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CONTINUA X FOR WHICH $C(X)$ HAS A PLANE NEIGHBORHOOD AT THE TOP

SERGIO LÓPEZ AND NORBERTO ORDOÑEZ

Abstract. Let X be a metric continuum and $C(X)$ be the hyperspace of subcontinua of X. In this paper we prove that if X has a planar neighborhood in $C(X)$, then X has a neighborhood in $C(X)$ which is a 2-cell. This answers a question by Sergio López.

1. Introduction

A continuum is a compact connected metric and nondegenerate space. Given a continuum X, we consider the hyperspace of subcontinua $C(X)$ of X defined by the collection of all nonempty connected and closed subsets of X, endowed with the Hausdorff metric H.

The element X has a special position in $C(X)$ and it is natural to ask what local properties X has in $C(X)$. For example, it is known that $C(X)$ is always locally connected at X ([5, Corollary 15.5]); however, it is easy to construct continua X for which $C(X)$ is locally connected only at X. Answering a question stated by Anne Marie Dilks in [2, Question 111], Hsao Kato, in [7, Example 3.3], and Alejandro Illanes, in [4], gave examples of continua X such that $C(X)$ is not locally contractible at X. In [10], Luis Montejano-Peimbert and Isabel Puga-Espinosa gave conditions under which a smooth dendroid X has a neighborhood in $C(X)$ which is homeomorphic to a topological cone of some continuum. Sergio López, in [8], obtained characterizations of the continua X such that there exist closed neighborhoods around X in $C(X)$ which are 2-cells. In [8, Corollary 2], he showed that $C(X)$ has closed neighborhoods around X in $C(X)$

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which are 2-cells if and only if there exists a Whitney level for $C(X)$ that is either an arc or a simple closed curve.

In this paper we show that if there exists a neighborhood $\mathcal{D}$ of X in $C(X)$ such that $\mathcal{D}$ is embeddable in $\mathbb{R}^2$, then X has a neighborhood in $C(X)$ which is a 2-cell, giving an affirmative answer to [8, Question 10].

2. Preliminary Results

In this section we introduce some definitions and results which we will use in this paper.

If X is a continuum, A and B are subcontinua of X, and $\varepsilon > 0$, then

- $N(\varepsilon, A) = \bigcup\{B_\varepsilon(a) : a \in A\}$;
- $H(A, B) < \varepsilon$ if and only if $A \subset N(\varepsilon, A)$ and $B \subset N(\varepsilon, B)$ (see [5, Theorem 2.2 and Exercise 2.9]);
- $B^H(\varepsilon, A) = \{B \in C(X) : H(A, B) < \varepsilon\}$ is the ball induced by the Hausdorff metric in $C(X)$ of radius ε and center at A;
- if C is a subset of X, then $int(C)$ and $\overline{C}$ denote the interior and the closure of C in X, respectively.

Definition 2.1. Let X be a continuum, a *Whitney map* for $C(X)$ is a continuous map $\mu : C(X) \mapsto [0, 1]$ such that

(1) $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for all $x \in X$,
(2) if $A \subsetneq B$, then $\mu(A) < \mu(B)$, and
(3) $\mu(X) = 1$.

It is well known that if X is a continuum, then there exists a Whitney map for $C(X)$ (see [5, Theorem 13.4]).

Definition 2.2. Let X be a continuum and let μ be a Whitney map for $C(X)$. A *Whitney level* for $C(X)$ is a subset of the form $\mu^{-1}(t)$, where $t \in [0, 1]$.

Remark 2.3. If μ is a Whitney map for $C(X)$, by [5, Theorem 19.9], $\mu^{-1}(t)$ is a subcontinuum of $C(X)$ for each $t \in [0, 1]$; this implies that μ is monotone. Thus, by [13, Chapter VIII, Theorem (2.2)], we obtain that $\mu^{-1}([t, s])$ is a subcontinuum of $C(X)$ for every $t, s \in [0, 1]$ and $t < s$.

Definition 2.4. Let X be a continuum and let A and B be subcontinua of X such that $A \subsetneq B$. An *order arc* from A to B is an embedding $\alpha : [0, 1] \mapsto C(X)$ such that

(1) $\alpha(0) = A$ and $\alpha(1) = B$ and
(2) $\alpha(s) \subsetneq \alpha(t)$, if $0 \leq s < t \leq 1$.

3. **Local Connectedness of Whitney Levels**

The purpose of this section is to show that if X has a planar neighborhood in $C(X)$, then Whitney levels $\mu^{-1}(t)$, for t close to 1, are locally connected.

Let X be a continuum and let μ be a Whitney map for $C(X)$. If $t \in [0,1)$ and $\mathcal{F} \subset C(X)$, we define

$$T[\mathcal{F},t] = \{B \in \mu^{-1}([t,1]) : F \subset B \text{ for some } F \in \mathcal{F}\}.$$

Lemma 3.1. *Let X be a continuum and let μ be a Whitney map for $C(X)$. If $t \in [0,1)$ and $\mathcal{F}$ is a nonempty closed subset of $C(X)$, then $T[\mathcal{F},t]$ is a subcontinuum of $C(X)$ contained in $\mu^{-1}([t,1])$.*

Proof. First we show that if $M \in C(X)$, then $T[\{M\},t]$ is a subcontinuum of $C(X)$ contained in $\mu^{-1}([t,1])$.

Since $X \in T[\{M\},t]$, we obtain that $T[\{M\},t]$ is nonempty.

If $N \in T[\{M\},t] - \{X\}$, then $M \subset N$. By [5, Theorem 14.6], there exists an order arc $\alpha : [0,1] \mapsto C(X)$ from N to X. So, for each $s \in [0,1]$, $M \subset \alpha(s)$, which implies that $\mu(\alpha(s)) \geq \mu(M) \geq t$. Therefore, $\alpha([0,1]) \subset T[\{M\},t]$, and hence $T[\{M\},t]$ is connected. Clearly, $T[\{M\},t]$ is closed in $C(X)$. This shows that $T[\{M\},t]$ is a subcontinuum of $C(X)$ contained in $\mu^{-1}([t,1])$.

Notice that $T[\mathcal{F},t] = \bigcup_{M\in\mathcal{F}} T[\{M\},t]$. Since $T[\{M\},t]$ is a subcontinuum of $C(X)$ contained in $\mu^{-1}([t,1])$ and $X \in T[\{M\},t]$ for each $M \in \mathcal{F}$, we have that $T[\mathcal{F},t]$ is connected. Since $\mathcal{F}$ is closed, it follows that $T[\mathcal{F},t]$ is closed. This ends the proof of this lemma. □

Definition 3.2. A continuum X is said to have *property (b)* if every mapping of X into S^1 is homotopic to a constant mapping.

In [1, Lemma 13], José G. Anaya showed the following.

Lemma 3.3. *Let X be a continuum, let $A \in C(X)$, and let $\mathcal{R}_A = \{B \in C(X) : A \subset B\}$. If $\mathcal{R}$ is a nonempty subset of $C(X)$ and $\mathcal{R}_A \subset \mathcal{R}$ for each $A \in \mathcal{R}$, then $\mathcal{R}$ has property (b).*

Definition 3.4. A continuum X is *aposyndetic* provided that, for every two different elements $a, b \in X$, there exists a subcontinuum $A \in C(X)$ such that $a \in int(A)$ and $b \notin A$.

Theorem 3.5. *Let X be a continuum and let μ be a Whitney map for $C(X)$. If $t < 1$, then*

(1) $\mu^{-1}([t,1])$ *has property (b) and*
(2) $\mu^{-1}([t,1])$ *is aposyndetic.*

Proof. (1) follows from Lemma 3.3.

Now we prove (2). Let $A, B \in \mu^{-1}([t,1])$ be two different elements. We consider three cases.

Case 1: $\mu(A) < \mu(B)$.

By Remark 2.3, $\mathcal{D} = \mu^{-1}([t, \frac{\mu(A)+\mu(B)}{2}])$ is a subcontinuum of $C(X)$ contained in $\mu^{-1}([t,1])$ which contains A in its interior and $B \notin \mathcal{D}$.

Case 2: $\mu(A) = \mu(B)$.

We can choose $a \in A - B$. Let U be an open set of X such that $a \in U$ and $\overline{U} \cap B = \emptyset$. Let $\mathcal{F} = \{E \in C(X) : E \cap \overline{U} \neq \emptyset\}$, by Lemma 3.1, $T[\mathcal{F}, t]$ is a subcontinuum of $C(X)$ and $B \notin T[\mathcal{F}, t]$. Since U is open in X, we have that $\mathcal{W} = \{E \in C(X) : E \cap U \neq \emptyset\}$ is an open subset of $C(X)$ and $A \in \mathcal{W}$. Thus, $\mathcal{W} \cap \mu^{-1}([t,1])$ is an open subset of $\mu^{-1}([t,1])$ and $\mathcal{W} \cap \mu^{-1}([t,1]) \subset T[\mathcal{F}, t]$. This shows that $T[\mathcal{F}, t]$ contains A in its interior.

Case 3: $\mu(A) > \mu(B)$.

By Remark 2.3, $\mathcal{G} = \mu^{-1}([\frac{\mu(A)+\mu(B)}{2}, 1])$ is a subcontinuum of $C(X)$ contained in $\mu^{-1}([t,1])$ which contains A in its interior and $B \notin \mathcal{G}$.

This ends the proof of the theorem. □

Corollary 3.6. *Let X be a continuum and let μ be a Whitney map for $C(X)$. Suppose that $\mu^{-1}([t,1])$ is embeddable in $\mathbb{R}^2$ for some $t < 1$, then $\mu^{-1}([t,1])$ is locally connected.*

Proof. By (1) of Theorem 3.5 and [3, Theorem VI 13], $\mu^{-1}([t,1])$ does not separate $\mathbb{R}^2$. In [6, Theorem 1], F. Burton Jones showed that an aposyndetic continuum in the plane which does not separate $\mathbb{R}^2$ is locally connected. Therefore, applying (2) of Theorem 3.5, we conclude that $\mu^{-1}([t,1])$ is locally connected. □

Theorem 3.7. *Let X be a continuum and let μ be a Whitney map for $C(X)$. If $\mu^{-1}([t,1])$ is locally connected for some $t < 1$, then $\mu^{-1}(t)$ is locally connected.*

Proof. We only need to prove that $\mu^{-1}(t)$ is connected im kleinen at each one of its elements. Let $A \in \mu^{-1}(t)$ and let $\varepsilon > 0$. By [5, Exercise 66.8], there exists $\delta > 0$ such that if $B \in \mu^{-1}(t)$ and $B \subset N(\delta, A)$, then $H(A,B) < \varepsilon$.

Let $\mathcal{C}$ be the component of $\overline{B^H(\frac{\delta}{2}, A)} \cap \mu^{-1}([t,1])$ such that $A \in \mathcal{C}$. Since $\mu^{-1}([t,1])$ is connected im kleinen at A, A lies in the interior of $\mathcal{C}$ in $\mu^{-1}([t,1])$. On the other hand, by [12, Lemma 1.43], $E = \bigcup\{D : D \in \mathcal{C}\}$ is a subcontinuum of X, and by [12, Theorem 14.11.1], $\mathcal{B} = \mu^{-1}(t) \cap C(E)$ is a subcontinuum of $\mu^{-1}(t)$.

Now we show that $\mathcal{B}$ is contained in $B^H(\varepsilon, A)$. Since $\mathcal{C}$ is a component of $\overline{B^H(\frac{\delta}{2}, A)}$, we have that $E \subset N(\delta, A)$. So, if $M \in \mathcal{B}$, then $M \in \mu^{-1}(t)$

and $M \subset N(\delta, A)$. By the choice of δ, we obtain that $H(M, A) < \varepsilon$. Therefore, $\mathcal{B}$ is contained in $B^H(\varepsilon, A)$. Since $\mathcal{C} \subset C(E)$, we have that $\mathcal{C} \cap \mu^{-1}(t) \subset \mathcal{B}$. Since A is in the interior of $\mathcal{C}$ in $\mu^{-1}([t, 1])$, then A is in the interior of $\mathcal{B}$ in $\mu^{-1}(t)$. Since $\mathcal{B}$ is a subcontinuum of $\mu^{-1}(t)$ contained in $B^H(\varepsilon, A)$, we conclude that $\mu^{-1}(t)$ is connected im kleinen at A. Thus, $\mu^{-1}(t)$ is locally connected. $\square$

4. Simple Triods in Whitney Levels

Definition 4.1. A *simple triod* is a continuum which is the union of three arcs L_1, L_2, and L_3 such that $L_i \cap L_j = \{p\}$ for every $i \neq j$ and p is an end point of each one of the arcs.

Lemma 4.2. *Let X be a continuum and let $x_0, x_1, x_2, x_3 \in X$ be four different points. For each $i \in \{1, 2, 3\}$, suppose that there exist embeddings $\gamma_i : [0, 1] \mapsto X$ with $\gamma_i(0) = x_0$ and $\gamma_i(1) = x_i$ for each $i \in \{1, 2, 3\}$ and $x_i \notin \gamma_j([0, 1])$ if $i \neq j$. Then X contains a simple triod.*

Proof. Let $t_2 = \max\{t \in [0, 1] : \gamma_2(t) \in \gamma_1([0, 1])\}$. Since $x_2 = \gamma(1) \notin \gamma_1([0, 1])$, we have that $t_2 < 1$. We consider two cases.

Case 1: $t_2 > 0$.

Let $t_1 \in [0, 1]$ be such that $\gamma_1(t_1) = \gamma_2(t_2)$. Since γ_2 is an embedding and $x_1 \notin \gamma_2([0, 1])$, we have that $t_1 \in (0, 1)$. Therefore, $\gamma_1([0, t_1]) \cup \gamma_1([t_1, 1]) \cup \gamma_2([t_2, 1])$ is a simple triod.

Case 2: $t_2 = 0$.

We have that $\gamma_1([0, 1]) \cup \gamma_2([0, 1])$ is an arc. Let $t_3 = \max\{t \in [0, 1] : \gamma_3(t) \in \gamma_1([0, 1]) \cup \gamma_2([0, 1])\}$. Since $x_3 = \gamma_3(1) \notin \gamma_1([0, 1]) \cup \gamma_2([0, 1])$, we have that $t_3 < 1$. Suppose, without loss of generality, that $\gamma_3(t_3) \in \gamma_1([0, 1])$. Let $t_1 \in [0, 1]$ be such that $\gamma_1(t_1) = \gamma_3(t_3)$. Since $x_1 \notin \gamma_3([0, 1])$, we have that $t_1 < 1$. Thus, $\gamma_1([t_1, 1]) \cup [\gamma_1([0, t_1]) \cup \gamma_2([0, 1])] \cup \gamma_3([t_3, 1])$ is a simple triod.

This ends the proof of the lemma. $\square$

Construction 4.3. *Let X be a continuum, let μ be a Whitney map for $C(X)$, and let $t_0 \in [0, 1]$. Suppose that $A_0, A_1 \in \mu^{-1}(t_0)$ and there exists an arc, denoted by $[A_0, A_1]$, joining A_0 and A_1 in $\mu^{-1}(t_0)$. We suppose that $[A_0, A_1]$ is endowed by the natural order $<$, where $A_0 < A_1$.*

Given $B, C \in [A_0, A_1]$, where $B \leq C$, $[B, C]$ denotes the subinterval of $[A_0, A_1]$ with end points B and C. As usual, $\bigcup[B, C]$ denotes the union of the elements of $[B, C]$. By [5, Exercise 11.5]*, the function $\varphi : \{[B, C] \in C([A_0, A_1]) : B, C \in [A_0, A_1]$ and $B \leq C\} \mapsto C(X)$ given by $\varphi([B, C]) = \bigcup[B, C]$ is well defined and continuous.*

Given $D \in [A_0, A_1] - \{A_1\}$ and $s \in [t_0, \mu(\bigcup[A_0, D])]$, for each $B \in [D, A_1]$, we have $\mu(\bigcup[A_0, B]) \geq \mu(\bigcup[A_0, D]) \geq s \geq t_0 = \mu(\bigcup[B, B])$.

Thus, there exists $C_B \in [A_0, B]$ *such that* $\mu([C_B, B]) = s$. *Define* $\beta : [D, A_1] \mapsto \mu^{-1}(s)$ *by* $\beta(B) = \bigcup[C_B, B]$.

Lemma 4.4. *The function* β *defined in Construction 4.3 is well defined and continuous.*

Proof. For each $B \in [D, A_1]$, we have that $\beta(B) = \bigcup[C_B, B] \in \mu^{-1}(s)$. Now suppose that there exists $C \in [A_0, B]$ such that $\mu(\bigcup[C, B]) = s$. We can suppose, without loss of generality, that $C \leq C_B$. So, $[C, B] \subset [C_B, B]$, which implies that $\bigcup[C_B, B] \subset \bigcup[C, B]$, but $\bigcup[C_B, B], \bigcup[C, B] \in \mu^{-1}(s)$, then $\beta(B) = \bigcup[C_B, B] = \bigcup[C, B]$. Therefore, β is well defined.

To show the continuity of β, let $\{B_n\}_{n=1}^{\infty}$ be a sequence in $[D, A_1]$ converging to an element $B \in [D, A_1]$. For each $n \in \mathbb{N}$, let $C_{B_n} \in [A_0, B_n]$ be an element such that $\beta(B_n) = \bigcup[C_{B_n}, B_n]$ and we suppose that $\lim C_{B_n} = C$ for some $C \in [A_0, A_1]$. By the continuity of the union, we have that $\lim \bigcup[C_{B_n}, B_n] = \bigcup[C, B]$; thus, by the continuity of μ, we have that $\bigcup[C, B] \in \mu^{-1}(s)$. By the paragraph above, $\bigcup[C_B, B] = \bigcup[C, B]$. This shows that β is continuous. □

Lemma 4.5. *Let* X *be a continuum and let* μ *be a Whitney map for* $C(X)$. *Suppose that there exists* $t_0 \in [0, 1)$ *such that* $\mu^{-1}(t_0)$ *contains a simple triod, then there exists* $t_1 \in (t_0, 1)$ *such that* $\mu^{-1}(t)$ *contains a simple triod for all* $t \in [t_0, t_1]$.

Proof. Suppose that $T = [A_0, A_1] \cup [A_0, A_2] \cup [A_0, A_3]$ is a simple triod in $\mu^{-1}(t_0)$, where $[A_0, A_i]$ is an arc joining A_0 to A_i and $[A_0, A_i] \cap [A_0, A_j] = \{A_0\}$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Let $\varepsilon > 0$ be such that $2\varepsilon < \min\{H(A_i, A) : A \in [A_0, A_j],\ i, j \in \{1, 2, 3\} \text{ and } i \neq j\}$. For every $i \in \{1, 2, 3\}$, notice that $A_0 \subsetneq \bigcup[A_0, A_i]$, which implies that $t_0 > \mu(\bigcup[A_0, A_i]) \leq 1$. Let $t_1 = \min\{\mu(\bigcup[A_0, A_i]) : i \in \{1, 2, 3\}\}$. By [5, Lemma 17.3], we can choose $\delta > 0$ with the following properties:

(1) $t_0 + \delta < t_1$ and
(2) if $A, B \in C(X)$ are such that $A \subset B$ and $\mu(B) - \mu(A) < \delta$, then $H(A, B) < \varepsilon$.

Let $t \in [t_0, t_0 + \delta)$; we are going to construct a simple triod in $\mu^{-1}(t)$.

Since $\mu([A_0, A_i]) > t > \mu(\bigcup[A_i, A_i])$, for each $i \in \{0, 1, 2, 3\}$, we can fix elements $L_i \in [A_0, A_i]$ and $D_1 \in [A_0, A_1]$ such that $\bigcup[L_i, A_i], \bigcup[A_0, D_1] \in \mu^{-1}(t)$.

Claim 1. The elements $\bigcup[A_0, D_1]$, $\bigcup[L_1, A_1]$, $\bigcup[L_2, A_2]$, and $\bigcup[L_3, A_3]$ are all different.

We divide the proof of Claim 1 into two steps.

Step 1. $\bigcup[A_0, D_1] \neq \bigcup[L_i, A_i]$ for each $i \in \{1, 2, 3\}$.

Let $i \in \{1, 2, 3\}$ and suppose that $\bigcup[A_0, D_1] = \bigcup[L_i, A_i]$. Then $A_0, A_i \subset \bigcup[A_0, D_1]$ and $\mu(\bigcup[A_0, D_1]) - \mu(A_0) = \mu(\bigcup[A_0, D_1]) - \mu(A_i) <$

$(t_0+\delta)-t_0=\delta$. By the choice of δ, we conclude that $H(\bigcup[A_0,D_1],A_0)<\varepsilon$ and $H(\bigcup[A_0,D_1],A_i)<\varepsilon$. Thus, $H(A_0,A_i)<2\varepsilon$, a contradiction with the choice of ε.

Step 2. $\bigcup[L_i,A_i]\neq\bigcup[L_j,A_j]$ for all $i\neq j$ where $i,j\in\{1,2,3\}$.

Let $i,j\in\{1,2,3\}$ with $i\neq j$ and suppose that $\bigcup[L_i,A_i]=\bigcup[L_i,A_j]$, then $A_i,A_j\subset\bigcup[L_i,A_i]$ and we obtain, as in Step 1, that $H(A_i,A_j)<2\varepsilon$, a contradiction with the choice of ε. This ends the proof of Claim 1.

We use Construction 4.3 for the arc $[A_0,A_1]$, $s=t$ and D_1. Then there exists a mapping $\beta_1:[D_1,A_1]\mapsto\mu^{-1}(t)$ such that $\bigcup[A_0,D_1],\bigcup[L_1,A_1]\in\beta_1([D_1,A_1])$. Now, we can choose an embedding $\alpha_1:[0,1]\mapsto\beta_1([D_1,A_1])$ such that $\alpha_1(0)=\bigcup[A_0,D_1]$ and $\alpha_1(1)=\bigcup[L_1,A_1]$.

We use Construction 4.3 for the arc $[D_1,A_0]\cup[A_0,A_2]=[D_1,A_2]$ and $s=t$. Then there exists a mapping $\beta_2:[A_0,A_2]\mapsto\mu^{-1}(t)$ such that $\bigcup[D_1,A_0],\bigcup[L_2,A_2]\in\beta_2([A_0,A_2])$. So, we can choose an embedding $\alpha_2:[0,1]\mapsto\beta_2([A_0,A_2])$ such that $\alpha_2(0)=\bigcup[D_1,A_0]$ and $\alpha_2(1)=\bigcup[L_2,A_2]$.

Finally, we use Construction 4.3 for the arc $[D_1,A_0]\cup[A_0,A_3]=[D_1,A_3]$ and $s=t$. Then there exists a mapping $\beta_3:[A_0,A_3]\mapsto\mu^{-1}(t)$ such that $\bigcup[D_1A_0],\bigcup[L_3,A_3]\in\beta_3([A_0,A_3])$. So, we can choose an embedding $\alpha_3:[0,1]\mapsto\beta_3([A_0,A_3])$, such that $\alpha_3(0)=\bigcup[D_1,A_0]$ and $\alpha_3(1)=\bigcup[L_3,A_3]$.

Claim 2. $\bigcup[L_i,A_i]\notin\alpha_j([0,1])$ if $i\neq j$ and $i,j\in\{1,2,3\}$.

We prove Claim 2. Suppose that there exist $i,j\in\{1,2,3\}$ with $i\neq j$ such that $\bigcup[L_i,A_i]\in\alpha_j([0,1])$. Since $\alpha_j([0,1])\subset\beta_j([A_0,A_j])$, there exists $E\in[A_0,A_j]$ such that $\beta_j(E)=\bigcup[L_i,A_i]$. By definition of β_j, we have that $\beta_j(E)=\bigcup[C_E,E]$; thus, $E,A_i\subset\bigcup[L_i,A_i]$. Now notice that $\mu(\bigcup[L_i,A_i])-\mu(E)=t-t_0<\delta$ and $\mu(\bigcup[L_i,A_i])-\mu(A_i)=t-t_0<\delta$, which implies, by the election of δ, that $H(\bigcup[L_i,A_i],E)<\varepsilon$ and $H(\bigcup[L_i,A_i],A_i)<\varepsilon$. Therefore, $H(E,A_i)<2\varepsilon$, a contradiction with the choice of ε. This ends the proof of Claim 2.

To finish the proof of this lemma, we apply Lemma 4.2 at the elements $\bigcup[L_1,A_1]$, $\bigcup[L_2,A_2]$, $\bigcup[L_3,A_3]$, and $\bigcup[A_0,D_1]$, and the embeddings α_1, α_2, and α_3. □

5. **Main Theorem**

The following theorem answers Question 10 of [8].

Theorem 5.1. *Let X be a continuum. Suppose that there is a plane neighborhood of X in $C(X)$. Then there exists a neighborhood of X in $C(X)$ which is a 2-cell.*

Proof. Let μ be a Whitney map for $C(X)$. Let $\mathcal{D}$ be a neighborhood of X in $C(X)$ such that it is embeddable in $\mathbb{R}^2$. By [8, Lemma 1.28], there exists $t_0 \in [0,1)$ such that $\mu^{-1}([t_0,1]) \subset \mathcal{D}$. So, $\mu^{-1}([t_0,1])$ is embeddable in $\mathbb{R}^2$. By Corollary 3.6, $\mu^{-1}([t_0,1])$ is locally connected, and, by Theorem 3.7, $\mu^{-1}(t_0)$ is locally connected.

Now suppose that $\mu^{-1}(t_0)$ contains a simple triod. By Lemma 4.5, there exists $t_1 \in (t_0,1)$ such that $\mu^{-1}(t)$ contains a simple triod for all $t \in [t_0,t_1]$. Thus, there exists a family of uncountably many pairwise disjoint simple triods contained in $\mathcal{D}$, which implies, by [11, Theorem 1], that $\mathcal{D}$ cannot embedded in $\mathbb{R}^2$, a contradiction. Since $\mu^{-1}(t_0)$ does not contain simple triods and it is a locally connected continuum, by [5, Exercise 8.40(b)], it is either an arc or a simple closed curve.

Finally, by [8, Corollary 2], there exists a neighborhood of X in $C(X)$, which is a 2-cell. $\square$

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