(X)) = (q_X^n)^{-1}(\mathcal{U}) \cap (\mathcal{L}_n(X) - F_n(X)) = (q_X^n)^{-1}(\mathcal{U}) \cap \mathcal{L}_n(X)$. Hence, $(q_X^n)^{-1}(\mathcal{U}) \cap \mathcal{L}_n(X)$ is arcwise connected. Therefore, $\{(q_X^n)^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{B}\}$ is a basis of neighborhoods of A in $C_n(X)$, satisfying the definition of $\mathcal{D}_n(X)$. Thus, $A \in \mathcal{D}_n(X)$. By Lemma 2.8, $A \in C(X)$ and $A \cap R(X) = \emptyset$. We have shown that $\mathcal{HD}_n(X) - \{F_X^n\} \subset \{q_X^n(A) \in HS_n(X) : A \in C(X) - F_n(X)$ and $A \cap R(X) = \emptyset\}$. The other inclusion can be proved with similar arguments.

(b) In this case, $R(X) \neq \emptyset$. By Lemma 2.4, $F_X^n \notin \mathcal{HD}_n(X)$. Thus, (b) follows from (a).

(c) Let $B \in HS_2(X)$ be such that there exists $A \in (q_X^2)^{-1}(B)$ such that $A \cap R(X) \neq \emptyset$. If $A \notin F_2(X)$, by Lemma 2.3, $B \notin \mathcal{HL}_2(X)$. If $A \in F_2(X)$, by Lemma 2.4, $B \notin \mathcal{HL}_2(X)$.

(d) Take $B \in HS_2(X)$ such that for each $A \in (q_X^2)^{-1}(B)$, $A \cap R(X) = \emptyset$. In the case that $B \neq F_X^2$, there exists a unique A in the set $(q_X^2)^{-1}(B)$ and $A \cap R(X) = \emptyset$. Let A_1 and A_2 be the components of A, where $A_1 = A_2 = A$ in the case that A is connected. Given $i \in \{1, 2\}$, since $A_i \cap R(X) = \emptyset$ and X is not a simple closed curve, A_i is a subarc of X. Thus, we can enlarge A_i, a little bit, to an arc (not necessarily an edge) containing A_i in its interior. Hence, there exists an arc J_i in X such that $A_i \subset J_i^{\circ}$ and $J_i \cap R(X) = \emptyset$. We may ask that $J_1 \cap J_2 = \emptyset$, if $A_1 \neq A_2$, and $J_1 = J_2$, if $A_1 = A_2$. From [13, Example 5.1] and [9, Lemma 2.2], it follows that $\langle J_1, J_2 \rangle_2$ is a 4-cell and it is a neighborhood of A in $C_2(X)$. Since $A \notin F_2(X)$, there exists a 4-cell $\mathcal{M}$ such that $A \in \text{int}_{C_2(X)}(\mathcal{M})$ and $\mathcal{M} \subset \langle J_1, J_2 \rangle_2 - F_2(X)$. Thus, $q_X^n(\mathcal{M})$ is a 4-cell that is a neighborhood of B in $HS_2(X)$. Hence, $B \in \mathcal{HL}_2(X)$. Now suppose that $B = F_X^2$. In this case, for each $p \in X$, p is not a ramification point. Thus, X is an arc. By [15, Theorem 4.6], $HS_2(X)$ is a 4-cell. In any case, $B \in \mathcal{HL}_2(X)$. This completes the proof of (d). □

Lemma 2.10. *Let X be a finite graph. Then*

(a) *if $n \geq 3$ and $R(X) \neq \emptyset$, then the components of $\mathcal{HD}_n(X)$ are the sets of the form $q_X^n(\langle J^o \rangle_n \cap C(X)) - \{F_X^n\}$, where $J \in \mathfrak{A}_S(X)$,*

(b) *if $R(X) \neq \emptyset$, then the components of $\mathcal{HL}_1(X)$ are the sets of the form $q_X^1(\langle J^o \rangle_1) - \{F_X^1\}$, where $J \in \mathfrak{A}_S(X)$,*

(c) *if $R(X) \neq \emptyset$, then the components of $\mathcal{HL}_2(X)$ are the sets of the form $q_X^2(\langle J^o, K^o \rangle_2) - \{F_X^2\}$, where $J, K \in \mathfrak{A}_S(X)$,*

(d) *if $R(X) \neq \emptyset$, then the components of $\mathcal{HE}_n(X)$ are the sets of the form $q_X^n(\langle J_1^o, \ldots, J_m^o \rangle_n) - \{F_X^n\}$, where $J_1, \ldots, J_m \in \mathfrak{A}_S(X)$ and $m \leq n$.*

(e) *if $n \geq 3$ and $R(X) = \emptyset$, then $\mathcal{HD}_n(X) \cup \{F_X^n\} = q_X^n(C(X))$.*

Proof. (a) By Lemma 2.9, $\mathcal{HD}_n(X) = \bigcup\{q_X^n(\langle J^o \rangle_n \cap C(X)) - \{F_X^n\} : J \in \mathfrak{A}_S(X)\}$. Since the sets of the form $q_X^n(\langle J^o \rangle_n \cap C(X)) - \{F_X^n\}$, where $J \in \mathfrak{A}_S(X)$, are connected, open in $\mathcal{HD}_n(X)$, and pairwise disjoint, we conclude that they are the components of $\mathcal{HD}_n(X)$.

Using Lemma 2.3 and Lemma 2.4, (b) can be proved in a similar way as (a). The proof of (c) is also similar to that of (a), but it is important to notice that since $R(X) \neq \emptyset$, then $F_X^2 \neq B$ for each $B \in \mathcal{HL}_2(X)$.

(d) By (MV), $\dim_A[C_n(X)] = 2n$ if and only if $A \cap R(X) = \emptyset$. By Lemma 2.4, $F_X^n \notin \mathcal{HE}_n(X)$. Now, the proof of (d) is easy. Finally, the proof of (e) follows from Lemma 2.9(a). □

We are going to obtain topological models for the sets $\mathcal{HD}(J, K)$. We consider the different possibilities for J and K (in $\mathfrak{A}_S(X)$).

Lemma 2.11. *Let X be a finite graph such that $R(X) \neq \emptyset$ and $p \in X$. Then, for each $J \in \mathfrak{A}_S(X)$, $\{q_X^2(\{p\} \cup A) : A \in \mathcal{E}(J)\}$ is a 2-cell in $HS_2(X)$.*

Proof. Notice that the mapping $A \to \{p\} \cup A$ from $C(X)$ into $C_2(X)$ is an embedding. In the case that J is an arc, by [13, Example 5.1], there is a homeomorphism $h : \mathcal{E}(J) \to T$, where T is the convex triangle in $\mathbb{R}^2$, with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$ and $h(F_1(J))$ is the convex segment L that joins the points $(0, 0)$ and $(1, 1)$. Then $q_X^2(\{\{p\} \cup A : A \in \mathcal{E}(J)\})$ is homeomorphic to the quotient space obtained from T by shrinking L to a one-point set, which is a 2-cell. In the case that J is a cycle, using [13, Example 5.2], it is possible to show that there exists a homeomorphism $g : \mathcal{E}(J) \to W$, where W is the compact subset limited by two tangent circles R_1 and R_2, and $g(F_1(J)) = R_1$. Thus, $q_X^2(\{\{p\} \cup A : A \in \mathcal{E}(J)\})$ is homeomorphic to the quotient space obtained from W by shrinking R_1 to a one-point set, which is a 2-cell. □

Lemma 2.12. *Let X be a finite graph such that $R(X) \neq \emptyset$ and let $n \in \mathbb{N}$. Then*

(a) $\partial\mathcal{HL}_n(X) = q_X^n(\partial\mathcal{L}_n(X) - F_n(X))$,

(b) $\partial\mathcal{HL}_2(X) = \{q_X^2(A) \in HS_2(X) : A \in \mathcal{L}_2(X) - F_2(X)$ *and either A is connected or A has a degenerate component or A contains an end point of X*$\}$.

Proof. By Lemma 2.4, $F_X^n \notin \partial\mathcal{HL}_n(X)$. Thus, the equality in (a) follows from the fact that $q_X^n|_{C_n(X)-F_n(X)} : C_n(X) - F_n(X) \to HS_n(X) - \{F_X^n\}$ is a homeomorphism. (b) follows from (a) and Lemma 2.1. □

Lemma 2.13. *Let X be a finite graph such that $R(X) \neq \emptyset$. Then for every $J \in \mathfrak{A}_S(X) - \mathfrak{A}_R(X)$, $\mathcal{HF}_2(J) = \{F_X^2\} \cup \{q_X^2(A) \in HS_2(X) : A \subset J$, $A \cap R(X) \neq \emptyset$, $A \notin F_2(X)$, and either A is connected, or A has a degenerate component, or A contains an end point of $X\}$.*

Proof. ($\subset$) Given $B \in \mathcal{HF}_2(J)$, we have that $B = F_X^2$ or $B \notin q_X^2(\langle J^o \rangle_2)$ and there exists a sequence $\{B_m\}_{m=1}^{\infty}$ in $\partial\mathcal{HL}_2(X) \cap q_X^2(\langle J^o \rangle_2)$ such that $\lim B_m = B$. In the case that $B = F_X^2$ we are done. In the case that $B \neq F_X^2$, we may assume that $B_m \neq F_X^2$ for each $m \in \mathbb{N}$. Then for every $m \in \mathbb{N}$, there exists $A_m \subset J^o - F_2(X)$ such that $q_X^2(A_m) = B_m$. Then $\lim A_m = A$, where A is the only element of $C_2(X) - F_2(X)$ such that $q_X^2(A) = B$. Since $\langle J \rangle_2$ is closed, $A \subset J$. By Lemma 2.12(b), each A_m is connected or it has a degenerate component or it contains an end point of X. This implies that A is connected or it has a degenerate component or it contains an end point of X. Since $B \notin q_X^2(\langle J^o \rangle_2)$, A is not contained in J^o. Hence, A contains some ramification point of X. This completes the proof of the first inclusion.

($\supset$) Let $B = q_X^2(A)$ be such that $A \subset J$, $A \cap R(X) \neq \emptyset$, $A \notin F_2(X)$, and either A is connected or it has a degenerate component or it contains an end point of X. Since J is an arc, it is easy to show that A can be approximated by elements D in $\langle J^o \rangle_2 - F_2(X)$ satisfying that D is connected or it has a degenerate component or it contains an end point of X. By Lemma 2.12(b) for such elements D, $q_X^2(D) \in \partial\mathcal{HL}_2(X) \cap q_X^2(\langle J^o \rangle_2)$. Hence, $B \in \operatorname{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X) \cap q_X^2(\langle J^o \rangle_2))$. If $B = F_X^2$, then $B \in \mathcal{HF}_2(J)$, and we are done. So, suppose that $B \neq F_X^2$. In this case, $A \notin \langle J^o \rangle_2$ and $B \notin q_X^2(\langle J^o \rangle_2)$. Hence, $B \in \mathcal{HF}_2(J)$. This completes the proof of the other inclusion. □

Lemma 2.14. *Let X be a finite graph such that $R(X) \neq \emptyset$ and let $J, K \in \mathfrak{A}_S(X)$ be such that $J \in \mathfrak{A}_E(X)$ and $K \in \mathfrak{A}_R(X)$. Then $\mathcal{HF}_2(J)$ is a 2-sphere and it is not homeomorphic to $\mathcal{HF}_2(K)$.*

Proof. We are going to show that $\mathcal{HF}_2(J)$ is a 2-sphere, while $\mathcal{HF}_2(K)$ is not a 2-sphere. In order to prove this, we identify J with the interval

$[0,1]$ and we identify q_J to 0. According to Lemma 2.13, $\mathcal{HF}_2(J) = \{F_X^2\} \cup \{q_X^2(A) \in HS_2(X) : A \subset [0,1],\, 0 \in A,\, A \notin F_2(X)$ and either A is connected or it has a degenerate component or $1 \in A\}$. This implies that $\mathcal{HF}_2(J)$ is homeomorphic to $\{q^2_{[0,1]}([0,b] \cup \{c\}) : b, c \in [0,1]\} \cup \{q^2_{[0,1]}(\{0\} \cup [a,b]) : 0 \le a \le b \le 1\} \cup \{q^2_{[0,1]}([0,b] \cup [d,1]) : 0 \le b \le d \le 1\}$.

Let $\mathcal{D}_1 = \{[0,b] \cup \{c\} : b, c \in [0,1]$ and $b \le c \le 1\}$, $\mathcal{D}_2 = \{\{0\} \cup [a,b] : 0 \le a \le b \le 1\}$, and $\mathcal{D}_3 = \{[0,b] \cup [d,1] : 0 \le b \le d \le 1\}$. Let $\varphi_1 : \mathcal{D}_1 \to \mathbb{R}^2$ be given by $\varphi_1([0,b] \cup \{c\}) = (b, c-b)$. Clearly, φ_1 is an embedding and its image is the convex triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$. Thus, $\mathcal{D}_1$ is a 2-cell with manifold boundary $\{\{0,c\} : c \in [0,1]\} \cup \{[0,b] : b \in [0,1]\} \cup \{[0,b] \cup \{1\} : b \in [0,1]\}$. Hence, $q^2_{[0,1]}(\mathcal{D}_1)$ is a 2-cell with manifold boundary $\{q^2_{[0,1]}([0,b]) : b \in [0,1]\} \cup \{q^2_{[0,1]}([0,b] \cup \{1\}) : b \in [0,1]\}$. Since $\mathcal{D}_2$ is homeomorphic to $C([0,1])$, from [13, Example 5.1], $q^2_{[0,1]}(\mathcal{D}_2)$ is a 2-cell with manifold boundary $\{q^2_{[0,1]}([0,b]) : b \in [0,1]\} \cup \{q^2_{[0,1]}(\{0\} \cup [a,1]) : a \in [0,1]\}$. Let T be the convex triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$, and $(1,1)$ and let R be the quotient space obtained from T by shrinking the segment joining $(0,0)$ and $(1,1)$ to a point. Let $q : T \to R$ be the quotient map. Notice that R is a 2-cell. Let $\varphi_3 : \mathcal{D}_3 \to R$ be given by $\varphi_3([0,b] \cup [d,1]) = q(b,d)$. Then φ_3 is a homeomorphism. This implies that $q^2_{[0,1]}(\mathcal{D}_3)$ is a 2-cell with manifold boundary $\{q^2_{[0,1]}([0,b] \cup \{1\}) : b \in [0,1]\} \cup \{q^2_{[0,1]}(\{0\} \cup [a,1]) : a \in [0,1]\}$. From here, it is easy to show that $\mathcal{HF}_2(J)$ is a 2-sphere.

In order to show that $\mathcal{HF}_2(K)$ is not a 2-cell, we will show that the element $K \in \mathcal{HF}_2(K)$ does not have a neighborhood in $\mathcal{HF}_2(K)$ that is a 2-cell. For simplicity, we suppose that K is the unit circle S^1 in $\mathbb{R}^2$ and q_K is the element $(1,0)$. Given $t \in [0,1]$, let $e(t) = (\cos 2\pi t, \sin 2\pi t)$. Given $t \in [0, \frac{1}{4}]$, let $S_t = \{e([t,1]) \cup \{e(s)\} : s \in [0,t]\}$ and let $g_t : S_t \to \mathbb{R}^2$ be the mapping given by $g_t(e([t,1]) \cup \{e(s)\} = te(\frac{s}{t})$, if $t \neq 0$; and $g_0(e([0,1])) = (0,0)$. Let $S = \bigcup\{S_t : t \in [0, \frac{1}{4}]\}$ and $g : S \to \mathbb{R}^2$ be the mapping that is the common extension of all mappings g_t. It is easy to show that g is an embedding and Im g is a disk with center at $g(S^1) = (0,0)$. Given $t \in [\frac{3}{4}, 1]$, let $R_t = \{e([0,t]) \cup \{e(s)\} : s \in [t,1]\}$. Proceeding in a similar way, it can be proved that $R = \bigcup\{R_t : t \in [\frac{3}{4}, 1]\}$ is a 2-cell having S^1 as a manifold interior point and $R \cap S = \{S^1\}$. Using Lemma 2.12(b), it can be shown that $S \cup R \subset \mathcal{HF}_2(K)$. Hence, K does not have a neighborhood in $\mathcal{HF}_2(K)$ that is a 2-cell. Therefore, $\mathcal{HF}_2(J)$ is not homeomorphic to $\mathcal{HF}_2(K)$. □

Lemma 2.15. *Let X be a finite graph such that $R(X) \neq \emptyset$. Let $J, K \in \mathfrak{A}_S(X)$. Then*

$$\mathcal{HD}(J,K)=\{q_X^2(\{p\}\cup G): p\in Fr_X(J) \text{ and } G\in\mathcal{E}(K) \text{ or } p\in Fr_X(K) \text{ and } G\in\mathcal{E}(J)\}.$$

Proof. ($\subset$) Let $B\in\mathcal{HD}(J,K)$. If $B=F_X^2$, choose a point $p\in \mathrm{Fr}_X(J)$ and a point $q\in K$, then $\{q\}\in\mathcal{E}(K)$ and $q_X^2(\{p\}\cup\{q\})=B$. Thus, we may assume that $B\neq F_X^2$. Let A be the unique element in $C_2(X)-F_2(X)$ such that $q_X^2(A)=B$. Since $B\in \mathrm{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X)\cap q_X^2(\langle J^o,K^o\rangle_2))$, there exists a sequence $\{A_m\}_{m=1}^\infty$ of elements of $\langle J^o,K^o\rangle_2$ such that $\lim q_X^2(A_m)=B$, $A_m\notin F_2(X)$ and $q_X^2(A_m)\in\partial\mathcal{HL}_2(X)$ for each $m\in\mathbb{N}$. By the continuity of $(q_X^2)^{-1}$ at B, $\lim A_m=A$. By Lemma 2.12(a), $A_m\in\partial\mathcal{L}_2(X)$ for each $m\in\mathbb{N}$. Hence, $A\in \mathrm{cl}_{C_2(X)}(\partial\mathcal{L}_2(X)\cap\langle J^o,K^o\rangle_2)$. On the other hand, since $B\in \mathrm{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X)-q_X^2(\langle J^o,K^o\rangle_2))$, there exists a sequence $\{E_m\}_{m=1}^\infty$ of elements of $\partial\mathcal{HL}_2(X)-q_X^2(\langle J^o,K^o\rangle_2)$ such that $\lim E_m=B$ and $E_m\neq F_X^2$ for each $m\in\mathbb{N}$. Given $m\in\mathbb{N}$, let D_m be the unique element of $C_2(X)-F_2(X)$ such that $q_X^2(D_m)=E_m$. Then $\lim D_m=A$. By Lemma 2.12(a), $D_m\in\partial\mathcal{L}_2(X)-\langle J^o,K^o\rangle_2$ for each $m\in\mathbb{N}$. Hence, $A\in \mathrm{cl}_{C_2(X)}(\partial\mathcal{L}_2(X)-\langle J^o,K^o\rangle_2)$. We have shown that $A\in\mathcal{D}(J,K)$. By Lemma 2.2, $A=\{p\}\cup G$ for some $p\in \mathrm{Fr}_X(J)$ and $G\in\mathcal{E}(K)$ or some $p\in \mathrm{Fr}_X(K)$ and $G\in\mathcal{E}(J)$. This completes the proof of the first inclusion.

($\supset$) Take $B=q_X^2(\{p\}\cup G)$, where $p\in \mathrm{Fr}_X(J)$ and $G\in\mathcal{E}(K)$ or $p\in \mathrm{Fr}_X(K)$ and $G\in\mathcal{E}(J)$. By Lemma 2.2, the set $A=\{p\}\cup G$ belongs to $\mathcal{D}(J,K)$. In the case that $B=F_X^2$, choose points $p\in J^o$ and $q\in K^o$, where $p\neq q$. Let $\{Q_m\}_{m=1}^\infty$ be a sequence in $C(K)$ such that $\lim Q_m=\{q\}$, $Q_m\subset K^o$ and Q_m is nondegenerate for each $m\in\mathbb{N}$. Given $m\in\mathbb{N}$, since $Q_m\cup\{p\}\in\partial\mathcal{L}_2(X)\cap\langle J^o,K^o\rangle_2$, by Lemma 2.12(b), $q_X^2(Q_m\cup\{p\})\in\partial\mathcal{HL}_2(X)\cap q_X^2(\langle J^o,K^o\rangle_2)$, so $B=q_X^2(\{p,q\})=\lim q_X^2(Q_m\cup\{p\})\in \mathrm{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X)\cap q_X^2(\langle J^o,K^o\rangle_2))$. Let $J',K'\in\mathfrak{A}_S(X)$ be such that $\{J',K'\}\neq\{J,K\}$ (in the case that $J=K$, we can take $J'\neq J$ and $K'=J$, and in the case that $J\neq K$, we can take $J'=K'=K$). Taking a point $p\in J'$ and a sequence $\{Q_m\}_{m=1}^\infty$ in K' as before, we can prove that $B\in \mathrm{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X)-q_X^2(\langle J^o,K^o\rangle_2)$. Hence, $F_X^2\in \mathrm{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X)\cap q_X^2(\langle J^o,K^o\rangle_2))\cap \mathrm{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X)-q_X^2(\langle J^o,K^o\rangle_2))$. Thus, we can suppose that $B\neq F_X^2$.

Since $A\in\mathcal{D}(J,K)$, there exists a sequence $\{A_m\}_{m=1}^\infty$ in $\partial\mathcal{L}_2(X)\cap\langle J^o,K^o\rangle_2$ such that $\lim A_m=A$ and $A_m\notin F_2(X)$ for each $m\in\mathbb{N}$. Hence, $q_X^2(A_m)\in\partial\mathcal{HL}_2(X)\cap q_X^2(\langle J^o,K^o\rangle_2)$. Thus, $B\in \mathrm{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X)\cap q_X^2(\langle J^o,K^o\rangle_2))$. Similarly, $B\in \mathrm{cl}_{HS_2(X)}(\partial\mathcal{HL}_2(X)-q_X^2(\langle J^o,K^o\rangle_2))$. Therefore, $B\in\mathcal{HD}(J,K)$. □

Now, we are prepared to describe models for the different sets of the form $\mathcal{HD}(J,K)$ where $J,K\in\mathfrak{A}_S(X)$ (for a finite graph X such that $R(X)\neq\emptyset$). We consider nine cases.

Case A: $J = K$, J is an arc, and $J \notin \mathfrak{A}_E(X)$.

According to Lemma 2.15, $\mathcal{HD}(J,J) = \{q_X^2(\{p_J\} \cup A) : A \in C(J)\} \cup \{q_X^2(\{q_J\} \cup A) : A \in C(J)\}$. By Lemma 2.11, each one of these sets is a 2-cell and they intersect in the set $\{F_X^2, q_X^2(J)\}$.

Case B: $J = K$ and $J \in \mathfrak{A}_E(X)$.

Here, $\mathcal{HD}(J,J) = \{q_X^2(\{q_J\} \cup A) : A \in C(J)\}$ is a 2-cell.

Case C: $J = K$ and $J \in \mathfrak{A}_R(X)$.

Here, $\mathcal{HD}(J,J) = \{q_X^2(\{q_J\} \cup A) : A \in \mathcal{E}(J)\}$ is a 2-cell.

For the remaining cases we suppose that $J \neq K$.

Case D: Both J and K are arcs and $J, K \notin \mathfrak{A}_E(X)$.

Let $\mathcal{D}_1 = \{q_X^2(\{p_J\} \cup A) : A \in C(K)\}$, $\mathcal{D}_2 = \{q_X^2(\{q_J\} \cup A) : A \in C(K)\}$, $\mathcal{D}_3 = \{q_X^2(\{p_K\} \cup A) : A \in C(J)\}$, and $\mathcal{D}_4 = \{q_X^2(\{q_K\} \cup A) : A \in C(J)\}$. Note that $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ are 2-cells and $\mathcal{HD}(J,K) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$. Here, we consider three subcases.

D.1: $J \cap K = \emptyset$.

In this subcase, $\mathcal{D}_i \cap \mathcal{D}_j = \{F_X^2\}$, if $i \neq j$.

D.2: $J \cap K$ is a one-point set.

In this subcase, $\mathcal{D}_i \cap \mathcal{D}_j = \{F_X^2\}$, if $i \neq j$.

D.3: $J \cap K$ is a set with exactly two points. We may assume that $p_J = p_K$ and $q_J = q_K$. Then $\mathcal{D}_1 \cap \mathcal{D}_2 = \{F_X^2, q_X^2(K)\}$, $\mathcal{D}_1 \cap \mathcal{D}_3 = \{F_X^2\} = \mathcal{D}_1 \cap \mathcal{D}_4$, $\mathcal{D}_2 \cap \mathcal{D}_3 = \{F_X^2\} = \mathcal{D}_2 \cap \mathcal{D}_4$, and $\mathcal{D}_3 \cap \mathcal{D}_4 = \{F_X^2, q_X^2(J)\}$.

Case E: Both J and K are arcs and $J \notin \mathfrak{A}_E(X)$ and $K \in \mathfrak{A}_E(X)$.

Let $\mathcal{D}_1 = \{q_X^2(\{p_J\} \cup A) : A \in C(K)\}$, $\mathcal{D}_2 = \{q_X^2(\{q_J\} \cup A) : A \in C(K)\}$, and $\mathcal{D}_3 = \{q_X^2(\{q_K\} \cup A) : A \in C(J)\}$. Note that $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are 2-cells; $\mathcal{HD}(J,K) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$; and $\mathcal{D}_i \cap \mathcal{D}_j = \{F_X^2\}$, if $i \neq j$.

Case F: J is an arc, $J \notin \mathfrak{A}_E(X)$, and K is a simple closed curve.

Let $\mathcal{D}_1 = \{q_X^2(\{p_J\} \cup A) : A \in \mathcal{E}(K)\}$, $\mathcal{D}_2 = \{q_X^2(\{q_J\} \cup A) : A \in \mathcal{E}(K)\}$, and $\mathcal{D}_3 = \{q_X^2(\{q_K\} \cup A) : A \in C(J)\}$. Note that $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are 2-cells; $\mathcal{HD}(J,K) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$; and $\mathcal{D}_i \cap \mathcal{D}_j = \{F_X^2\}$, if $i \neq j$.

Case G: J and K are arcs and $J, K \in \mathfrak{A}_E(X)$.

Let $\mathcal{D}_1 = \{\{q_J^2\} \cup A : A \in C(K)\}$ and $\mathcal{D}_2 = \{\{q_K^2\} \cup A : A \in C(J)\}$. Then $\mathcal{HD}(J,K) = \mathcal{D}_1 \cup \mathcal{D}_2$; $\mathcal{D}_1$ and $\mathcal{D}_2$ are 2-cells; and $\mathcal{D}_1 \cap \mathcal{D}_2 = \{F_X^2\}$.

Case H: $J \in \mathfrak{A}_E(X)$ and $K \in \mathfrak{A}_R(X)$.

Let $\mathcal{D}_1 = \{\{q_J^2\} \cup A : A \in \mathcal{E}(K)\}$ and $\mathcal{D}_2 = \{\{q_K^2\} \cup A : A \in C(J)\}$. Then $\mathcal{HD}(J,K) = \mathcal{D}_1 \cup \mathcal{D}_2$; $\mathcal{D}_1$ and $\mathcal{D}_2$ are 2-cells; and $\mathcal{D}_1 \cap \mathcal{D}_2 = \{F_X^2\}$.

Case I: J and K belong to $\mathfrak{A}_R(X)$.

Let $\mathcal{D}_1 = \{\{q_J^2\} \cup A : A \in \mathcal{E}(K)\}$ and $\mathcal{D}_2 = \{\{q_K^2\} \cup A : A \in \mathcal{E}(J)\}$. Then $\mathcal{HD}(J,K) = \mathcal{D}_1 \cup \mathcal{D}_2$; $\mathcal{D}_1$ and $\mathcal{D}_2$ are 2-cells; and $\mathcal{D}_1 \cap \mathcal{D}_2 = \{F_X^2\}$.

Observe the following. Let J, K, J', and K' in $\mathfrak{A}_S(X)$ be such that $\mathcal{D}(J,K)$ is homeomorphic to $\mathcal{D}(J',K')$. Then (a) J and K are as in Case

A if and only if J' and K' are as in Case A and (b) J and K are as in one of the cases B and C if and only if J' and K' are as in one of the cases B or C.

3. Main Results

Theorem 3.1. *Let X and Y be finite graphs such that $R(X) \neq \emptyset \neq R(Y)$, let $n \in \mathbb{N}$, and let $h : HS_n(X) \to HS_n(Y)$ be a homeomorphism. Suppose that for each $J \in \mathfrak{A}_S(X)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^n(\langle J^o \rangle_n \cap C(X)) - \{F_X^n\}) \subset q_Y^n(\langle J_h^o \rangle_n)$ and $\mathfrak{A}_S(Y) = \{J_h : J \in \mathfrak{A}_S(X)\}$. Then*

(A) *for each $J \in \mathfrak{A}_S(X)$, $h(q_X^n(\langle J^o \rangle_n) - \{F_X^n\}) = q_Y^n(\langle J_h^o \rangle_n) - \{F_Y^n\}$,*

(B) *for each $J \in \mathfrak{A}_S(X)$, $h^{-1}(q_Y^n(\langle J_h^o \rangle_n) \cap C(Y)) - \{F_Y^n\}) \subset q_X^n(\langle J^o \rangle_n) - \{F_X^n\}$,*

(C) *the association $J \to J_h$ is a bijection between $\mathfrak{A}_S(X)$ and $\mathfrak{A}_S(Y)$,*

(D) *$h(F_X^n) = F_Y^n$.*

If we also suppose that

(1) *if $J \in \mathfrak{A}_R(X)$, then $J_h \in \mathfrak{A}_R(Y)$ and*

(2) *if $J \in \mathfrak{A}_E(X)$, then $J_h \in \mathfrak{A}_E(Y)$, then X is homeomorphic to Y.*

Proof. By Lemma 2.10(d), the components of $\mathcal{HE}_n(X)$ are the sets of the form $q_X^n(\langle J_1^o, \ldots, J_m^o \rangle_n) - \{F_X^n\}$, where $J_1, \ldots, J_m \in \mathfrak{A}_S(X)$ and $m \leq n$. Given $J \in \mathfrak{A}_S(X)$, $h(q_X^n(\langle J^o \rangle_n) - \{F_X^n\})$ is a component of $\mathcal{HE}_n(Y)$. Take two different subarcs A and B of J^o. Then $h(q_X^n(A)), h(q_X^n(B)) \in h(q_X^n(\langle J^o \rangle_n) - \{F_X^n\})$. We may assume that $h(q_X^n(A)) \neq F_Y^n$. By hypothesis $h(q_X^n(A)) \in q_Y^n(\langle J_h^o \rangle) - \{F_Y^n\}$. Hence, $h(q_X^n(A))$ belongs to the sets $h(q_X^n(\langle J^o \rangle_n) - \{F_X^n\})$ and $q_Y^n(\langle J_h^o \rangle) - \{F_Y^n\}$. Since these two sets are components of $\mathcal{HE}_n(Y)$, we obtain that $h(q_X^n(\langle J^o \rangle_n) - \{F_X^n\}) = q_Y^n(\langle J_h^o \rangle_n) - \{F_Y^n\}$. This proves (A).

Note that (B) is immediate from (A).

To prove (C), suppose that $J, L \in \mathfrak{A}_S(X)$ are such that $J_h = L_h$. By (A), $h(q_X^n(\langle J^o \rangle_n) - \{F_X^n\}) = h(q_X^n(\langle L^o \rangle_n) - \{F_X^n\})$. Thus, $q_X^n(\langle J^o \rangle_n) - \{F_X^n\} = q_X^n(\langle L^o \rangle_n) - \{F_X^n\}$. If A is an arc such that $A \subset J^o$, then $q_X^n(A) \in q_X^n(\langle L^o \rangle_n) - \{F_X^n\}$, so there exists $B \in \langle L^o \rangle_n - F_n(X)$ such that $q_X^n(A) = q_X^n(B)$. Hence, $A = B \in \langle J^o \rangle_n \cap \langle L^o \rangle_n$ and $\emptyset \neq A \subset J^o \cap L^o$. This implies that $J = L$. This proves (C).

We claim that $\{F_X^n\} = \bigcap\{\mathrm{cl}_{HS_n(X)}(q_X^n(\langle J^o \rangle_n) - \{F_X^n\}) : J \in \mathfrak{A}_S(X)\}$. Given $J \in \mathfrak{A}_S(X)$, since each element of the form $\{p\}$ where $p \in J^o$ can be approximated by elements in $\langle J^o \rangle_n - F_n(X)$, we have

$$\{p\} \in \mathrm{cl}_{\,C_n(X)}(\langle J^o \rangle_n - F_n(X)).$$

This implies that $F_X^n \in \mathrm{cl}_{HS_n(X)}(q_X^n(\langle J^o \rangle_n) - \{F_X^n\})$. Suppose that there exists an element $B \in \bigcap\{\mathrm{cl}_{HS_n(X)}(q_X^n(\langle J^o \rangle_n) - \{F_X^n\}) : J \in \mathfrak{A}_S(X)\} -$

$\{F_X^n\}$. Fix $J_0 \in \mathfrak{A}_S(X)$. Then there exists a sequence $\{A_m\}_{m=1}^\infty$ of elements in $\langle J_0^\circ\rangle_n$ such that $\lim q_X^n(A_m) = B$. We may assume that $\lim A_m = A$ for some $A \in \langle J_0\rangle_n \cap C_n(X)$. Since $B \neq F_X^n$ and $q_X^n(A) = B$, we obtain that $A \in C_n(J_0) - F_n(J_0)$. Thus, A has a nondegenerate component and $A \subset J_0$. This implies that $A \cap J_0^\circ \neq \emptyset$. Since $R(X) \neq \emptyset$, there exists $J \in \mathfrak{A}_S(X)$ such that $J \neq J_0$. By an analogous argument, there exists $D \in C_n(J) - F_n(J)$ such that $q_X^n(D) = B$. Since $B \neq F_X^n$, $A = D$. Thus, $J_0^\circ \cap J \neq \emptyset$. This is a contradiction that ends the proof that $\{F_X^n\} = \bigcap\{\mathrm{cl}_{HS_n(X)}(q_X^n(\langle J^\circ\rangle_n) - \{F_X^n\}) : J \in \mathfrak{A}_S(X)\}$.

An analogous argument shows that $\{F_Y^n\} = \bigcap\{\mathrm{cl}_{HS_n(Y)}(q_Y^n(\langle J_h^\circ\rangle_n) - \{F_Y^n\}) : J_h \in \mathfrak{A}_S(Y)\}$. Thus,

$$\begin{aligned} h(\{F_X^n\}) = &\bigcap\{\mathrm{cl}_{HS_n(Y)}(h(q_X^n(\langle J^\circ\rangle_n) - \{F_X^n\})) : J \in \mathfrak{A}_S(X)\} = \\ &\bigcap\{\mathrm{cl}_{HS_n(Y)}(q_Y^n(\langle J_h^\circ\rangle_n) - \{F_Y^n\}) : J \in \mathfrak{A}_S(X)\} = \\ &\bigcap\{\mathrm{cl}_{HS_n(Y)}(q_Y^n(\langle J_h^\circ\rangle_n) - \{F_Y^n\}) : J_h \in \mathfrak{A}_S(Y)\} = \{F_Y^n\}. \end{aligned}$$

Hence, $h(F_X^n) = F_Y^n$.

Given $A \in C_n(X) - F_n(X)$, $q_X^n(A) \neq F_X^n$. Then $h(q_X^n(A)) \neq F_Y^n$ and there exists a unique $D_A \in C_n(Y) - F_n(Y)$ such that $h(q_X^n(A)) = q_Y^n(D_A)$. Notice that the association $A \to D_A$ is a homeomorphism between $C_n(X) - F_n(X)$ and $C_n(Y) - F_n(Y)$. Hence, $\dim_A[C_n(X)] = \dim_{q_X^n(A)}[HS_n(X)] = \dim_{h(q_X^n(A))}[HS_n(Y)] = \dim_{D_A}[C_n(Y)]$.

Given $J \in \mathfrak{A}_S(X)$, let $\mathcal{K}_n(J, X) = \mathrm{cl}_{C_n(X)}(\langle J^\circ\rangle_n) - F_n(X)$.

Claim 1. Let $J \in \mathfrak{A}_S(X)$. Then
(a) $\mathcal{K}_n(J_h, Y) = \{D_A \in C_n(Y) - F_n(Y) : A \in \mathcal{K}_n(J, X)\}$,
(b) $\{\dim_A[C_n(X)] : A \in \mathcal{K}_n(J, X)\} = \{\dim_B[C_n(Y)] : B \in \mathcal{K}_n(J_h, Y)\}$,
(c) $|J \cap R(X)| = |J_h \cap R(Y)|$,
(d) if $A \in \mathcal{K}_n(J, X)$, then $|A \cap R(X)| = |D_A \cap R(Y)|$.

Proof of Claim 1. For (a), let $A \in \mathcal{K}_n(J, X)$. Then there exists a sequence $\{A_m\}_{m=1}^\infty$ in $\langle J^\circ\rangle_n - F_n(X)$ such that $A = \lim A_m$. Since $h(q_X^n(\langle J^\circ\rangle_n) - F_n(X)) = q_Y^n(\langle J_h^\circ\rangle_n) - \{F_Y^n\}$, for each $m \in \mathbb{N}$, $D_{A_m} \in \langle J_h^\circ\rangle_n - F_n(Y)$ and $D_A \in \mathcal{K}_n(J_h, Y)$. This proves the inclusion $\supset$. The proof of the other one is similar. Property (b) follows from (a).

To prove (c), it is enough to note that, by (MV),
$|\{\dim_A[C_n(X)] : A \in \mathcal{K}_n(J, X)\}| \geq 3$ if and only if $|J \cap R(X)| = 2$ and $|\{\dim_A[C_n(X)] : A \in \mathcal{K}_n(J, X)\}| = 2$ if and only if $|J \cap R(X)| = 1$.

Finally, we prove (d). Take $A \in \mathcal{K}_n(J, X)$. If $|A \cap R(X)| = 2$, then $|J \cap R(X)| = 2$. Thus, $|J_h \cap R(Y)| = 2$ and, by (MV),

$$\dim_A[C_n(X)] = \max\{\dim_E[C_n(X)] : E \in \mathcal{K}_n(J, X)\}.$$

Hence, $\dim_{D_A}[C_n(Y)] = \max\{\dim_B[C_n(Y)] : B \in \mathcal{K}_n(J_h, Y)\}$. This implies that $|D_A \cap R(Y)| = 2$. Similarly, if $|D_A \cap R(Y)| = 2$, then $|A \cap R(X)| = 2$. If $|A \cap R(X)| = 0$, by (MV), $2n = \dim_A[C_n(X)] =$

$\dim_{D_A}[C_n(Y)]$. Applying again (MV), we obtain that $|D_A \cap R(Y)| = 0$. Similarly, if $|D_A \cap R(Y)| = 0$, then $|A \cap R(X)| = 0$. Finally, if $|A \cap R(X)| = 1$, then $|D_A \cap R(Y)| \notin \{0, 2\}$. Thus, $|D_A \cap R(Y)| = 1$. This completes the proof of Claim 1.

Claim 2. Let $J \in \mathfrak{A}_S(X)$ and $v \in J \cap R(X)$. Then the set $\mathcal{K}(v, J) = \{A \in \mathcal{K}_n(J, X) : A \cap R(X) = \{v\}\}$ is arcwise connected.

Proof of Claim 2. Let $A \in \mathcal{K}_n(J, X)$ be such that $A \cap R(X) = \{v\}$. In the case that J is an arc, it is easy to show that there exists a subarc L of J such that $A \subset L$ and $L \cap R(X) = \{v\}$. By [13, Theorem 14.6], there exists a mapping $\alpha : [0, 1] \to C_n(L)$ such that $\alpha(0) = A$, $\alpha(1) = L$, and $\alpha(s) \subset \alpha(t)$, if $s \leq t$. Clearly, $\operatorname{Im} \alpha \subset \mathcal{K}_n(J, X)$ and $\alpha(s) \cap R(X) = \{v\}$ for each $s \in [0, 1]$. Since the set $\{N : N$ is a subarc of J and $N \cap R(X) = \{v\}\}$ is arcwise connected, we conclude that $\mathcal{K}(v, J)$ is arcwise connected. Now suppose that J is a cycle in X. We identify J with the unit circle S^1 in $\mathbb{R}^2$ and v with the point $(1, 0)$. Let $e : [0, 1] \to S^1$ be given by $e(t) = (\cos(2\pi t), \sin(2\pi t))$. Let $A = A_1 \cup \cdots \cup A_r$, where $A_1, \ldots, A_r$ are the different components of A. Suppose that $v \in A_1$. If $r < n$ or ($r = n = 1$ and $A = J$), then for each $s \in (0, 1]$, the set $B(s) = e(\{sx : x \in e^{-1}(A)\})$ belongs to $\mathcal{K}(v, J)$. Notice that the association $s \to B(s)$ is continuous and $B(\frac{1}{2}) \subset e([0, \frac{1}{2}])$. Since $e([0, \frac{1}{2}]) \in \mathcal{K}(v, J)$ and $e([0, \frac{1}{2}])$ is a subarc of J, we can complete the proof of this case as we did in the case where J is an arc. If $r = n > 1$ (or $r = n = 1$ and A_1 is a subarc of J), since $A \in \mathcal{K}_n(J, X)$, v is not in the interior as manifold of A_1. Then we may assume that there exists $s_0 \in [0, 1)$ such that $A \subset e([0, s_0])$. Since $e([0, s_0]) \in \mathcal{K}(v, J)$ and $e([0, s_0])$ is a subarc of J, again we can complete the proof of this case as we did in the case that J is an arc. This completes the proof of Claim 2.

Given $v \in R(X)$, let $J \in \mathfrak{A}_S(X)$ be such that $v \in J$. Let $\mathcal{K}(v, J)$ be as in Claim 2 and let $A \in \mathcal{K}(v, J)$. By Claim 1, $D_A \in \mathcal{K}_n(J_h, Y)$ and there exists a unique point $v_h(A) \in R(Y) \cap D_A$. Notice that $v_h(A) \in J_h$.

We claim that $v_h(A)$ does not depend on A and, in fact, it does not depend on the choice of J. That is, if $K \in \mathfrak{A}_S(X)$ and $E \in \mathcal{K}(v, K)$, then $v_h(A) = v_h(E)$. In order to prove this, take J, K, A, and E as described. Take a subarc A_1 of J such that $A_1 \neq J$ and v is an end point of A_1. Note that $A_1 \in \mathcal{K}(v, J)$. Similarly, there exists $E_1 \in \mathcal{K}(v, K)$ such that E_1 is connected. By Claim 2, $\mathcal{K}(v, J)$ and $\mathcal{K}(v, K)$ are arcwise connected. Thus, there exist mappings $\alpha_A : [0, 1] \to \mathcal{K}(v, J)$ and $\alpha_E : [0, 1] \to \mathcal{K}(v, K)$ such that $\alpha_A(0) = A$, $\alpha_A(1) = A_1$, $\alpha_E(0) = E_1$, and $\alpha_E(1) = E$. It is easy to show that there exists a map $\alpha_0 : [0, 1] \to C(A_1 \cup E_1)$ such that $\alpha_0(0) = A_1$, $\alpha_0(1) = E_1$, and, for each $t \in [0, 1]$, $\alpha_0(t) \cap R(X) = \{v\}$ and $\alpha_0(t) \notin F_n(X)$. Define $\alpha : [0, 1] \to C(A_1 \cup E_1) \cup \mathcal{K}(v, J) \cup \mathcal{K}(v, K)$ by

$$\alpha(t) = \begin{cases} \alpha_A(3t), & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \alpha_0(3t-1), & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \alpha_E(3t-2), & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Clearly, $\alpha(0) = A$, $\alpha(1) = E$, α is continuous, and for each $t \in [0,1]$, $\alpha(t) \cap R(X) = \{v\}$. Let i_0 be the order of v in X.

By (MV) for each $t \in [0,1]$, $2n < 2n + (i_0 - 2) = \dim_{\alpha(t)}[C_n(X)] = \dim_{q_X^n(\alpha(t))}[HS_n(X)] = \dim_{h(q_X^n(\alpha(t)))}[HS_n(Y)] = \dim_{(q_Y^n)^{-1}(h(q_X^n(\alpha(t)))}[C_n(Y)]$. Let $T = \{t \in [0,1] : v_h(A) \in (q_Y^n)^{-1}(h(q_X^n(\alpha(t))))\}$. It is easy to show that T is a closed subset of $[0,1]$ and $0 \in T$. We need to show that $T = [0,1]$. Suppose to the contrary that $T \neq [0,1]$. Let R be a component of $[0,1] - T$. Then $t_0 = \inf R \in T$ and there exists a sequence $\{r_m\}_{m=1}^{\infty}$ of elements of R such that $\lim r_m = t_0$. Suppose that the order of the ramification point $v_h(A)$ in Y is j. Then $j \geq 3$. Applying again (MV) we obtain that $(q_Y^n)^{-1}(h(q_X^n(\alpha(t))))$ intersects $R(Y)$ for each $t \in [0,1]$. Since $R(Y)$ is finite, we may assume that there exists $w \in R(Y)$ such that $w \in (q_Y^n)^{-1}(h(q_X^n(\alpha(r_m))))$ for each $m \in \mathbb{N}$. This implies that $w \in (q_Y^n)^{-1}(h(q_X^n(\alpha(t_0))))$. Since $\dim_{(q_Y^n)^{-1}(h(q_X^n(\alpha(0))))}[C_n(Y)] = 2n + (i_0 - 2)$ and the only ramification point of Y in $(q_Y^n)^{-1}(h(q_X^n(A))) = (q_Y^n)^{-1}(h(q_X^n(\alpha(0)))) = D_A$ is $v_h(A)$, by (MV) the order of $v_h(A)$ in Y is i_0. Thus, $j = i_0$. Since $r_1 \notin T$, $v_h(A) \notin (q_Y^n)^{-1}(h(q_X^n(\alpha(r_1))))$, so $v_h(A) \neq w$. Let $j_0 \geq 3$ be the order of w in Y. Since $v_h(A), w \in (q_Y^n)^{-1}(h(q_X^n(\alpha(t_0))))$, by (MV), $\dim_{(q_Y^n)^{-1}(h(q_X^n(\alpha(t_0))))}[C_n(Y)] \geq 2n + (i_0 - 2) + (j_0 - 2) > 2n + (i_0 - 2)$, a contradiction. This completes the proof that $T = [0,1]$. Hence, for each $t \in [0,1]$, $v_h(A) \in (q_Y^n)^{-1}(h(q_X^n(\alpha(t))))$. Proceeding as previously, it is easy to prove that for each $t \in [0,1]$, $v_h(A)$ is the only ramification point of Y in the set $(q_Y^n)^{-1}(h(q_X^n(\alpha(t))))$. Since $v_h(E) \in (q_Y^n)^{-1}(h(q_X^n(E))) = (q_Y^n)^{-1}(h(q_X^n(\alpha(1))))$, we conclude that $v_h(A) = v_h(E)$. Therefore, $v_h(A)$ does not depend on the choice of either J or A.

From now on, we simply write v_h instead of $v_h(A)$. In this way we have a function $\varphi : R(X) \to R(Y)$ given by $\varphi(v) = v_h$. Notice that φ satisfies the following property.

If v is a ramification point of X in the edge J of X, then $\varphi(v)$ is a ramification point of Y in the edge J_h of Y.

By (B) and (C), X and Y satisfy symmetric conditions. Thus, we define a function $\varphi^{-1} : R(Y) \to R(X)$ with a similar procedure. That is, φ^{-1} can be defined as follows. Given $w \in R(Y)$, let $K \in \mathfrak{A}_S(Y)$ be such that $w \in K$. Let $J \in \mathfrak{A}_S(X)$ be such that $K = J_h$. Let B be an arc such that $B \subset K$, $B \neq K$, and w is an end point of B. Then there exists a unique $A \in C_n(J_h)$ such that $q_X^n(A) = h^{-1}(q_Y^n(B))$ and $\varphi^{-1}(w)$ is the unique point in $A \cap R(X)$. Clearly, φ^{-1} is the inverse of φ.

Notice that φ has the following property.

A ramification point v of X belongs to the edge J of X if and only if the ramification point $\varphi(v)$ of Y belongs to the edge J_h of Y.

Let $J \in \mathfrak{A}_S(X)$ be such that $|J \cap R(X)| = 1$. By hypothesis (2) in Theorem 3.1, if J is an arc, then J_h is an arc. On the other hand, if $J \in \mathfrak{A}_S(X)$ is such that $|J \cap R(X)| = 1$ and J_h is an arc, by hypothesis (1) in Theorem 3.1, J is not a cycle. Thus, J is also an arc. Therefore, if $J \in \mathfrak{A}_S(X)$ and $|J \cap R(X)| = 1$, then J is an arc if and only if J_h is an arc. Similarly, we obtain that if $J \in \mathfrak{A}_S(X)$ satisfies $|J \cap R(X)| = 1$, then J is a cycle if and only if J_h is a cycle.

Now we can extend φ to a homeomorphism between X and Y. Take $J \in \mathfrak{A}_S(X)$. In the case that $|J \cap R(X)| = 2$, we have that J is an arc. Let u and v be the end points of J. Then $\{u, v\} = J \cap R(X)$. Since J_h is an arc with end points $\varphi(u)$ and $\varphi(v)$, we can consider a homeomorphism $\varphi_J : J \to J_h$ such that $\varphi_J(u) = \varphi(u)$ and $\varphi_J(v) = \varphi(v)$. In the case that $|J \cap R(X)| = 1$, assuming that $J \cap R(X) = \{w\}$, we know that $J_h \cap R(Y) = \{\varphi(w)\}$. If J is an arc, we know that w is an end point of J and $\varphi(w)$ is an end point of the arc J_h. Hence, we can take a homeomorphism $\varphi_J : J \to J_h$ such that $\varphi_J(w) = \varphi(w)$. Finally, if J is a cycle, since J_h is a cycle such that $\varphi(w) \in J_h$, there exists a homeomorphism $\varphi_J : J \to J_h$ such that $\varphi_J(w) = \varphi(w)$.

Hence, we can take a common extension φ of the maps φ_J (where $J \in \mathfrak{A}_S(X)$). Clearly, φ is a homeomorphism between X and Y. □

Theorem 3.2. *Let X be a finite graph and $n \in \mathbb{N}$. Then X has unique hyperspace $HS_n(X)$.*

Proof. Let Y be a continuum such that $HS_n(X)$ is homeomorphic to $HS_n(Y)$. Since X is a finite graph, by [15, Theorem 3.2], Y is a finite graph. Let $h : HS_n(X) \to HS_n(Y)$ be a homeomorphism. If $R(X) \neq \emptyset$, let $A \in C_n(X) - F_n(X)$ be such that $A \cap R(X) \neq \emptyset$ and let $B \in C_n(Y) - F_n(Y)$ be such that $h(q_X^n(A)) = q_Y^n(B)$. By (MV), $2n < \dim_A[C_n(X)] = \dim_B[C_n(Y)]$. Applying (MV) again, we obtain that $B \cap R(Y) \neq \emptyset$. Hence, $R(Y) \neq \emptyset$. From here, $R(X) \neq \emptyset$ if and only if $R(Y) \neq \emptyset$.

We consider three cases.

Case 1: $R(X) \neq \emptyset$ and $n \neq 2$.

We check that X, Y, and h satisfy the hypothesis of Theorem 3.1.

Since h is a homeomorphism, $h(\mathcal{HL}_n(X)) = \mathcal{HL}_n(Y)$. This implies that $h(\mathcal{HD}_n(X)) = \mathcal{HD}_n(Y)$. We apply Lemma 2.10(a) and (b) to obtain that for each $J \in \mathfrak{A}_S(X)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^n(\langle J^{\circ} \rangle_n \cap$

$C(X)) - \{F_X^n\}) = q_Y^n(\langle J_h^o \rangle_n \cap C(Y)) - \{F_Y^n\} \subset q_Y^n(\langle J_0^o \rangle_n) - \{F_Y^n\}$. Notice that the association $J \to J_h$ between $\mathfrak{A}_S(X)$ and $\mathfrak{A}_S(Y)$ is a bijection. By Theorem 3.1, $h(F_X^n) = F_Y^n$.

Let $J \in \mathfrak{A}_S(X)$ be such that $|J \cap R(X)| = 1$. Next, we show that if J is an arc, then J_h is an arc (and, by symmetry, the converse implication also holds). Suppose that J is an arc with end points v and p, where $v \in R(X)$. Let A be a subarc of J such that $p \in A$ and $v \notin A$. By the definition of J_h, we have that $h(q_X^n(\langle J^o \rangle_n \cap C(X)) - \{F_X^n\}) = q_Y^n(\langle J_h^o \rangle_n) \cap C(Y) - \{F_Y^n\}$. If J_h is not an arc, then it is a simple closed curve. Let $D = q_X^n(A)$ and $E = h(D) \in q_Y^n(\langle J_h^o \rangle_n \cap C(Y)) - \{F_Y^n\}$. Then there exists $B \in \langle J_h^o \rangle_n \cap C(Y) - F_1(Y)$ such that $q_Y^n(B) = E$. Notice that B is a nondegenerate subarc of J_h. By (MV), $2n = \dim_A[C_n(X)] = \dim_D[HS_n(X)] = \dim_E[HS_n(Y)] = \dim_B[C_n(Y)]$. Thus, by (MV), $B \cap R(Y) = \emptyset$. From [13, Example 5.2], it follows that B has a neighborhood $\mathcal{M}$ in $\langle J_h^o \rangle_n \cap C(Y) \subset C(J_h)$ such that $\mathcal{M}$ is a 2-cell, contains B in its interior as a manifold, and $\mathcal{M} \cap F_1(Y) = \emptyset$. Hence, $q_Y^n(\mathcal{M})$ is a 2-cell, it is a neighborhood of E in $q_Y^n(\langle J_h^o \rangle_n \cap C(Y)) - \{F_Y^n\}$, and it contains E in its interior as a manifold. Since $h(F_X^n) = F_Y^n$, $(q_X^n)^{-1}h^{-1}(q_Y^n(\mathcal{M}))$ is a neighborhood of A in $\langle J^o \rangle_n \cap C(X) - F_1(X) \subset C(J)$ that is a 2-cell and has the element A in its interior as a manifold. It is easy to show that this contradicts [13, Example 5.1] and completes the proof that J_h is an arc.

As a consequence of the fact proven in the previous paragraph, we obtain that if $J \in \mathfrak{A}_S(X)$ satisfies $|J \cap R(X)| = 1$, then J is a cycle if and only if J_h is a cycle. Therefore, by Theorem 3.1, we obtain that X is homeomorphic to Y.

Case 2: $R(X) \neq \emptyset$ and $n = 2$.

We know that $R(Y) \neq \emptyset$. We check that X, Y, and h satisfy the hypothesis of Theorem 3.1. By Lemma 2.10(d), the components of $\mathcal{HE}_2(X)$ are the sets of the form $q_X^2(\langle J^o, K^o \rangle_2) - \{F_X^2\}$, where $J, K \in \mathfrak{A}_S(X)$. Since the definition of $\partial\mathcal{HL}_2(X)$ is given in terms of topological properties, we have that $h(\partial\mathcal{HL}_2(X)) = \partial\mathcal{HL}_2(Y)$. Similarly, $h(\mathcal{HE}_2(X)) = \mathcal{HE}_2(Y)$. Since the components of $\mathcal{HE}_2(Y)$ are the sets of the form $q_Y^2(\langle J_h^o, K_h^o \rangle_2) - \{F_Y^2\}$, where $J_h, K_h \in \mathfrak{A}_S(Y)$, given $J, K \in \mathfrak{A}_S(X)$, there exist $J_h, K_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^2(\langle J^o, K^o \rangle_2) - \{F_X^2\}) = q_Y^2(\langle J_h^o, K_h^o \rangle_2) - \{F_Y^2\}$. By Lemma 2.4, $F_X^2 \notin \partial\mathcal{HL}_2(X)$ and $F_Y^2 \notin \partial\mathcal{HL}_2(Y)$. This implies that $h(\partial\mathcal{HL}_2(X) \cap q_X^2(\langle J^o, K^o \rangle_2)) = \partial\mathcal{HL}_2(Y) \cap q_Y^2(\langle J_h^o, K_h^o \rangle_2)$ and $h(\partial\mathcal{HL}_2(X) - q_X^2(\langle J^o, K^o \rangle)) = \partial\mathcal{HL}_2(Y) - q_Y^2(\langle J_h^o, K_h^o \rangle)$. Hence, $h(\mathcal{HD}(J, K)) = \mathcal{HD}(J_h, K_h)$.

Given $J \in \mathfrak{A}_S(X)$, we have $h(\mathcal{HD}(J, J)) = \mathcal{HD}(J_h, K_h)$, for some $J_h, K_h \in \mathfrak{A}_S(Y)$. By the description of the sets of the form $\mathcal{HD}(L, K)$ given after Lemma 2.15, we have that $\mathcal{HD}(J, J)$ lies in some of the cases A, B, or C. Since $\mathcal{HD}(J,J)$ is not homeomorphic to any of the sets $\mathcal{HD}(L,K)$ in cases D, E, F, G, H, and I, we obtain that $J_h = K_h$. Thus, we have $h(\mathcal{HD}(J,J)) = \mathcal{HD}(J_h, J_h)$ and $h(q_X^2(\langle J^{\circ}\rangle_2) - \{F_X^2\}) = q_Y^2(\langle J_h^{\circ}\rangle_2) - \{F_Y^2\}$. By symmetry, the association $J \to J_h$ from $\mathfrak{A}_S(X)$ onto $\mathfrak{A}_S(Y)$ is a bijection. Notice that $h(q_X^2(\langle J^{\circ}\rangle_2 \cap C(X)) - \{F_X^2\}) \subset q_Y^2(\langle J_h^{\circ}\rangle_2)$. If J is an arc and $J \notin \mathfrak{A}_E(X)$, then $\mathcal{HD}(J,J)$ lies in case A. Since $\mathcal{HD}(J,J)$ is not homeomorphic to any of the sets $\mathcal{HD}(L,L)$ in the cases B or C, we obtain that $\mathcal{HD}(J_h, J_h)$ also lies in case A. Thus, J_h is an arc and $J_h \notin \mathfrak{A}_E(Y)$.

By Theorem 3.1, $h(F_X^2) = F_Y^2$. This implies that $h(\mathcal{HF}_2(J)) = \mathcal{HF}_2(J_h)$. By Lemma 2.14, if $J \in \mathfrak{A}_E(X)$, then $J_h \in \mathfrak{A}_E(Y)$, and consequently, if $J \in \mathfrak{A}_R(X)$, then $J_h \in \mathfrak{A}_R(Y)$. Hence, X, Y, and h satisfy the hypothesis of Theorem 3.1; therefore, X is homeomorphic to Y.

Case 3: $R(X) = \emptyset$.

Since $R(X) = \emptyset$, we have that X is either an arc or a simple closed curve. By (MV), $\dim_A[C_n(X)] = 2n$ for each $A \in C_n(X)$. Thus, $\dim_A[HS_n(X)] = 2n$ for each $A \in HS_n(X) - \{F_X^n\}$. This implies that $\dim[HS_n(X)] = 2n$ [8, Theorem III 2]. Thus, $\dim[HS_n(Y)] = 2n$. Lemma 2.3 implies that $R(Y) = \emptyset$. Hence, Y also is an arc or a simple closed curve.

In the case that $n \geq 3$, by Lemma 2.10(e), $\mathcal{HD}_n(X) \cup \{F_X^n\} = q_X^n(C(X))$ and $\mathcal{HD}_n(Y) \cup \{F_Y^n\} = q_Y^n(C(Y))$. Since the definition of $\mathcal{HD}_n(X)$ is given in topological terms, $h(\mathcal{HD}_n(X)) = \mathcal{HD}_n(Y)$. Thus, $h(q_X^n(C(X))) \cup \{F_Y^n\} = h(\mathcal{HD}_n(X) \cup \{F_X^n\}) \cup \{F_Y^n\} = \mathcal{HD}_n(Y) \cup \{h(F_X^n)\} \cup \{F_Y^n\} = q_Y^n(C(Y)) \cup \{h(F_X^n)\}$.

Suppose that X is an arc with end points p and q and Y is a simple closed curve. Let A be a proper subarc of X containing either p or q. Then there exists a 2-cell $\mathcal{M}$ that is a neighborhood of A in $C(X) - F_1(X)$ and A is in the manifold boundary of $\mathcal{M}$. Thus, $q_X^n(\mathcal{M})$ is a neighborhood of $q_X^n(A)$ in $\mathcal{HD}_n(X)$ with the same characteristics. Hence, $h(q_X^n(\mathcal{M}))$ is a neighborhood of $h(q_X^n(A))$ in $\mathcal{HD}_n(Y) \cup \{F_Y^n\} = q_Y^n(C(Y))$ that is a 2-cell having $h(q_X^n(A))$ in its manifold boundary. If we take A with the additional property that $h(q_X^n(A)) \neq F_Y^n$ and $F_Y^n \notin h(q_X^n(\mathcal{M}))$ (we have uncountably many sets A to choose from), we have that $(q_Y^n)^{-1}(h(q_X^n(\mathcal{M})))$ is a neighborhood of $(q_Y^n)^{-1}(h(q_X^n(A)))$ in $C(Y)$ that is a 2-cell having $h(q_X^n(A))$ in its manifold boundary. Since $(q_Y^n)^{-1}(h(q_X^n(A))) \notin F_1(Y)$, $(q_Y^n)^{-1}(h(q_X^n(A)))$ is a nondegenerate subcontinuum of Y. Since $C(Y)$ is a 2-cell having $(q_Y^n)^{-1}(h(q_X^n(A)))$ in its manifold interior [13, Example 5.2], we obtain a contradiction. We have shown that if X is an arc, then Y is also an arc. By symmetry the converse also holds. Hence, X is homeomorphic to Y.

The cases that $n = 1$ and $n = 2$ were solved in [15, Example 2.3 and Theorem 4.8]. The case $n = 1$ follows from the fact that if X is an arc, then $HS_1(X)$ is a 2-cell and if X is a simple closed curve, then $HS_1(X)$ is homeomorphic to the 2-sphere in the Euclidean space $\mathbb{R}^3$. $\square$

Theorem 3.3. *For each $n \geq 3$, there is no finite-dimensional absolute n-fold hyperspace suspensions.*

Proof. Let $n \geq 3$. Suppose that there exists a continuum Z that is a finite-dimensional absolute n-fold hyperspace suspension. By [15, Theorem 4.2] Z is homeomorphic to $HS_n(Y)$, where Y is either an arc or a simple closed curve. Take two different elements A and B such that A and B have exactly n non-degenerate components of Y. By Lemma 2.5, there exist two $2n$-cells $\mathcal{M}$ and $\mathcal{N}$ contained in $C_n(Y) - F_n(Y)$ such that $A \in \text{int}_{C_n(Y)}(\mathcal{M})$ and $B \in \text{int}_{C_n(Y)}(\mathcal{N})$. Let $h : HS_n(Y) \to Z$ be a homeomorphism. Since Z is an absolute n-fold hyperspace suspension, there exist a continuum W and a homeomorphism $g : Z \to HS_n(W)$ such that $g(h(q_n^Y(A))) = q_W^n(W)$ and $g(h(q_n^Y(B))) = F_W^n$. By Theorem 3.2, W is homeomorphic to Y. Thus, W is either an arc or a simple closed curve. Let $\mathcal{R} = g(h(q_Y^n(\mathcal{N})))$ be a $2n$-cell in $HS_n(W)$ containing F_W^n in its interior. Let $C \in C_{n-1}(W)$ be such that $q_W^n(C) \in \text{int}_{HS_n(W)}(\mathcal{R})$. Thus, C has a neighborhood in $C_n(W)$ that is a $2n$-cell. This contradicts Lemma 2.6 and proves that there is no such Z. $\square$

The following result is a consequence of [15, Theorem 4.2, Corollary 4.4, and Theorem 4.9] and Theorem 13.

Corollary 3.4. *The 2-sphere is the only finite-dimensional absolute 1-fold hyperspace suspension and for each $n \geq 2$ there is no finite-dimensional absolute n-fold hyperspace suspensions.*

The referee has suggested that we consider the question whether the uniqueness of $C_n(X)$ implies the uniqueness of $HS_n(X)$ and/or conversely. Recall that $C([0,1])$ and $C(S^1)$ are 2-cells, while $HS_1([0,1])$ is a 2-cell and $HS_1(S^1)$ is a 2-sphere. In fact, by Theorem 3.2, $[0,1]$ has unique hyperspace $HS_1([0,1])$ and it does not have unique hyperspace $C([0,1])$. It would be interesting to know if there are more examples for which the mentioned implications do not hold. As it can be seen in [12], there are many families of continua X for which the uniqueness of $C_n(X)$ has been determined. For all these families it would be also interesting to see if they also have unique hyperspace $HS_n(X)$.

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References

[1] Gerardo Acosta, David Herrera-Carrasco, Fernando Macías-Romero, *Local dendrites with unique hyperspace $C(X)$*, Topology Appl. **157** (2010), no. 13, 2069–2085.

[2] Rodrigo Hernández-Gutiérrez, Alejandro Illanes, and Verónica Martínez-de-la-Vega, *Uniqueness of hyperspaces for Peano continua.* To appear in Rocky Mountain Journal of Mathematics.

[3] ______, *Rigidity of hyperspaces.* To appear in Topology and its Applications. Available at http://www.matem.unam.mx/∼rod/research/arrigifinal.pdf

[4] David Herrera-Carrasco, *Dendrites with unique hyperspace*, Houston J. Math. **33** (2007), no. 3, 795–805.

[5] David Herrera-Carrasco, Alejandro Illanes, María de J. López, and Fernando Macías-Romero, *Dendrites with unique hyperspace $C_2(X)$*, Topology Appl. **156** (2009), no. 3, 549–557.

[6] David Herrera-Carrasco and Fernando Macías-Romero, *Dendrites with unique n-fold hyperspace*, Topology Proc. **32** (2008), Spring, 321–337.

[7] ______, *Local dendrites with unique n-fold hyperspace*, Topology Appl. **158** (2011), no. 2, 244–251.

[8] Witold Hurewicz and Henry Wallman, *Dimension Theory* (revised). Princeton Mathematical Series, v. 4. Princeton, N. J.: Princeton University Press, 1969.

[9] Alejandro Illanes, *The hyperspace $C_2(X)$ for a finite graph X is unique*, Glas. Mat. Ser. III **37(57)** (2002), no. 2, 347–363.

[10] ______, *Finite graphs X have unique hyperspaces $C_n(X)$*, Topology Proc. **27** (2003), no. 1, 179–188.

[11] ______, *Dendrites with unique hyperspace $C_2(X)$, II*, Topology Proc. **34** (2009), 77–96.

[12] ______, *Uniqueness of hyperspaces*, Questions Answers Gen. Topology **30** (2012), no. 1, 21–44.

[13] Alejandro Illanes and Sam B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent Advances*. Monographs and Textbooks in Pure and Applied Mathematics, 216. New York: Marcel Dekker, Inc., 1999.

[14] Sergio Macías, *On the n-fold hyperspace suspension of continua*, Topology Appl. **138** (2004), no. 1-3, 125–138.

[15] Sergio Macías and Sam B. Nadler, Jr., *Absolute n-fold hyperspace suspensions*, Colloq. Math. **105** (2006), no. 2, 221–231.

[16] Verónica Martínez-de-la-Vega, *Dimension of n-fold hyperspaces of graphs*, Houston J. Math. **32** (2006), no. 3, 783–799.

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