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NON-IRREDUCIBILITY IS NOT A WHITNEY REVERSIBLE PROPERTY

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Abstract. In this paper we give an example of a hereditarily decomposable irreducible continua X such that none of its positive Whitney levels are irreducible. This shows that not being irreducible is not a Whitney-reversible property for the class of hereditarily decomposable continua.

1. Introduction

The notion of Whitney-reversible property was introduced in [4] by Sam B. Nadler, Jr. Since then many properties have been shown to be Whitney-reversible. However, there are still some properties for which it is not known if they are Whitney-reversible or not. An excellent survey on this topic is [2, Chapter 8].

In [2, Question 49.9], Alejandro Illanes and Nadler ask if the property of not being irreducible is a Whitney-reversible property. In [1, Example 3.1], Carl Eberhart and Nadler give a decomposable irreducible continuum X and an indecomposable continuum Y for which none of their positive Whitney levels are irreducible. However, both of these examples contain an indecomposable continuum. So, it is natural to ask if the property of not being irreducible is a Whitney-reversible property for the class of hereditarily decomposable continua. In this paper we construct a hereditarily decomposable irreducible continuum X for which none of its positive Whitney levels are irreducible. We modify this example to obtain

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a continuum with the same relevant properties, but which has only two points of irreducibility.

We note that our continuum is rational. In the classical hierarchy of structures of continua, arcs are the only irreducible locally connected continua, regular continua are locally connected, and regular continua are rational; see [5]. Since being an arc is a Whitney property, see [2, Theorem 31.1], an arc does not have the properties of our example. Thus, from the point of view of this hierarchy, our example is the simplest possible example.

2. Preliminaries

A *continuum* is a compact connected metric space. Given a continuum X,

$$C(X) = \{A \subset X : A \neq \emptyset \text{ and } A \text{ is a continuum}\}$$

is called the *hyperspace of subcontinua of* X; $C(X)$ is equipped with the Hausdorff metric. Given $A, B \in C(X)$, $H(A, B)$ will denote the Hausdorff distance between A and B.

A continuous map $\mu : C(X) \to [0, 1]$ is called a *Whitney map for* $C(X)$ if (1) $\mu(\{x\}) = 0$ for all $x \in X$, and (2) if $A \subsetneq B$, then $\mu(A) < \mu(B)$. A *Whitney level* for $C(X)$ is any subset in $C(X)$ that is the inverse image of a point in $[0, 1]$ under any Whitney map. Note that a non-degenerate Whitney level is any subset in $C(X)$ that is the inverse image of a point in $[0, 1)$ under any Whitney map. A *positive* Whitney level is any subset in $C(X)$ that is the inverse image of a point in $(0, 1]$ under any Whitney map; see [2, Definition 24.17].

A topological property $\mathcal{P}$ is called

(a) a *Whitney property* if whenever a continuum X has property $\mathcal{P}$, so does every positive non-degenerate Whitney level and
(b) a *Whitney-reversible property* if whenever X is a continuum such that all positive non-degenerate Whitney levels have property $\mathcal{P}$, then X has property $\mathcal{P}$.

Let X be a continuum and let $A \subset X$. The continuum X is said to be *irreducible about* A if no proper subcontinuum of X contains A. If a continuum is irreducible about two points p and q, then we say that X is *irreducible*.

A continuum X is called *decomposable* if $X = A \cup B$, where A and B are proper subcontinua of X. If every subcontinuum of X is decomposable, then X is called *hereditarily decomposable*. If a continuum is not decomposable, then it is called *indecomposable*.

A continuum X is called *regular* if every point in X has a local base whose members have finite boundary; X is called *rational* if every point in X has a local base whose members have boundary at most countable.

3. The Examples

To aid in understanding the examples, we first give intuitive arguments showing that there is a hereditarily decomposable irreducible continuum X such that none of its positive Whitney levels are irreducible. We then give the formal argument.

To construct X, place a copy of the $\sin\left(\frac{1}{x}\right)$ continuum on each square of the form $\left[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}\right] \times [-1,1]$, where n is a positive integer. Do this such that the right-hand endpoint of $\sin\left(\frac{1}{x}\right)$ coincides with the point $(\frac{2^n-1}{2^n}, 0)$; label these points p_n. The continuum X is the union of these continua and the line segment $\{1\} \times [-1,1]$; see Figure 1.

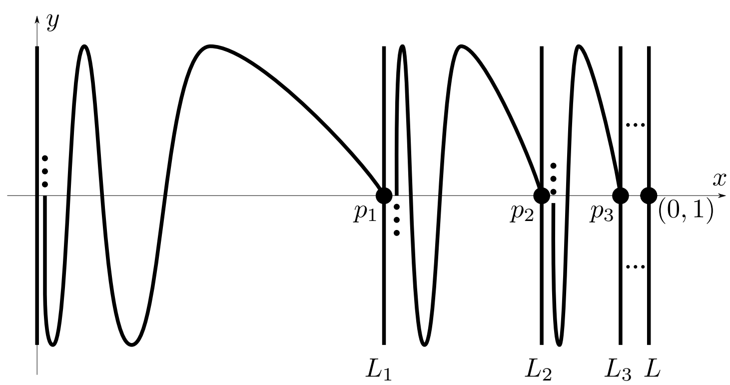

Figure 1. The continuum X

Note that X is irreducible between any point of $\{0\} \times [-1,1]$ and any point of $\{1\} \times [-1,1]$.

To see that no positive Whitney level is irreducible, it is enough to show that each positive Whitney level contains a 2-cell with nonempty interior. To see this, let μ be a Whitney map for $C(X)$ and let $0 < t < \mu(X)$. For each n, using order arcs, we can construct a proper subcontinuum of X containing the set $\{0\} \times [-1,1]$ and having p_n as endpoint. (Start the order arc at $\{p_n\}$.) Then, since $p_n \to (1,0)$, there is k and a proper subcontinuum A of X such that $\mu(A) > t$, $\{0\} \times [-1,1] \subset A$, and p_k is an endpoint of A. Now, using an order arc in $C(A)$ starting at $\{p_k\}$, we can find a subcontinuum B of A such that $\mu(B) = t$, and $B \cap (\{0\} \times [-1, 1]) = \emptyset$. By construction, p_k lies on the vertical line segment $L = \{\frac{2^k-1}{2^k}\} \times [-1, 1]$ and $B \cap L = \{p_k\}$; hence, $B \cup L$ is a triod with $\mu(B \cup L) > t$. Therefore, we can construct a 2-cell in $\mu^{-1}(t)$ by "moving" B in $B \cup L$ in three different directions from p_k (up, down, and left). To see that the 2-cell has nonempty interior, just observe that any continuum C, with $\mu(C) = t$, containing p_k and a point in $\left[\frac{2^k-1}{2^k}, \frac{2^{k+1}-1}{2^{k+1}}\right] \times [-1, 1]$ must contain L.

A more detailed construction of X is given below and it will be used to show the properties of X.

For every positive integer n, consider the set in $\mathbb{R}^2$ defined by

$$R_n = \left\{ \left(x, \sin\left(\frac{\pi}{2^{n-1}x - 2^{n-1} + 1} \right) \right) \in \mathbb{R}^2 : \frac{2^{n-1} - 1}{2^{n-1}} < x \leq \frac{2^n - 1}{2^n} \right\}.$$

Let X be the continuum given by

$$X = \mathrm{cl}\left(\bigcup_{n=1}^{\infty} R_n \right),$$

where $\mathrm{cl}(A)$ denotes the closure of A in $\mathbb{R}^2$; see Figure 1.

By construction, X is hereditarily decomposable and irreducible (between any point of $\{0\} \times [-1, 1]$ and any point of $\{1\} \times [-1, 1]$).

Proposition 3.1. *No positive Whitney level of X is irreducible.*

Since no irreducible continuum can contain a 2-cell with nonempty interior, the proof of Proposition 3.1 follows directly from Proposition 3.2. First, we introduce the following notation.

For $k \in \mathbb{N}$, define $X_k = \mathrm{cl}\left(\bigcup_{n=1}^{k} R_n \right)$, $L_k = \left\{ \frac{2^k-1}{2^k} \right\} \times [-1, 1]$, and $p_k = \left(\frac{2^k-1}{2^k}, 0 \right)$.

Note that X_k is irreducible between any point of $\{0\} \times [-1, 1]$ and p_k. Also note that $\{X_k\}_{k=1}^{\infty}$ is a strictly increasing sequence (with respect to inclusion) in $C(X)$ such that $X_k \to X$. Hence, for any Whitney map $\mu : C(X) \to [0, 1]$, $\mu(X_k) \to 1$.

Now, we prove the following proposition.

Proposition 3.2. *Every positive non-degenerate Whitney level of X contains a 2-cell with nonempty interior.*

Proof. Take $0 < t < 1$. By construction, there is a positive integer N such that $t < \mu(X_N)$. Using an order arc from $\{p_N\}$ to X_N, we can find $A \in C(X_N)$ such that $p_N \in A$ and $\mu(A) = t$.

Let $0 < a < 1$ be such that $\mu\left(\left\{\frac{2^N-1}{2^N}\right\} \times [-a,a]\right) < t$. Then, for every $u, v \in [0,a]$, consider

$$Z_{u,v} = X_N \cup \left(\left\{\frac{2^N-1}{2^N}\right\} \times [-u,v]\right).$$

Using an order arc from $\left\{\frac{2^N-1}{2^N}\right\} \times [-u,v]$ to $Z_{u,v}$, there is an $A_{u,v} \in C(Z_{u,v})$ such that $\mu(A_{u,v}) = t$. Note that, given $u, v \in [0,a]$, such continuum $A_{u,v}$ is unique. Then the set

$$\Delta = \{A_{u,v} : u, v \in [0,a]\}$$

is a 2-cell in $\mu^{-1}(t)$.

Now, let $u_0, v_0 \in [0,a)$ and let $\varepsilon = \min\{a - u_0, a - v_0\}$. Then the set

$$\Lambda = \{B \in \mu^{-1}(t) \,:\, H(A_{u_0,v_0}, B) < \varepsilon\}$$

is contained in Δ. To see this, note that if $B \in \mu^{-1}(t)$ and it intersects R_{N+1}, then either $L_N \subset B$ or $B \subset R_{N+1}$; in any case, $H(A_{u_0,v_0}, B) > \varepsilon$. So, if $B \in \mu^{-1}(t)$ and $H(A_{u_0,v_0}, B) < \varepsilon$, then $B = A_{u_1,v_1}$ for some $u_1, v_1 \in [0,a]$. Therefore, $\Lambda \subset \Delta$. Note that, by definition, Λ is open in $\mu^{-1}(t)$. Hence, Δ has nonempty interior in $\mu^{-1}(t)$. □

In the continuum X, let $L = \{1\} \times [-1,1]$. Irreducibility about only two points can be obtained by using the decomposition space obtained by shrinking L and L_0 to two points.

As noted in the introduction, there cannot be a locally connected example showing that not being irreducible is not a Whitney reversible property. Also, since being chainable is a Whitney property, see [2, Theorem 37.4], and every chainable continuum is irreducible, there cannot be a chainable example. Observe that our examples contain triods. So, there is a natural question to ask.

Question 3.3. Is there a hereditarily decomposable atriodic irreducible continuum such that none of its positive Whitney levels are irreducible? In other words, is not being irreducible a Whitney reversible property for the class of atriodic hereditarily decomposable continua?

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