**TOPOLOGY PROCEEDINGS**
**Volume 44, 2014**
Pages 197–206

**http://topology.auburn.edu/tp/**

# Box Products of One Point Compactifications and Related Results

by

Judith Roitman

Electronically published on August 19, 2013

**Topology Proceedings**

| | |
|---|---|
| **Web:** | http://topology.auburn.edu/tp/ |
| **Mail:** | Topology Proceedings<br>Department of Mathematics & Statistics<br>Auburn University, Alabama 36849, USA |
| **E-mail:** | topolog@auburn.edu |
| **ISSN:** | 0146-4124 |

http://topology.auburn.edu/tp/

**TOPOLOGY PROCEEDINGS**

Volume 44 (2014)
Pages 197-206

E-Published on August 19, 2013

# BOX PRODUCTS OF ONE POINT COMPACTIFICATIONS AND RELATED RESULTS

JUDITH ROITMAN

Abstract. Let $A+\infty$ be the one-point compactification of a discrete space $A$. If $\square(\omega+1)^\omega$ is basic ultraparacompact, i.e., every open cover has a pairwise disjoint subcover by canonical basic sets, and $|A|<\aleph_\omega$, then $\square(A+\infty)^\omega$ is paracompact. If all but finitely many $|A_n|\geq\aleph_2$, then $\nabla_{n<\omega}(A_n+\infty)^\omega$ fails to be hereditarily normal. We also give an elementary submodel proof of the following theorem by Scott W. Williams: The box product of countably many compact $T_3$ spaces each of weight at most $\omega_1$ is paracompact.

## 1. Preliminaries

**Definition 1.1.** The *box product* $\square_{i\in I}X_i$ of topological spaces $\{X_i : i\in I\}$ is the topology on $\Pi_{i\in I}X_i$ whose basis is all sets of the form $\Pi_{i\in I}u_i$ where each $u_i$ is open in $X_i$.

Box products have been intensively studied. Two main questions are asked about box products: Which ones are normal? Which ones are paracompact? There are many models in which $\square(\omega+1)^\omega$ is known to be paracompact; the biggest unsolved problem in this area is whether "$\square(\omega+1)^\omega$ is not paracompact" is consistent. For various reasons (see [9] for a recent survey), much of the focus has been on the index set $I$ being countable and the $X_i$'s being compact (or some version of compact). Note that we do not assume spaces are Hausdorff or $T_1$.

A related topology is defined below.

2010 *Mathematics Subject Classification.* Primary 54X10, 58Y30, 18D35; Secondary 55Z10.

*Key words and phrases.* box product, paracompact.

**Definition 1.2.** The *nabla product* $\nabla_{i \in I} X_i$ of topological spaces $\{X_i : i \in I\}$ is the quotient topology induced by the following equivalence relation on $\square_{i \in I} X_i$: $x =^* y$ if and only if $\{i : x(i) \neq y(i)\}$ is finite.

We will use the standard notation $=^*$ for partial functions as well. Similarly, we write $a \subseteq^* b$ if and only if $a \setminus b$ is finite, and so on. "$\forall^\infty$" means "for all but finitely many"; "$\exists^\infty$" means "there exists infinitely many."

When $I$ is countable, the study of box products is made a little easier by results of Mary Ellen Rudin [10] (Fact 1.3) and K. Kunen [8] (Theorem 1.5).

**Fact 1.3.** *For any $\nabla_{i<\omega} X_i$, the countable intersection of open sets is open.*

**Definition 1.4.** A space $X$ is *ultraparacompact* if and only if every open cover has a pairwise disjoint open refinement covering $X$.

**Theorem 1.5.** *If each $X_i$ is compact, then $\square_{i<\omega} X_i$ is paracompact if and only if $\nabla_{i<\omega} X_i$ is paracompact if and only if $\nabla_{i<\omega} X_i$ is ultraparacompact.*

Fact 1.3 and the first equivalence in Theorem 1.5 appeared in [8]; the second equivalence is also due to Kunen, but we know of no written reference. It follows from the fact that a regular paracompact space in which every countable intersection of open sets is open is ultraparacompact.

Theorem 1.5 allows us to focus on the nabla product.

**Definition 1.6.** A space is *scattered* if and only if every subset has an isolated point.

Kunen [8] also proved the following theorem.

**Theorem 1.7.** *Assume* CH. *If each $X_n$ is compact scattered, then $\square_{n<\omega} X_n$ is paracompact.*

We ask the basic question of this paper: In the context of compact scattered spaces, what can we do without CH?

In particular, under axioms weaker than CH, we consider the box product of Fort spaces[1], i.e., one-point compactifications of discrete spaces. We ask, " Under which set theoretic hypotheses is the box product of various Fort spaces paracompact?" We also investigate the hereditary normality of nabla products of Fort spaces. And, finally, we give an alternate proof of Scott W. Williams' theorem [11, Theorem 5.7(i)] that if $\mathfrak{d} = \omega_1$, then the box product of countably many compact regular spaces with weight at most $\omega_1$ is paracompact.

[1]so called because these spaces were first systematically investigated by M. K. Fort, Jr. [5].

## 2. The Nabla Topology for Fort Spaces

For any set $A$, we will denote the Fort space of cardinality $|A|$ by $A + \infty$.[2] As a set, $A + \infty = A \cup \{\infty\}$, where $A$ has the discrete topology and neighborhoods of $\infty$ are cofinite. We only consider infinite $A$.

Note that if $A \subset B$, then $A + \infty$ embeds continuously into $B + \infty$ by the identity mapping.

The point $\overline{\infty}$ is the function in $\square(A+\infty)^\omega$ constantly equal to $\infty$; in an abuse of notation, we also denote its image in the $\nabla$ product by $\overline{\infty}$. In a similar abuse of notation, we will refer to objects such as $\{n : x(n) = \infty\}$, $x \restriction a$, etc., when $x \in \nabla(A + \infty)^\omega$.

A basic open set for any $\nabla(A + \infty)^\omega$ has the form $N(g, G)$ where $g \in A^{\subseteq\omega}$ (i.e., $g$ is a partial function from $\omega$ to $A$), $G : \omega \to [A]^{<\omega}$ (recall that $[A]^{<\omega}$ is the set of finite subsets of $A$), and $x \in N(g, G)$ if and only if $x \restriction \text{dom } g =^* g$ and $\forall^\infty n \notin \text{dom } g\ x(n) \notin G(n)$. By "basic set," we mean a set of the form $N(g, G)$. Each basic set is clopen. If $N = N(g, G)$, we write $g = g_N$ and $G = G_N$.

**Fact 2.1.** (a) *Let $N$ and $N^*$ be basic sets. $N \subseteq N^*$ if and only if $g_N \supseteq g_{N^*}$, $\forall^\infty n \notin \text{dom } g_N\ G_N(n) \supseteq G_{N^*}(n)$, and $\forall^\infty n \in \text{dom}(g_N \setminus g_{N^*})\ g_N(n) \notin G_{N^*}(n)$.*

(b) *If $A \subseteq B$ and $N$ is basic for $\nabla(B+\infty)^\omega$, then $\nabla(A+\infty)^\omega \cap N \neq \emptyset$ if and only if $\forall^\infty n \in \text{dom } g_N\ g_N(n) \in A$.*

Let $A \subseteq B$. Given $N = N(g, G)$, basic in $\nabla(B+\infty)^\omega$ with $N \cap \nabla(A+\infty)^\omega \neq \emptyset$, we write $N_A = N(g, G_A)$ where $G_A(n) = G(n) \cap A$ for all $n$. A collection of basic open sets $\mathcal{V}$ is basic for $A$ if and only if for all $V \in \mathcal{V}$, $V \cap \nabla(A + \infty)^\omega \neq \emptyset$ and $\bigcup \mathcal{V} \supseteq \nabla(A + \infty)^\omega$.

For $x \in \nabla(B + \infty)^\omega$, we define $\underline{x} = x \restriction \{n : x(n) \neq \infty\}$, and if $A \subset B$, we write $\underline{x}_A = x \restriction \{n : x(n) \in A\}$. For $g \in B^{\subseteq\omega}$, we define $\overline{g}$ as

$$\overline{g}(n) = \begin{cases} g(n) & \text{if } n \in \text{dom } g \\ \infty & \text{if otherwise.} \end{cases}$$

**Definition 2.2.** An open family $\mathcal{U}$ is *basic* if and only if all of its elements are basic sets.

**Definition 2.3.** $\nabla(A+\infty)^\omega$ is *basic ultraparacompact* if and only if every open cover has a pairwise disjoint basic covering refinement.

In every model in which we know $\nabla(\omega+1)^\omega$ to be ultraparacompact, it is basic ultraparacompact, in fact, hereditarily basic ultraparacompact.[3]

---

[2] This is not standard notation, but we need to distinguish, e.g., the Fort space $\delta + \infty$ from the ordinal space $\delta + 1$; if $\delta = \omega$, then these spaces are homeomorphic.

[3] So a technical open question is if $\nabla(\omega + 1)^\omega$ is ultraparacompact, must it be basic ultraparacompact? Hereditarily basic ultraparacompact?

In particular, $\nabla(\omega+1)^\omega$ is hereditarily basic ultraparacompact under the combinatorial principle $\Delta$ (see [9]) which holds in every model in which we know $\nabla(\omega+1)^\omega$ is ultraparacompact. $\Delta$ is consistent; it is not known if $\neg\Delta$ is consistent. For completeness, we state $\Delta$ here.

**Definition 2.4.** $\Delta$ is the following statement: For all $f \in \omega^{\subseteq\omega}$, there is $h_f : \omega \to \omega$ so that if $f, g \in \omega^{\subseteq\omega}$ and $(f \setminus g) \neq^* \emptyset \neq^* (g \setminus f)$, then either (1) $\{n \in \text{dom } f \cap \text{ dom } g : f(n) \neq g(n)\}$ is infinite, or (2) $\exists^\infty n \in (\text{dom } f \setminus \text{ dom } g)\ f(n) \leq h_g(n)$, or (3) $\exists^\infty n \in (\text{dom } g \setminus \text{ dom } f)\ g(n) \leq h_f(n)$

By contrast, in section 4 we show that, without any extra hypotheses, a nabla product of countably many Fort spaces of size $> \aleph_1$ is not hereditarily normal, even when the nabla product is basic paracompact. The status of the hereditary normality of $\nabla(\aleph_1+\infty)^\omega$ is open.

## 3. $\nabla(\aleph_n+\infty)^\omega$ Can Be Paracompact

**Lemma 3.1.** *Let $A \subset B$ and suppose $\nabla(A+\infty)^\omega$ is basic ultraparacompact. If $\mathcal{U}$ is an open cover of $\nabla(B+\infty)^\omega$, then there is a basic pairwise disjoint family refining $\mathcal{U}$ and covering $\nabla(A+\infty)^\omega$.*

*Proof.* We may assume $\mathcal{U}$ is a collection of basic sets. Let $\mathcal{U}_A = \{N \in \mathcal{U} : N \cap \nabla(A+\infty)^\omega \neq \emptyset\}$. Let $\mathcal{V}$ be a pairwise disjoint basic refinement of $\{N_A : N \in \mathcal{U}_A\}$ covering $\nabla(A+\infty)^\omega$. Given $N \in \mathcal{V}$, there is $N^* \in \mathcal{U}_A$ with $N \subseteq N^*_A$. We define $N^\dagger$: Let $g_{N^\dagger} = g_N$, $G_{N^\dagger}(n) = G_N(n) \cup G_{N^*}(n)$ for all $n$. Then $N^\dagger \subseteq N \cap N^*$ and $N^\dagger \cap \nabla(A+\infty)^\omega = N \cap \nabla(A+\infty)^\omega$. So $\{N^\dagger : N \in \mathcal{V}\}$ is the desired refinement. □

**Lemma 3.2.** *Let $A \subset B$. If $\mathcal{V}$ is a disjoint collection of basic sets in $\nabla(B+\infty)^\omega$ which is basic for A, then $\bigcup \mathcal{V}$ is closed.*

*Proof.* Given $\mathcal{V}$ as in the hypothesis, suppose $x \notin \bigcup \mathcal{V}$.

Let $N(f,F)$ be the unique element of $\mathcal{V}$ with $\overline{x_A} \in N(f,F)$. Since $\underline{x}_A \supset f$ and $x \notin N(f,F)$, for infinitely many $n \in \text{dom } (\underline{x} \setminus f)$, $x(n) \in F(n)$. Hence, every $N(\underline{x},H) \cap N(f,F) = \emptyset$.

Now let $N(g,G) \in \mathcal{V}$ with $N(f,F) \neq N(g,G)$. Since $N(g,G) \cap N(f,F) = \emptyset$, either

(1) $\exists^\infty n \in \text{dom } (f \cap g)\ f(n) \neq g(n)$, or
(2) case (1) fails and $\exists^\infty n \in \text{dom } (f \setminus g)\ f(n) \in G(n)$, or
(3) cases (1) and (2) fail and $\exists^\infty n \in \text{dom } (g \setminus f)\ g(n) \in F(n)$.

In cases (1) and (2), it is immediate that $N(\underline{x},F) \cap N(g,G) = \emptyset$. In case (3), let $E = \{n \in \text{dom } (g \setminus f) : g(n) \in F(n)\}$. If $E \setminus \text{ dom } \underline{x} \neq^* \emptyset$, again it is immediate that $N(\underline{x},F) \cap N(g,G) = \emptyset$. So we may assume that $E \subseteq^* \text{dom } \underline{x}$. Since each $N \in \mathcal{V}$ has $N \cap \nabla(A+\infty)^\omega \neq \emptyset$, $E \subseteq^*$ dom

$\underline{x}_A$. We may assume $g \restriction E =^* \underline{x}_A \restriction E$. Hence, $\exists^\infty n \in$ dom $(\underline{x}_A \setminus f)$ with $\underline{x}_A(n) \in F(n)$. So $\overline{\underline{x}_A} \notin N(f, F)$, a contradiction.

Thus, in all cases, $N(\underline{x}, F) \cap N(g, G) = \emptyset$, so $N(\underline{x}, F) \cap \bigcup \mathcal{V} = \emptyset$. $\square$

**Lemma 3.3.** *Suppose $B \supseteq \bigcup_{n<\omega} A_n$, each $A_n \subset A_{n+1}$, each $A_{n+1} \setminus A_n$ infinite, and there are basic families $\mathcal{V}_n$ in $\nabla(B + \infty)^\omega$ where each $\mathcal{V}_n$ is pairwise disjoint, basic for $A_n$, and each $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$. Let $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$. Then $\bigcup \mathcal{V}$ is closed in $\nabla(B + \infty)^\omega$.*

*Proof.* By Lemma 3.2, each $\bigcup \mathcal{V}_n$ is closed. By Fact 1.3, $\bigcup \mathcal{V}$ is closed. $\square$

**Lemma 3.4.** *If $A \subset B$, $\nabla(A+\infty)^\omega$ is basic ultraparacompact, and $|B| = |A|$, then, for every open cover $\mathcal{U}$ of $\nabla(B+\infty)^\omega$, there is a basic pairwise disjoint refinement $\mathcal{V}_1 \cup \mathcal{V}_2$ of $\mathcal{U}$ covering $\nabla(B + \infty)^\omega$ where $\mathcal{V}_1$ is basic for $A$.*

*Proof.* First note that, because of the uniform way we have defined basic open sets, if $|A| = |B|$ and $\nabla(A + \infty)^\omega$ is basic ultraparacompact, then so is $\nabla(B + \infty)^\omega$.

Given $\mathcal{U}$ a cover of $\nabla(B + \infty)^\omega$, by Lemma 3.1, we construct a basic pairwise disjoint $\mathcal{V}_1$ refining $\mathcal{U}$, covering $\nabla(A+\infty)^\omega$, and if $N \in \mathcal{V}_1$, then $N \cap \nabla(A + \infty)^\omega \neq \emptyset$. By Lemma 3.2, $\bigcup \mathcal{V}_1$ is closed in $\nabla(B + \infty)^\omega$. Construct a basic cover $\mathcal{U}_1$ of $\nabla(B + \infty)^\omega \setminus \bigcup \mathcal{V}_1$ refining $\mathcal{U}$ so that if $N \in \mathcal{U}_1$, then $N \cap \bigcup \mathcal{V}_1 = \emptyset$.

Since $|B| = |A|$ and $\bigcup \mathcal{V}_1$ is clopen, there is $\mathcal{V}_2$, a basic pairwise disjoint refinement of $\mathcal{U}_1$ covering $\nabla(B+\infty)^\omega \setminus \bigcup \mathcal{V}_1$. Then $\mathcal{V}_1 \cup \mathcal{V}_2$ is the desired basic disjoint refinement of $\mathcal{U}$ covering $\nabla(B + \infty)^\omega$. $\square$

**Lemma 3.5.** *Suppose $B \supseteq \bigcup_{n<\omega} A_n$, each $A_n \subset A_{n+1}$ and $|B| = |A_n|$ for some $n$. If $\nabla(A_n+\infty)^\omega$ is basic ultraparacompact, then there is a sequence $\{\mathcal{V}_n : n \leq \omega\}$ where if $n < \omega$, then $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ and $\bigcup_{i\leq n} \mathcal{V}_i$ is a basic pairwise disjoint cover of $\nabla(A_n + \infty)^\omega$ refining $\mathcal{U}$, $\bigcup \mathcal{V}_\omega \cap \bigcup_{n<\omega} \mathcal{V}_n = \emptyset$, and $\bigcup_{n\leq\omega} \mathcal{V}_n$ covers $\nabla(B + \infty)^\omega$.*

*Proof.* Let $\mathcal{U}$ be an open cover of $\nabla(B + \infty)^\omega$. Using the technique of Lemma 3.4, we construct $\mathcal{V}_n, n < \omega$ by induction. Let $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$. By Lemma 3.3, $\bigcup \mathcal{V}$ is closed. Let $\mathcal{U}_1$ be a basic refinement of $\mathcal{U}$ with $\bigcup \mathcal{U}_1 \cap \bigcup \mathcal{V} = \emptyset$. Since $|B| = |A_n|$ for some $n$, we can find a basic pairwise disjoint refinement $\mathcal{V}_\omega$ of $\mathcal{U}_1$ covering $\nabla(B + \infty)^\omega \setminus \bigcup \mathcal{V}$. Then $\mathcal{V} \cup \mathcal{V}_\omega$ is the desired basic disjoint refinement of $\mathcal{U}$ covering $\nabla(B + \infty)^\omega$. $\square$

**Lemma 3.6.** *Suppose $\kappa$ has uncountable cofinality and $B = \bigcup_{\alpha<\kappa} A_\alpha$ where if $\alpha < \beta < \kappa$, then $A_\alpha \subset A_\beta$. If each $\nabla(A_\alpha + \infty)^\omega$ is basic ultraparacompact, so is $\nabla(B + \infty)^\omega$.*

*Proof.* Let $\mathcal{U}$ be an open cover of $\nabla(B+\infty)^\omega$. By induction, construct $\mathcal{V}_\alpha$ basic pairwise disjoint refinements of $\mathcal{U}$ so that if $\alpha<\beta$, then $\mathcal{V}_\alpha \subset \mathcal{V}_\beta$ and $\mathcal{V}_\alpha$ is basic for $A_\alpha$. If $\beta$ has uncountable cofinality, then at stage $\beta$, we define $\mathcal{V}_\beta = \bigcup_{\alpha<\beta} \mathcal{V}_\alpha$. This works because $\nabla(A_\beta+\infty)^\omega = \bigcup_{\alpha<\beta}\nabla(A_\alpha+\infty)^\omega$. □

**Theorem 3.7.** *Assume $\nabla(\omega+1)^\omega$ is basic ultraparacompact. If $|A|<\aleph_\omega$, then $\nabla(A+\infty)^\omega$ is basic ultraparacompact.*

*Proof.* Without loss of generality, $|A|=\kappa<\aleph_\omega$ and $\kappa>\omega$. Let $\Lambda=\{\delta_\alpha : \alpha<\kappa\}$. List the limit ordinals below $\kappa$ in increasing order. By hypothesis, if $\alpha$ has countable cofinality and $\alpha>\omega$, then for some $\beta<\alpha$, $|\delta_\alpha|=|\delta_\beta|$, so by induction and lemmas 3.4, 3.5, and 3.6, each $\nabla(\delta_\alpha+\infty)^\omega$ is basic ultraparacompact. By Lemma 3.6, $\nabla(\kappa+\infty)^\omega$ is basic ultraparacompact. □

## 4. Not Hereditarily Normal

Kunen [7] showed that $\square(\omega+1)^\omega$ is not hereditarily normal. This is a fundamental difference from $\nabla(\omega+1)^\omega$, since we know of no models in which $\nabla(\omega+1)^\omega$ fails to be hereditarily ultraparacompact. By contrast, in this section, we show that if all but finitely many $\kappa_n>\omega_1$, then $\nabla_{n<\omega}(\kappa_n+\infty)$ is not hereditarily normal. We do not know the consistency of the (non-)hereditary normality of $\nabla(\aleph_1+\infty)^\omega$.

Since $\nabla(\omega_2+\infty)$ is a closed subset of any $\nabla_{n<\omega}(\kappa_n+\infty)$ in which all but finitely many $\kappa_n>\omega_1$, it suffices to prove that $\nabla(\omega_2+\infty)$ is not hereditarily normal.

**Theorem 4.1.** $\nabla(\omega_2+\infty)\setminus\{\overline{\infty}\}$ *is not normal.*

*Proof.* Let $a$ be an infinite co-infinite subset of $\omega$. Let $H=\{x\in\nabla(\omega_2+\infty)^\omega : \text{dom } \underline{x}=a, \exists\alpha<\omega_2\, \forall^\infty n\in a\; x(n)=\alpha\}$. We write $x_\alpha=$ the unique elements in $H$ so that $\forall^\infty n\in a\; x(n)=\alpha$. Let $K=\{y\in\nabla(\omega_2+\infty)^\omega$: if $i<j$ and $i,j\in\text{dom } \underline{y}$, then $y(i)<y(j)\}$.

$H\cap K=\emptyset$ and cl $H\,\cap$ cl $K=\{\overline{\infty}\}$.

Fix $U$ open with $H\subset U$ and $V$ open with $K\subset V$. We may assume $U=\bigcup\mathcal{U}$ where $\mathcal{U}=\{u_x : x\in H\}$ and each $u_x=N(\underline{x},F_x)$. We may assume $V=\bigcup\mathcal{V}$ where $\mathcal{V}=\{v_y : y\in K\}$ and each $v_y=N(\underline{y},G_y)$. We show $U\cap V\neq\emptyset$.

For each $\alpha<\omega_2$, write $F_\alpha=\bigcup_{n<\omega}F_{x_\alpha}(n)$. Each $F_\alpha$ is countable. So there is a sequence $\{\delta_\gamma : \gamma<\omega_1\}$ where if $\forall^\infty n\in\text{dom } \underline{x}\; \underline{x}(n)=\alpha<\delta_\gamma$, then $\bigcup_{n<\omega}F_x(n)\subset\delta_\gamma$. Let $\delta=\sup\{\delta_\gamma : \gamma<\omega_1\}\in[\omega_1,\omega_2)$. Let $H_\delta=\{x\in H : \forall^\infty n$ if $x(n)\neq\infty_n$ then $x(n)<\delta\}$.

Let $y\in K$ with dom $\underline{y}\cap a=\emptyset$ and $\forall^\infty n\in\text{dom } \underline{y}\; \underline{y}(n)>\delta$. There is $x\in H_\delta$ with range $x\cap\bigcup_{n<\omega}G_y(n)=\emptyset$. Then $v_y\cap u_x\neq\emptyset$. □

## 5. An Alternate Proof of a Theorem of Williams

**Definition 5.1.** $w(X) = \inf\{\kappa : \exists \mathcal{B}$ a base of open sets for $X$ with $|\mathcal{B}| = \kappa\}$.

In [11, Theorem 5.7(i)], Williams proves the following theorem.

**Theorem 5.2.** *If $\mathfrak{d} = \omega_1$ and each $X_n$ is a compact $T_3$ space with $w(X_n) \leq \omega_1$, then $\nabla_{n<\omega} X_n$ is ultraparacompact.*[4]

Both Williams' proof and our proof proceed by showing that, under the hypothesis of Theorem 5.2, $\nabla_{n<\omega} X_n$ is $\omega_1$-metrizable; $\kappa$-metrizability implies paracompactness (see [3]).

**Definition 5.3.** $X$ is *$\kappa$-metrizable* if and only if every point $x$ has a neighborhood base $\mathcal{V}_x = \{V_{x,\alpha} : \alpha < \kappa\}$ where, for each $x$ and $y$ and for each $\beta \geq \alpha$, if $y \notin V_{x,\alpha}$, then $V_{x,\alpha} \cap V_{y\beta} = \emptyset$, and if $y \in V_{x,\alpha}$, then $V_{y,\beta} \subseteq V_{x,\alpha}$.[5]

Williams proved $\omega_1$-metrizability by using uniformities, necessitating familiarity with machinery which is not in everyone's toolbox. Here we give an elementary submodel proof. Elementary submodels are also not in everyone's toolbox, but hopefully they are in the toolboxes of a large proportion of people unfamiliar with uniformities.

First, some preliminaries.

**Definition 5.4.** $\mathfrak{d} = \inf\{\kappa :$ there exists $\mathcal{F} \subset \omega^\omega$ $|\mathcal{F}| = \kappa$ and for all $g \in \omega^\omega$, there exists $f \in g$ $g \leq^* f\}$.[6]

The theorem below is a corollary to the proof of Theorem 10.6 in [3].

**Theorem 5.5.** *If $\mathfrak{d} = \omega_1$ and each $X_n$ is compact and pseudometrizable, then $\nabla_{n<\omega} X_n$ is $\omega_1$-metrizable.*[7]

Note that a regular space of countable weight is pseudometrizable. Using countable models, we will define an inverse limit sequence $\{X_\alpha : \alpha < \omega_1\}$ of compact completely regular spaces of countable weight, so each $X_\alpha$ will be pseudometrizable and Theorem 5.5 will apply.

**Definition 5.6.** Let $X$ be a completely regular space and $X \in M$ where $M$ is a model of enough set theory. For $x, y \in X$, define $x \approx_M y$ if and only if $\forall f \in C(X) \cap M$ $f(x) = f(y)$. $X/M$ is the quotient space defined by $\approx_M$.

---

[4]This theorem is stated incorrectly in [11] due to a typographical error: "$\mathfrak{b}$" was written instead of "$\mathfrak{d}$."

[5]This is not the usual definition of $\kappa$-metrizability, but it is equivalent if cf $\kappa > \omega$.

[6]$g \leq^* f$ if and only if $\forall^\infty n$ $g(n) < f(n)$.

[7]This theorem was originally stated for compact metrizable spaces, but the proof goes through for pseudometrizable.

Every completely regular space $X$ has a base $\mathcal{B} = \{f^{\leftarrow}(p,q) : f \in C(X), p,q \in \mathbb{Q}\}$, and if $w(X) = \kappa$, then there is $\mathcal{F} \subset C(X)$ with $|\mathcal{F}| = \kappa$ and a base $\mathcal{B} = \{f^{\leftarrow}(p,q) : f \in \mathcal{F}, p,q \in \mathbb{Q}\}$. We will call such an $\mathcal{F}$ a generating family.

**Definition 5.7.** $X(M)$ is the topology on $X$ where $u$ is open if and only if there exists $v$ open in $X/M$ $u = \{x : x/M \in v\}$.

Note that $X(M)$ is generally very badly not Hausdorff.

**Fact 5.8.** *If $X$ is completely regular, $X \in M$ where $M$ is a model of enough set theory, then $X/M$ is completely regular.*

*Proof.* Let $X$ and $M$ be as in Definition 5.6 and suppose $x/M \notin H/M$ where $H/M$ is closed in $X/M$. We may assume $H \in M$. By elementarity, there is $f \in C(X) \cap M, r,s \in \mathbb{Q}$ with $f(x) \in (r,s)$, and for all $h \in H$, $f(h) \notin (r,s)$. □

Note that if $X/M$ is completely regular, so is $X(M)$.

With these preliminaries, here is the proof of Theorem 5.2.

*Proof.* Without loss of generality, we consider $\nabla X^\omega$ where $w(X) = \omega_1$. Let $\mathcal{F}$ be a generating family for the topology on $X$, $|\mathcal{F}| = \omega_1$.

Let $\{M_\alpha : \alpha < \omega_1\}$ be an increasing sequence of countable models of enough set theory with $X, \mathcal{F} \in M_0$, $\mathcal{F} \subset \bigcup_{\alpha<\omega_1} M_\alpha$. Let $X_\alpha = X(M_\alpha)$. Each $X_\alpha$ is completely regular and has a countable base, so each $X_\alpha$ is pseudometrizable. By Theorem 5.5, each $\nabla(X_\alpha)^\omega$ is $\omega_1$-metrizable. Let $\{V_{x,\beta,\alpha} : \beta < \omega_1, x \in \nabla(X_\alpha)^\omega\}$ witness this.

For each $\alpha, \beta < \omega_1$, let $T_{\beta,\alpha} = \{V_{x,\beta,\alpha} : x \in \nabla(X_\alpha)^\omega\}$. Each $T_{\beta,\alpha}$ covers $\nabla X^\omega$.

For $x \in \nabla X^\omega$, let $W_{x,\alpha} = \bigcap_{\beta\leq\alpha} V_{x,\beta,\alpha}$. Then each $W_{x,\alpha}$ is open, and $\{W_{x,\alpha} : \alpha < \omega_1\}$ witnesses the $\omega_1$-metrizability of $\nabla X^\omega$ □

## 6. Open Questions

The following questions, closely related to the results of this paper, remain open.

**Question 6.1.** Without CH, is $\nabla(\kappa + \infty)^\omega$ consistently paracompact for $\kappa \geq \aleph_\omega$?

**Question 6.2.** Is $\nabla(\aleph_1 + \infty)^\omega$ hereditarily normal? Consistently hereditarily normal?

A question which deserves more notice than it has gotten relates to a result of William G. Fleissner and Adrienne M. Stanley [4]. First, their result.

**Theorem 6.3.** *The box product of arbitrarily many scattered spaces of Cantor-Bendixson height 2 is a $D$-space. (A scattered space $X$ has height 2 if and only if $X \setminus I$ is discrete, where $I$ is the set of isolated points of $X$.)*

"$D$-space" is a covering property: $X$ is a $D$-*space* if and only if for every $\{u_x : x \in X\}$ where $x \in u_x$ is open, there is a closed discrete set $Y$ so that $\{u_y : y \in Y\}$ covers $X$.

In particular, Fort spaces have Cantor-Bendixson height 2, so Theorem 6.3 has as a corollary that the box product of arbitrarily many Fort spaces is a $D$-space. For a survey of $D$-spaces, see [6]. In particular, it is not known whether every ultraparacompact space is a $D$-space, which leads to the following question.

**Question 6.4.** Is there a scattered space of finite height with a box product which fails to be a $D$-space? What about compact scattered spaces of arbitrary height?

**Question 6.5.** Is every ultraparacompact space a $D$-space?

Finally, going back to Definition 2.4, we have the following question.

**Question 6.6.** Is $\neg\Delta$ consistent?

## References

[1] Tomek Bartoszynski and Haim Judah, *Set Theory: On the Structure of the Real Line*. Wellesley, MA: A. K. Peters, Ltd., 1995.

[2] Eric K. van Douwen, *The integers and topology* in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 111–167.

[3] ———, *Covering and separation properties of box products* in Surveys in General Topology. Ed. George M. Reed. New York-London-Toronto, Ont.: Academic Press, 1980. 55–129.

[4] William G. Fleissner and Adrienne M. Stanley, *D-spaces*, Topology Appl. **114** (2001), no. 3, 261–271.

[5] M. K. Fort, Jr., *Solution to Problem 4577 [1954, 126]*, Amer. Math. Monthly **62** (1955), no. 5, 372–373.

[6] Gary Gruenhage, *A survey of D-spaces* in Set Theory and Its Applications. Ed. L. Babinkostova, A. E. Caicedo, S. Geschke, and M. Scheepers. Contemporary Mathematics, 533. Providence, RI: Amer. Math. Soc., 2011. 13–28.

[7] K. Kunen, *Some comments on box products* in Infinite and Finite Sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II. Ed. A. Hajnal, R. Rado, and Vera T. Sós. Colloquia Mathematica Societatis János Bolyai, Vol. 10. Amsterdam: North-Holland, 1975. 1011–1016.

[8] ———, *Paracompactness of box products of compact spaces*, Trans. Amer. Math. Soc. **240** (1978), 307–316.

[9] Judith Roitman, *Paracompactness and normality in box products: Old and new* in Set Theory and Its Applications. Ed. L. Babinkostova, A. E. Caicedo, S. Geschke, and M. Scheepers. Contemporary Mathematics, 533. Providence, RI: Amer. Math. Soc., 2011. 157–181.

[10] Mary Ellen Rudin, *The box product of countably many compact metric spaces*, General Topology and Appl. **2** (1972), 293–298.

[11] Scott W. Williams, *Box products* in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 169–200.

Department of Mathematics; University of Kansas; Lawrence, KS 66049
*E-mail address*: roitman@math.ku.edu