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WHEN HAUSDORFF CONTINUA HAVE NO GAPS

PAUL BANKSTON

Abstract. An interpretation of betweenness on a set is *gap free* if each two distinct points of the set have a third point between them. In this paper we are interested in gap free betweenness relations naturally induced by the topology of Hausdorff continua. In particular, we say *c lies between a and b in the K-interpretation* precisely when every subcontinuum that contains both a and b also contains c. We explore the connection between K-gap freeness and hereditary unicoherence.

1. Introduction

If $[\ ,\ ,\ ]$ is a ternary relation on a set X interpreting a notion of betweenness, then we say the structure $\langle X, [\ ,\ ,\ ]\rangle$ is *gap free* if each two elements of X always have a third element between them. This amounts to satisfying the universal-existential sentence

$$\text{Gap Freeness: } \forall\, ab\ \exists\, x\, (a \neq b \rightarrow ([a, x, b] \wedge x \neq a \wedge x \neq b))$$

in the appropriate first-order language L_t (see, e.g., [5]).

This paper is a continuation of [1], in which road systems are introduced as a means of unifying the majority of known interpretations of the intuitive notion of betweenness. Briefly, a *road system* is an ordered pair $\langle X, \mathcal{R}\rangle$, where X is a nonempty set of points and $\mathcal{R}$ is a collection of nonempty subsets of X—the *roads*—satisfying (1) every singleton subset of X is a road and (2) every doubleton subset of X is contained in at least one road. The road system is *additive* if the union of two overlapping roads is a road; the system is *separative* if for any $a, b, c \in X$, with

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$b \neq c$, there is a road containing a that also contains exactly one of b and c. If $a, b \in X$, $\mathcal{R}(a,b)$ comprises the roads *connecting* a and b; i.e., the family $\{R \in \mathcal{R} : a, b \in R\}$. Then the $\mathcal{R}$-*interval* with *bracket points* a and b is $[a,b]_{\mathcal{R}} := \bigcap \mathcal{R}(a,b)$. We write $[a,c,b]_{\mathcal{R}}$ to indicate that point c lies *between* a and b; i.e., when $c \in [a,b]_{\mathcal{R}}$. The relation $[\ ,\ ,\ ]_{\mathcal{R}}$ is then said to be *induced* by the road system $\mathcal{R}$; a ternary relation induced by a road system on a set is called an *R-relation.* So we refer to a road system as being gap free just in case the same goes for its induced R-relation.

Theorem 2.0.1 in [1] lays down first-order criteria that characterize when a ternary relation is an R-relation, namely the universal L_t-sentences

Symmetry: $\forall abc\,([a,c,b] \to [b,c,a])$;
Reflexivity: $\forall ab\,[a,b,b]$;
Minimality: $\forall ab\,([a,b,a] \to b = a)$; and
Strong Transitivity: $\forall abcdx\,(([a,c,b] \wedge [a,d,b] \wedge [c,x,d]) \to [a,x,b])$.

A ternary relation satisfying the first three of these sentences is called *basic*; in interval terms, basic ternary relations are characterized by saying $[a,b] = [b,a]$, $[a,b] \supseteq \{a,b\}$, and $[a,a] = \{a\}$. When we add strong transitivity, we get $[c,d] \subseteq [a,b]$ whenever $c, d \in [a,b]$. (*Transitivity* is just strong transitivity with each occurrence of d replaced by a; see [1].) Gap freeness in basic ternary structures $\langle X, [\ ,\ ,\ ]\rangle$ just says that $[a,b]$ always properly contains $\{a,b\}$ when $a \neq b$. It is a kind of "density" property; indeed, when $[\ ,\ ,\ ]$ is induced by a total ordering in the classical way, gap freeness and order density are the same.

Separativity in a road system is easily seen to be equivalent to its induced R-relation satisfying

Antisymmetry: $\forall abc\,(([a,c,b] \wedge [a,b,c]) \to b = c)$.

And a road system (see [1, Theorem 4.0.5]) is contained within an additive road system with the same induced R-relation if and only if its induced R-relation satisfies

Disjunctivity: $\forall abcx\,([a,x,b] \to ([a,x,c] \vee [c,x,b]))$.

In [1] we discuss three topological interpretations of betweenness, each induced by an additive road system reflecting an aspect of the topology of a connected space. Of the three, the most restrictive and extensively studied is the *Q-interpretation*, defined by saying $[a,c,b]_Q$ holds precisely when either $c \in \{a,b\}$ or a and b lie in separate quasicomponents of $X \setminus \{c\}$. By replacing "quasicomponents" in this definition with "components," we obtain the slightly weaker *C-interpretation* $[\ ,\ ,\ ]_C$. It is not very difficult to show that the collection $\mathcal{C}$ of connected subsets of a connected space X provides a separative, additive road system inducing $[\ ,\ ,\ ]_C$. With a bit more effort, one can also show that $[\ ,\ ,\ ]_Q$ is induced by a separative, additive road system containing $\mathcal{C}$ (see [1, Theorem 6.1.2 and Corollary 6.2.2]).

When the space X is a Hausdorff continuum, Q-gap freeness makes the space a *dendron* and is known [7] to be equivalent to the *connected intersection property*: the intersection of any two connected subsets is connected.

Here we continue the study of the *K-interpretation* of betweenness in Hausdorff continua, defined by saying $[a, c, b]_K$ holds for $c \notin \{a, b\}$ precisely when a and b lie in separate continuum components of $X \setminus \{c\}$. As with the C-interpretation, it is straightforward to show that the collection $\mathcal{K}$ of connected closed subsets (i.e., *subcontinua*) of X provides an additive road system inducing $[\ ,\ ,\ ]_K$. However, $\mathcal{K}$ may not be separative: if X is the classic $\sin(1/x)$-continuum, a is a point on the "curvy bit," and b and c are points on the "straight bit," then every subcontinuum containing a and b (a and c, respectively) also contains c (b, respectively). (See [1, Example 6.3.2].)

So in first-order terms, both the Q- and the C-interpretations of betweenness satisfy the antisymmetry condition, while the K-interpretation may not.

K-gap freeness in Hausdorff continua is much weaker than C-gap freeness, but is still closely related to certain weakened versions of the connected intersection property. One such is *hereditary unicoherence*, saying that the intersection of any two subcontinua is connected. In this paper we show that (1) K-gap freeness—along with some related first-order properties—is strictly weaker than hereditary unicoherence; however, (2) strong K-gap freeness, a natural refinement of K-gap freeness in the absence of antisymmetry, is equivalent to the conjunction of hereditary unicoherence and hereditary decomposability (also a consequence of Q-gap freeness [7]). In a later paper [2], we focus on the role of antisymmetry. In particular, we show that the conjunction of gap freeness and antisymmetry in the K-interpretation of betweenness is equivalent to saying that all K-intervals with more than one point are Hausdorff arcs, i.e., Hausdorff continua with precisely two noncut points.

2. K-Gap Freeness and Hereditary Unicoherence

From here on, we will be considering only continua, i.e., compact connected spaces, that satisfy the Hausdorff separation axiom, and so *continuum* will be our nickname for *Hausdorff continuum.* Similarly, a subset of a topological space will be termed a *subcontinuum* if its subspace topology is that of a continuum. A continuum, or any topological space, is *nondegenerate* if it contains more than one point.

Also, unless otherwise stated, we will regard all betweenness notions arising from a continuum X as those related to the road system $\langle X, \mathcal{K}\rangle$. This will allow us to drop the letter K from most prefixes and subscripts.

We begin with a simple interval characterization of hereditary unicoherence.

Proposition 2.1. *A continuum is hereditarily unicoherent if and only if each of its intervals is connected.*

Proof. If M and N are subcontinua of X and $M \cap N$ is disconnected, then there are two points a and b in separate components of $M \cap N$. This provides a disconnection of $[a, b] \subseteq M \cap N$. Conversely, if X is hereditarily unicoherent and $a, b \in X$, then $\mathcal{K}(a, b)$ is a downwardly directed family of subcontinua; i.e., the intersection of each two of its members contains a third. By basic continuum theory (see, e.g., [6]), $[a, b] = \bigcap \mathcal{K}(a, b)$ is a subcontinuum. □

In the study of continua there are some important results that go under the rubric of *boundary bumping theorems*. The most useful of these for our purposes is the following.

Lemma 2.2 ([6, Corollary 5.5])**.** *If M is a subcontinuum of continuum Y and U is a proper open subset of Y that contains M, then there is a subcontinuum N with $M \subseteq N \subseteq U$ and $M \neq N$.*

For a direct application of this, we have the following proposition.

Proposition 2.3. *Let X be a continuum with $a, b \in X$ distinct and $[a, b]$ connected. Then there exists $c \in [a, b]$ with $c \neq a$ and $b \notin [a, c]$.*

Proof. Applying Lemma 2.2, let Y be $[a, b]$, let M be $\{a\}$, and let U be $[a, b] \setminus \{b\}$. Then we have a nondegenerate subcontinuum N of $[a, b]$, properly containing a and missing b. Pick $c \in N \setminus \{a\}$. □

Hereditarily unicoherent continua are clearly gap free, by Proposition 2.1. By Proposition 2.3, however, their K-interpretations satisfy an ostensibly more restrictive L_t-sentence, namely

Semi-strong Gap Freeness: $\forall\, ab\, \exists\, x\, (a \neq b \to ([a, x, b] \wedge x \neq a \wedge \neg[a, b, x]))$.

We remark that the first-order statement of semi-strong gap freeness differs from that of gap freeness only in the replacement of the subformula $x \neq b$ by the more restrictive $\neg[a, b, x]$. If we now similarly replace the other inequality, we obtain

Strong Gap Freeness: $\forall\, ab\, \exists\, x\, (a \neq b \to ([a, x, b] \wedge \neg[x, a, b] \wedge \neg[a, b, x]))$.

Of course, in the presence of antisymmetry, these three gap freeness notions collapse into one. And since both the Q- and the C-interpretations of betweenness are antisymmetric, it is only in the K-interpretation that there is an issue.

In this section we concentrate on showing that even semi-strong gap freeness is not enough to ensure hereditary unicoherence in a continuum. In section 3 we show that the existence of centroids, another consequence of hereditary unicoherence, is still not enough, and in section 4 we show that strong gap freeness in a continuum is actually equivalent to the conjunction of hereditary unicoherence and hereditary decomposability. We take up a deeper study of antisymmetry for the K-interpretation in [2].

If X is a continuum, recall that a *decomposition* of X is an ordered pair $\langle M, N\rangle$ of proper subcontinua whose union is X. X is (*hereditarily*) *decomposable* if (every nondegenerate subcontinuum of) X has a decomposition. X is (*hereditarily*) *indecomposable* if (every subcontinuum of) X fails to be decomposable. Hereditary indecomposability is equivalent to the property that the intersection of two overlapping subcontinua is one or the other of them; hence, by Proposition 2.1, all intervals in hereditarily indecomposable continua are subcontinua.

Lemma 2.4. *Let $\langle M, N\rangle$ be a decomposition of continuum X and suppose H is a subcontinuum of X intersecting both M and N. If C is a component of $H \cap M$, then C intersects N.*

Proof. Assume otherwise, that $C \subseteq H \setminus N$. Since H intersects N, we know that $H \setminus N$ is a proper open subset of H, and we may use Lemma 2.2 to obtain a new subcontinuum D with $C \neq D$, but $C \subseteq D \subseteq H \setminus N \subseteq H \cap M$. This contradicts the assumption that C is maximally connected in $H \cap M$. □

We define a continuum X to be an *annulus* if it has a decomposition $\langle M, N\rangle$ such that $M \cap N$ is a union of two disjoint nondegenerate subcontinua. An annulus $X = M \cup N$ is *crooked* if M and N are hereditarily indecomposable. (For example, glue two pseudo-arcs together appropriately [6].) Clearly, no annulus is unicoherent, let alone hereditarily so. We show that every crooked annulus is semi-strongly gap free nevertheless.

By way of notation: If Y is a topological space and $y \in Y$, then $C(y, Y)$ denotes the component of y in Y.

Lemma 2.5. *If X is a crooked annulus with defining decomposition $\langle M, N\rangle$ such that $M \cap N = A \cup B$ and H is a subcontinuum that intersects A, then either $H \subseteq A$ or $H \supseteq A$.*

Proof. If either $H \subseteq M$ or $H \subseteq N$, then the conclusion follows because both M and N are hereditarily indecomposable. So assume H intersects both $M \setminus N$ and $N \setminus M$. Then of course $H \not\subseteq A$, so it remains to show $H \supseteq A$.

If C is a component of $H \cap M$, then, by Lemma 2.4, C intersects A or C intersects B. If C also contains points of $M \setminus N$, then hereditary indecomposability for M ensures that $C \supseteq A$ or $C \supseteq B$. We are done if there is at least one component, either of $H \cap M$ or of $H \cap N$, that contains A.

So suppose $C \supseteq B$ and $C \cap A = \emptyset$ for every C that is either a component of $H \cap M$ containing points of $M \setminus N$ or a component of $H \cap N$ containing points of $N \setminus M$. Let $x \in H \setminus N$ and $y \in H \setminus M$. Then $B \subseteq C(x, H \cap M) \cap C(y, H \cap N)$; $C(x, H \cap M) \cup C(y, H \cap N)$ is disjoint from A; and all other components of either $H \cap M$ or $H \cap N$ lie in A. Since H intersects A, $\langle H \cap A, C(x, H \cap M) \cup C(y, H \cap N)\rangle$ forms a disconnection of H and gives us a contradiction. □

No annulus is hereditarily unicoherent, as mentioned above.

Theorem 2.6. *Every crooked annulus is semi-strongly gap free.*

Proof. Let X, M, N, A, and B be as in Lemma 2.5. It will suffice to show

($\star$) For any two distinct points $x, y \in X$, $[x, y]$ contains nondegenerate subcontinua S and T with $x \in S$ and $y \in T$.

Indeed, suppose ($\star$) holds. If $S \cap T = \emptyset$, then any $z \in S \setminus \{x\}$ ($z \in T \setminus \{y\}$, respectively) witnesses the statements $z \neq x$ and $\neg[x, y, z]$ ($z \neq y$ and $\neg[z, x, y]$, respectively). And if $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{K}(x, y)$. Hence, $[x, y] = S \cup T$ is connected and nondegenerate, so we may apply Proposition 2.3.

If Y is an arbitrary subcontinuum of X and $x, y \in Y$, we denote by $[x, y]^Y$ the interval in Y with bracket points x and y. Then, of course, $[x, y]^Y \supseteq [x, y]^X = [x, y]$. In the remainder of the proof, x and y are two distinct points of X, and H is a subcontinuum of X containing both x and y (in symbols, $H \in \mathcal{K}(x, y)$). We have two main cases dictated by whether or not the two points lie in the same member of the decomposition $\langle M, N\rangle$.

Case 1: ($x \in M \setminus N$ and $y \in N \setminus M$): Fix $a \in A$ and $b \in B$. Then $[x, a]^M \cap [x, b]^M \neq \emptyset$ and M is hereditarily indecomposable; so each interval is a subcontinuum, and hence one of them is contained in the other. In particular, $[x, a]^M \cap [x, b]^M$ is a nondegenerate subcontinuum of X that contains x. We show $[x, y] \supseteq ([x, a]^M \cap [x, b]^M) \cup ([y, a]^N \cap [y, b]^N)$, thereby establishing condition ($\star$).

Fixing H, let $C = C(x, H \cap M)$. Then, by Lemma 2.4, $C \cap (A \cup B) \neq \emptyset$. But C is not contained in $A \cup B$, A and B are also subcontinua of M, and M is hereditarily indecomposable. Hence, either $C \supseteq A$ or $C \supseteq B$. In the first instance, $C \supseteq [x,a]^M$; in the second, $C \supseteq [x,b]^M$. In either case, we have $H \supseteq C \supseteq [x,a]^M \cap [x,b]^M$.

By the same token, $H \supseteq [y,a]^N \cap [y,b]^N$; hence $[x,y] \supseteq ([x,a]^M \cap [x,b]^M) \cup ([y,a]^N \cap [y,b]^N)$, as desired.

Case 2: $(x, y \in M)$: This case has three subcases, dictated by how $[x,y]^M$ intersects $A \cup B$.

(2.1) $([x,y]^M \subseteq M \setminus N)$: We show $[x,y]^M = [x,y]$, and for this, it suffices to show that arbitrary $H \in \mathcal{K}(x,y)$ contains $[x,y]^M$. This is clearly true if $H \subseteq M$, so assume H intersects N and let $C = C(x, H \cap M)$. Then, by Lemma 2.4, C intersects N; hence, C intersects, but is not a subset of, subcontinuum $[x,y]^M$. Thus, $H \supseteq C \supseteq [x,y]^M$.

(2.2) $([x,y]^M$ intersects A, but not $B)$: If both x and y are in A, then, by Lemma 2.5, every subcontinuum of X containing x and y must either be contained in A or contain A. Hence, $[x,y] = [x,y]^M$.

So assume $y \in M \setminus A$; we show $H \supseteq [x,y]^M$. If $H \subseteq M$, we are done; so assume $H \cap (N \setminus M) \neq \emptyset$ and let $C_x = C(x, H \cap M)$ and $C_y = C(y, H \cap M)$. If both components intersect A, then at least we have $C_y \supseteq A$, since $y \notin A$; hence, $C_x \cap C_y \neq \emptyset$ and there is only one component after all. Since that component contains both x and y, it must contain $[x,y]^M$. If either component is disjoint from A, then it must intersect B, by Lemma 2.5. Since $[x,y]^M \cap B = \emptyset$, neither C_x nor C_y is contained in B. Thus, we have either $C_x \supseteq B$ or $C_y \supseteq B$, so either $C_x \supseteq [x,y]^M$ or $C_y \supseteq [x,y]^M$ (hereditary indecomposability of M). Hence, $H \supseteq [x,y]^M$, and therefore $[x,y] = [x,y]^M$.

(2.3) $([x,y]^M$ intersects both A and $B)$: If $x \in A$ and $y \in B$, then $H \supseteq A \cup B$, by Lemma 2.5; hence, $[x,y] \supseteq A \cup B$. Since $A \cup B = M \cap N$, we also know $[x,y] \subseteq A \cup B$. Hence, $[x,y] = A \cup B$, and we have $(\star)$ holding.

If $x \in A$ and $y \in M \setminus N$, then $[x,y]^M \supseteq B$, so pick $b \in B$. Then $b \in [x,y]^M$; hence, $[y,b]^M \subseteq [x,y]^M$. If $H \subseteq M$, then $H \supseteq [x,y]^M \supseteq A \cup [y,b]^M$. And if $H \cap (N \setminus M) \neq \emptyset$, then let $C = C(y, H \cap M)$. If $C \supseteq A$, then $C \supseteq [x,y]^M$ because $x \in A$. If $C \supseteq B$, then $C \supseteq [y,b]^M$. In any event, $C \supseteq [y,b]^M$, a nondegenerate subcontinuum containing y. Since H also contains A, we have $[x,y] \supseteq A \cup [y,b]^M$, and therefore $(\star)$.

Finally, if $x, y \in M \setminus N$, then $[x,y]^M \supseteq A \cup B$. Pick $a \in A$ and $b \in B$, with $C_x = C(x, H \cap M)$ and $C_y = C(y, H \cap M)$. If $H \subseteq M$, then $H \supseteq [x,y]^M \supseteq [x,a]^M \cup [y,b]^M$. If $H \cap (N \setminus M) \neq \emptyset$, then $C_x \supseteq A$ or $C_x \supseteq B$. In the first instance, $C_x \supseteq [x,a]^M$; in the second, $C_x \supseteq [x,b]^M$. Thus, $C_x \supseteq [x,a]^M \cap [x,b]^M$. Likewise, $C_y \supseteq [y,a]^M \cap [y,b]^M$, and we infer that $[x,y] \supseteq ([x,a]^M \cap [x,b]^M) \cup ([y,a]^M \cap [y,b]^M)$. The first of these intersections is a subcontinuum containing x and the second is a subcontinuum containing y. Thus, we have ($\star$) in this subcase too.

This completes the argument for Case 2, and hence the proof that X is semi-strongly gap free. □

3. Existence of Centroids and Hereditary Unicoherence

If $\langle X, \mathcal{R} \rangle$ is a road system, with $a, b, c \in X$, we denote by $[abc]_{\mathcal{R}}$ the intersection $[a,b]_{\mathcal{R}} \cap [a,c]_{\mathcal{R}} \cap [b,c]_{\mathcal{R}}$. Elements of $[abc]_{\mathcal{R}}$ are the $\mathcal{R}$-*centroids* of the ordered triple $\langle a,b,c \rangle$. The road system is *centroidal* if all of its centroid sets are nonempty; i.e., if the following L_t-sentence holds for $\langle X, [\ ,\ ,\ ]_{\mathcal{R}} \rangle$.

Centroid Existence: $\forall\, abc\ \exists\, x\, ([a,x,b] \wedge [a,x,c] \wedge [b,x,c])$.

A continuum X is *centroidal* (*C-centroidal, Q-centroidal*, respectively) if the corresponding road system is centroidal. Then, of course, $[abc]_Q \subseteq [abc]_C \subseteq [abc]_K = [abc]$ always holds where the subscripts have their obvious meanings.

Proposition 3.1. *Let X be a continuum, with $a, b \in X$. If $[a,b]$ is connected, then $[abc] \neq \emptyset$ for any $c \in X$. If X is also hereditarily unicoherent, then all centroid sets are subcontinua.*

Proof. Let $A = [a,c] \cap [a,b]$ and $B = [b,c] \cap [a,b]$. By disjunctivity, $[a,b] = A \cup B$. A and B are closed nonempty subsets of $[a,b]$, and $[a,b]$ is connected; so $A \cap B \neq \emptyset$. Any element of $A \cap B$ is a centroid for $\langle a,b,c \rangle$.

If X is hereditarily unicoherent, then intervals are connected and each centroid set is nonempty. Being the intersection of three subcontinua, that centroid set is therefore a subcontinuum. □

It is natural to ask whether being centroidal is enough to imply hereditary unicoherence in a continuum, and the answer is still no.

Theorem 3.2. *Every crooked annulus is centroidal.*

Proof. Let $X = M \cup N$, where M and N are hereditarily indecomposable subcontinua and $M \cap N$ is the union of disjoint nondegenerate subcontinua A and B. As in the proof of Theorem 2.6, we use superscripts to indicate intervals relative to a subcontinuum.

Let $x, y, z \in X$. We need to show that the centroid set $[xyz]$ is never empty, and, by Proposition 3.1, we may infer $[xyz] \neq \emptyset$ whenever at least one of the three contributing intervals is connected. We are done, therefore, once we show $[xyz]$ to be nonempty under the assumption that all three of $[x,y]$, $[x,z]$, and $[y,z]$ are disconnected.

Suppose $x, y \in M$ and $[x,y]$ is disconnected. Then, by the Case 2 argument in the proof of Theorem 2.6, it follows that $[x,y]^M$ intersects both A and B. If $x \in A$ and $y \in B$, then $[x,y] = A \cup B$, and if $x \in A$ and $y \in M \setminus N$, then $[y,b]^M \supseteq B$ for $b \in B$. Hence, $[x,y] \supseteq A \cup [y,b]^M \supseteq A \cup B$ here too.

If both x and y are in $M \setminus N$, then we have $[x,y] \supseteq ([x,a]^M \cap [x,b]^M) \cup ([y,a]^M \cap [y,b]^M)$, where $a \in A$ and $b \in B$. Using distributivity, this rewrites to $([x,a]^M \cup [y,a]^M) \cap ([x,a]^M \cup [y,b]^M) \cap ([x,b]^M \cup [y,a]^M) \cap ([x,b]^M \cup [y,b]^M)$. Since $a, b \in [x,y]^M$, the first and last of the four unions in the expression both equal $[x,y]^M$, which contains both the other unions. Thus, we have $[x,y] \supseteq ([x,a]^M \cup [y,b]^M) \cap ([x,b]^M \cup [y,a]^M)$. Since neither x nor y is in $A \cup B$, each of these unions contains $A \cup B$. Hence, we conclude that if both x and y are in either M or N and $[x,y]$ is disconnected, then $[x,y] \supseteq A \cup B$.

Now suppose $x, y, z \in X$ are such that all intervals contributing to the centroid are disconnected. We may assume that $x, y \in M$. If $z \in M$ also, then all three intervals contain $A \cup B$ and we are done. So we may further assume $z \in N \setminus M$. We show that $[x,z] \cap [y,z]$ contains either A or B, and this will prove the same for $[xyz]$. Assume first that both x and y are in $M \setminus N$, and pick $a \in A$ and $b \in B$. Then the Case 1 argument from the proof of Theorem 2.6 shows that $[x,z] \supseteq ([x,a]^M \cap [x,b]^M) \cup ([z,a]^N \cap [z,b]^N)$ and $[y,z] \supseteq ([y,a]^M \cap [y,b]^M) \cup ([z,a]^N \cap [z,b]^N)$.

Now since N is a hereditarily indecomposable continuum, we know $[z,a]^N \cap [z,b]^N$ equals $[z,a]^N \supseteq A$ or it equals $[z,b]^N \supseteq B$ (because $z \notin A \cup B$). Thus, $[x,z] \cap [y,z]$ contains A or B. The case where, say, $x \in M \setminus N$ and $y \in A \cup B$, is handled similarly, but is even easier, and the proof is complete. □

4. Strong Gap Freeness and Hereditary Unicoherence

In light of the results of sections 2 and 3, the following is a natural, but as yet unanswered, question.

Question 4.1. Is there an L_t-sentence φ such that for any continuum X, φ is true for $\langle X, [\ ,\ ,\ ]\rangle$ if and only if X is hereditarily unicoherent?

Our objective in this section is to answer a variation on this question by proving that strong gap freeness in a continuum is equivalent to that continuum's being both hereditarily unicoherent and hereditarily decomposable. And a major step in that direction lies in showing that strong gap freeness is equivalent to the property that every nondegenerate interval is a decomposable continuum.

If $[a,b]$ is an interval in continuum X, we call $c \in [a,b]$ a *decomposition point* of $[a,b]$ if it happens that $b \notin [a,c]$ and $a \notin [c,b]$. (So strong gap freeness is the property that every nondegenerate interval has a decomposition point.)

Proposition 4.2. *Let X be a continuum. Then any interval in X that is a decomposable subcontinuum must have a decomposition point. If X is hereditarily unicoherent, then every interval with a decomposition point must be a decomposable subcontinuum.*

Proof. Assume $[a,b]$ is a decomposable subcontinuum of X. Then it has a decomposition $\langle M,N\rangle$ into proper subcontinua. It cannot be the case that both a and b are contained in the same member of the decomposition; if it were, $[a,b]$ would be a proper subset of itself. So assume $a \in M \setminus N$ and $b \in N \setminus M$. Since $[a,b]$ is connected, there is some $c \in M \cap N$. Since a and c belong to M and $b \notin M$, we have $b \notin [a,c]$. Similarly, we infer that $a \notin [c,b]$; hence, c is a decomposition point for $[a,b]$.

For the second half, suppose X is hereditarily unicoherent and c is a decomposition point for $[a,b]$. Then, by disjunctivity in additive road systems, $[a,b]=[a,c]\cup[c,b]$. Since all intervals are continua, and both $[a,c]$ and $[c,b]$ are proper subsets of $[a,b]$, we conclude that $[a,b]$ is a decomposable subcontinuum of X. □

The following is a key step in the pursuit of our characterization theorem.

Lemma 4.3. *Let X be a continuum with A and B two disjoint nonempty closed subsets. Then there is an interval $[a_0,b_0]$, with $a_0 \in A$ and $b_0 \in B$, such that if $[a,b] \subseteq [a_0,b_0]$ where $a \in A$ and $b \in B$, then $[a,b]=[a_0,b_0]$.*

Proof. Let $\mathcal{S} = \{[a,b] : a \in A, b \in B\}$ be partially ordered by reverse inclusion. What we are looking for is a maximal element relative to this ordering, and, by Zorn's Lemma, all we need show is that every nonempty chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$. Indeed, suppose $\mathcal{L} \subseteq \mathcal{S}$ is a nonempty chain. Then both $\{A\}\cup\mathcal{L}$ and $\{B\}\cup\mathcal{L}$ are nonempty families of closed subsets of X, and both satisfy the finite intersection property. So we may find $a^* \in A \cap \bigcap\mathcal{L}$ and $b^* \in B \cap \bigcap\mathcal{L}$. Then $[a^*,b^*] \in \mathcal{S}$. Moreover, if $[a,b] \in \mathcal{L}$, then we have $\{a^*,b^*\} \subseteq [a,b]$; hence, $[a^*,b^*] \subseteq [a,b]$. This shows that $[a^*,b^*]$ is our desired upper bound and completes the proof. □

Theorem 4.4. *Let X be a continuum. The following are equivalent:*

(i) *Every nondegenerate interval in X is a decomposable continuum.*

(ii) *X is strongly gap free.*

Proof. Let $[a, b]$ be a nondegenerate interval in X and suppose it is a decomposable continuum. Then $[a, b]$ has a decomposition point, by the first half of Proposition 4.2.

For the converse, suppose X is strongly gap free. We first show X is hereditarily unicoherent. Indeed, suppose not. Then there are subcontinua M and N such that $M \cap N = A \cup B$, where A and B are nonempty disjoint closed sets. Applying Lemma 4.3, let $[a_0, b_0]$ be a maximal element of $\mathcal{S}$ (under reverse inclusion). Since $a_0 \neq b_0$, there is a decomposition point $c \in [a_0, b_0] \subseteq M \cap N$, so either $c \in A$ or $c \in B$. In the first case, we have $[c, b_0] = [a_0, b_0]$; in the second, $[a_0, c] = [a_0, b_0]$. This is a contradiction.

Since X is hereditarily unicoherent, the second half of Proposition 4.2 shows that each nondegenerate interval is a decomposable continuum. □

Recall that a continuum Y is *irreducible about* a subset A if no proper subcontinuum of Y contains A. Thus, in a hereditarily unicoherent continuum, each interval is (the unique subcontinuum that is) irreducible about its set of bracket points. Y is *irreducible* if it is irreducible about a two-point subset, i.e., if it equals one of its own nondegenerate intervals.

By [6, Corollary 11.20], a *metrizable* nondegenerate continuum is indecomposable if and only if it contains three points such that it is irreducible about any two of them. In the language of centroids, this is equivalent to saying that $[abc] = X$ for some $a, b, c \in X$. Metrizability is an important assumption because, by results in [4], there are nonmetrizable indecomposable continua that are not irreducible at all. However, by results in [3], every nondegenerate indecomposable continuum contains an indecomposable subcontinuum that is irreducible, and this is all we need for the following.

Corollary 4.5. *A continuum is strongly gap free if and only if it is hereditarily unicoherent and hereditarily decomposable.*

Proof. If X is hereditarily unicoherent and hereditarily decomposable, then every nondegenerate interval has a decomposition point, by Proposition 4.2. Thus, X is strongly gap free.

Conversely, if X is strongly gap free, then, by Theorem 4.4, each of its nondegenerate intervals is a decomposable continuum. Hence, the continuum is hereditarily unicoherent. If it were not hereditarily decomposable, it would have a nondegenerate indecomposable subcontinuum M which, by [3], could be taken to be irreducible about, say, the points a and b. But then $[a, b] \subseteq M$ is a subcontinuum, and that means $[a, b] = M$. But $[a, b]$ is decomposable, a contradiction. □

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