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by

Alejandro Ramírez-Páramo

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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THE κ-CLOSURE AND CARDINAL INEQUALITIES

ALEJANDRO RAMÍREZ-PÁRAMO

To Devora Melendez-Ramirez

Abstract. We show that for every T_1-space X such that
(1) $\psi(X) \leq 2^\kappa$;
(2) if $|A| \leq 2^\kappa$, then $|cl_\kappa(A)| \leq 2^\kappa$; and
(3) if $\mathcal{U}$ is a collection of open subsets of X and A is a κ-closed subset of X and $A \subseteq \bigcup \mathcal{U}$, then there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$, such that $A \subseteq \bigcup\{\Phi(V) : V \in \mathcal{V}\}$,

then $|X| \leq 2^\kappa$. This result is a common generalization of three cardinal inequalities recently published by Santi Spadaro (Topology Appl. **158** (2011), 2091–2093), Angelo Bella (Topology Appl. **159** (2012), 3640–3643), and Filippo Cammaroto, Andrei Catalioto, and Jack Porter, (Topology Appl. **160** (2013), 137–142).

1. Introduction

Among the best-known theorems concerning cardinal functions are those which give an upper bound on the cardinality of a space in terms of other cardinal invariants, for example, the well-known Arhangel'skii inequality:

For every Hausdorff space X, $|X| \leq 2^{L(X)\chi(X)}$.

Arhangel'skii's theorem above gave a boost to the area of cardinal invariants in topology by inspiring the development of new techniques and the discovery of refinements and variations.

Currently, there is a wide range of generalizations and variations of this result (a very good survey of Arhangel'skii's theorem is R. Hodel's paper [7]). For instance, recently, Santi Spadaro and István Juhász independently obtained a generalization of the inequality in question. They

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proved that if X is a T_2 space, then $|X| \leq 2^{L(X)F(X)\psi(X)}$ (see [10]). Later, Angelo Bella [2] improved the inequality of Juhász and Spadaro, establishing that $|X| \leq 2^{L(X)F_c(X)\psi(X)}$ (notation and definitions are given below) for every Hausdorff space X. More recently, Filippo Cammaroto, Andrei Catalioto, and Jack Porter [5] bettered Bella's inequality (and the Bella-Cammaroto inequality $|X| \leq 2^{aL_c(X)\psi_c(X)t(X)}$, for every Hausdorff space X), with the next: If X is a Hausdorff space, then $|X| \leq 2^{aL_\kappa(X)\psi_c(X)}$.

In this paper, following the ideas of Spadaro [10], Bella [2], and Cammaroto, Catalioto, and Porter [5], we will establish a common generalization of these four inequalities.

2. Notation and Definitions

For a topological space X and a subset A of X, we denote by $\overline{A}$ the closure of A in X. For any set X and an infinite cardinal κ, $[X]^{\leq\kappa}$ denotes the collection of all subsets of X with cardinality $\leq \kappa$; $[X]^{<\kappa}$ and $[X]^{\kappa}$ are defined analogously.

We refer the reader to [6] for definitions and terminology on cardinal functions not explicitly given here. We assume that the reader is familiar with the cardinal functions netweight, Lindelöf degree, character, pseudocharacter, closed pseudocharacter, and tightness, which will be denoted by the symbols nw, L, χ, ψ, ψ_c, and t, respectively.

For $Y \subseteq X$, the *almost Lindelöf degree of Y relative to X*, denoted by $aL(Y, X)$ (see [5]), is the smallest infinite cardinal κ such that for every open cover $\mathcal{U}$ of Y, by open subsets of X, there is a subcollection $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $Y \subseteq \bigcup\overline{\mathcal{V}} = \bigcup\{\overline{V} : V \in \mathcal{V}\}$. The *almost Lindelöf degree of* X, denoted by $aL(X)$, is $aL(X, X)$. The *almost Lindelöf degree relative to closed subsets of* X, denoted by $aL_c(X)$, is $\sup\{aL(C, X) : C \text{ is a closed subset of } X\}$. The κ-*almost Lindelöf degree of* X, denoted by $aL_\kappa(X)$, is $\sup\{aL(C, X) : C \text{ is a } \kappa\text{- closed subset of } X\}$.

We shall use the notation and terminology employed in [1] and [9]. For the convenience of the reader, we repeat some of the definitions contained in these papers.

Definition 2.1. Let X be a nonempty set and let τ and κ be infinite cardinals. An operator $c : \mathcal{P}(X) \to \mathcal{P}(X)$ will be called a (τ, κ)-*closure* if

(1) $A \subseteq c(A)$, for every $A \in \mathcal{P}(X)$;
(2) if $A \subseteq B$, then $c(A) \subseteq c(B)$, for every $A, B \in \mathcal{P}(X)$; and
(3) if $|A| \leq \tau^\kappa$, then $|c(A)| \leq \tau^\kappa$, for each $A \in \mathcal{P}(X)$.

If the operator $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies (1) and (3) only, we say that c is a *quasi*-(τ, κ)-*closure* operator.

Remark 2.2. It is clear that if $\kappa^+ = \tau$, then $\tau^\kappa = 2^\kappa$; so in this case, (3) establishes that

If $A \in \mathcal{P}(X)$ is such that $|A| \leq 2^\kappa$, then $|c(A)| \leq 2^\kappa$.

Clearly, every (τ, κ)-closure operator is a quasi-(τ, κ)-closure operator, but the converse is not true (see [9]).

Let X be a set and let Y be a nonempty subset of X. In what follows, let τ and κ be infinite cardinals such that $\kappa < cf(\tau)$, and let $\mu = \tau^\kappa$. We also denote $\mathcal{L} = [Y]^{\leq \mu}$.

A *τ-long increasing sequence in* $\mathcal{L}$ is a transfinite sequence $\{F_\alpha : \alpha < \tau\}$ of elements of $\mathcal{L}$ such that $F_\alpha \subseteq F_\beta$ if $\alpha < \beta < \tau$. A *τ-long sequence in* $\mathcal{L}$ is a transfinite sequence $\{F_\alpha : \alpha < \tau\}$ of elements of $\mathcal{L}$.

A *sensor* is a pair $(\mathcal{A}, \mathcal{F})$, where $\mathcal{A}$ is a family of subsets of Y and $\mathcal{F}$ is a collection of families of subsets of X.

We assume that with each sensor $s = (\mathcal{A}, \mathcal{F})$ a subset $\Theta(s)$ of X is associated, called *the* Θ-*closure of* s. If, in an argument below, we define the Θ-closure only for sensors of a particular kind, we mean that only such sensors are effectively involved in the argument.

Definition 2.3. A sensor $s = (\mathcal{A}, \mathcal{F})$ will be called *small* if

(1) $|\mathcal{A}| \leq \kappa$ and $|A| \leq \kappa$ for every $A \in \mathcal{A}$,
(2) $|\mathcal{F}| \leq \kappa$ and $|\mathcal{C}| \leq \kappa$ for every $\mathcal{C} \in \mathcal{F}$, and
(3) $Y \setminus \Theta(s) \neq \emptyset$.

Let H be a subset of Y and let $\mathcal{G}$ be a family of subsets of X. A sensor $(\mathcal{A}, \mathcal{F})$ is said to be *generated by the pair* $(H, \mathcal{G})$ if $A \subseteq H$ for each $A \in \mathcal{A}$ and $\mathcal{C} \subseteq \mathcal{G}$ for each $\mathcal{C} \in \mathcal{F}$.

Let $\mathcal{Q}$ be the set of all families $\mathcal{G}$ of subsets of X such that $|\mathcal{G}| \leq \mu$. If g is a mapping of $\mathcal{L}$ into $\mathcal{Q}$ and $\mathcal{E} \subseteq \mathcal{L}$, define $\mathcal{U}_g(\mathcal{E}) = \bigcup\{g(F) : F \in \mathcal{E}\}$.

Let g be a mapping of $\mathcal{L}$ into $\mathcal{Q}$ and let $\mathcal{E}$ be a subfamily of $\mathcal{L}$. A sensor s will be called *good* for $\mathcal{E}$ if it is generated by the pair $(\cup\mathcal{E}, \mathcal{U}_g(\mathcal{E}))$ and $\cup\mathcal{E} \subseteq \Theta(s)$.

A *propeller* (*quasi-propeller*), with respect to g and Θ, in $\mathcal{L}$ is a τ-long increasing sequence (τ-long sequence) $\mathcal{E}$ in $\mathcal{L}$ such that no small sensor s is good for $\mathcal{E}$ (for a quasi-propeller which is not a propeller, see [9]).

There are several cardinal inequalities which have a common construction inspired by Arhangel'skii's proof to the inequality (see [7]): If $X \in T_2$, then $|X| \leq 2^{L(X)\chi(X)}$. This suggests the general problem of finding a result which captures this common core. In [1, Theorem 1], Arhangel'skii establishes a result of this type. However, as he comments in [1, Page 322, 12], it is not true that all important cardinal inequalities can be proven just by following the algorithm described in this theorem, and, he also states that he does not know a proof for Gryzlov's inequality: $|X| \leq 2^{\psi(X)}$ $X \in T_1$.

In [9], we formulate (following the ideas in [1]) the next generalization of [1, Theorem 1], which also provides an algorithm for proving a wide range of cardinal inequalities and relative versions of cardinal inequalities, including Gryzlov's inequality noted above.

Theorem 2.4 ([9])**.** *Let X be a set, let Y be a nonempty subset of X, and let τ and κ be infinite cardinals such that $\kappa < cf(\tau)$. If $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is a quasi-(τ, κ)-closure operator, then for every function $g : \mathcal{L} \to \mathcal{Q}$, there exists a family $\{E_\alpha : \alpha \in \tau\} \subseteq \mathcal{L}$, such that*

1. *for each $0 < \alpha < \tau$, $\bigcup\{c(E_\beta) \cap Y : \beta < \alpha\} \subseteq E_\alpha$, and*
2. *$\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha \in \tau\}$ is a quasi-propeller in $\mathcal{L}$.*

Note that if in the above theorem, the operator c is a (τ, κ)-closure operator, then (from (2)) we have that $\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha \in \tau\}$ is a propeller in $\mathcal{L}$.

3. The Main Result

The proofs of the four inequalities mentioned in the introduction were originally obtained with different techniques. For instance, Spadaro [10] employed elementary submodels to show that $|X| \leq 2^{L(X)F(X)\psi(X)}$ for every Hausdorff space X. It is possible to use Theorem 2.4 to prove each one of these four inequalities, but instead of doing this, we will employ Theorem 2.4 to give a common generalization for them. But before we do this, we need a couple of definitions; the first is well known.

Definition 3.1. Let X be a topological space, and $Y \subseteq X$, the *κ-closure of Y in X*, denoted by $cl_\kappa(Y)$ or $[Y]_\kappa$, is the set $\bigcup\{\overline{D} : D \in [Y]^{\leq\kappa}\}$.

The next notion was introduced by Hodel in [8].

Definition 3.2. Let (X, τ) be a topological space. We will call a function $\Phi : \tau \to \mathcal{P}(X)$ a *Hodel operator* if this satisfies the following conditions for every open sets U and V in X:

1. $V \subseteq \Phi(V)$.
2. If $U \subseteq V$, then $\Phi(U) \subseteq \Phi(V)$.

We are in position to prove the main result of this paper.

Theorem 3.3. *Let X be a T_1-space, let κ be an infinite cardinal, and let Φ be a Hodel operator such that*

1. *$\psi(X) \leq 2^\kappa$;*
2. *for every $A \in \mathcal{P}(X)$, if $|A| \leq 2^\kappa$, then $|cl_\kappa(A)| \leq 2^\kappa$; and*
3. *if $\mathcal{U}$ is a collection of open subsets of X and A is a κ-closed subset of X and $A \subseteq \bigcup\mathcal{U}$, then there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$, such that $A \subseteq \bigcup\{\Phi(V) : V \in \mathcal{V}\}$.*

Then $|X| \leq 2^\kappa$.

Proof. Let $\tau = \kappa^+$ and let $\mu = 2^\kappa$. For every $x \in X$, let $\mathcal{B}_x$ be a local pseudobase of x in X with $|\mathcal{B}_x| \leq 2^\kappa$. For each $F \in \mathcal{L} = [X]^{\leq \mu}$, $g(F) = \bigcup\{\{\Phi(V) : V \in \mathcal{B}_x\} : x \in cl_\kappa(F)\}$, and for every sensor $s = (\emptyset, \mathcal{F})$, we put $\Theta(s) = \bigcup\{\bigcup \mathcal{C} : \mathcal{C} \in \mathcal{F}\}$. Let $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be defined as $c(A) = cl_\kappa(A)$. Since c is a (κ^+, κ)-closure operator, by Theorem 2.4 there exists a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$, such that for every $0 < \alpha < \kappa^+$, $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$, and $\mathcal{E} = \{c(E_\alpha) : \alpha \in \kappa^+\}$ is a propeller in $\mathcal{L}$. Let $H = \bigcup \mathcal{E}$ and note that $H = c(H)$.

Moreover, since $|H| = |\bigcup\{c(E_\alpha) : \alpha \in \kappa^+\}|$, using (2), we have $|H| \leq 2^\kappa$.

Claim. $X \subseteq H$.

Suppose not and let $p \in X \setminus H$. For each $x \in H$, let $V_x \in \mathcal{B}_x$ such that $p \notin V_x$. It is clear that collection $\{V_x : x \in H\}$ covers H; hence, by (3), there exists $H' \in [H]^{\leq \kappa}$ such that $H \subseteq \bigcup\{\Phi(V_x) : x \in H'\}$. Let $\mathcal{C} = \{\Phi(V_x) : x \in H'\}$, let $\mathcal{F} = \{\mathcal{C}\}$, and let $s = (\emptyset, \mathcal{F})$. It is clear that $p \notin \Theta(s)$ while $H \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$, which is a contradiction. Thus, $X \subseteq H$. □

Now, we will use Theorem 3.3 to deduce the four inequalities mentioned in our introduction. Before presenting the first of them, we need to make the following observation.

In corollaries 3.4–3.7, condition (2) in Theorem 3.3 follows from the next well-known facts for every Hausdorff space X:

(1) $\psi_c(X) \leq L(X)\psi(X)$ (see, for example, [7]).
(2) Let κ be an infinite cardinal. If $\psi_c(X) \leq \kappa$ and A is a subset of X such that $|A| \leq 2^\kappa$, then $|cl_\kappa(A)| \leq |A|^{\psi_c(X)}$ (see [5, Proposition 2(c)]).

Corollary 3.4 (Spadaro-Juhász (see [10])). *Let X be a Hausdorff space, then $|X| \leq 2^{L(X)F(X)\psi(X)}$.*

Proof. Let $\kappa = L(X)F(X)\psi(X)$ and consider the Hodel operator $\Phi(V) = V$ for every open set V.

Notice that $L(X)F(X) \leq \kappa$ implies $L(A) \leq \kappa$ for every κ-closed $A \subseteq X$ (see [5, Proposition 2(f)]). Thus, by Theorem 3.3, $|X| \leq 2^{L(X)F(X)\psi(X)}$. □

In the proofs of corollaries 3.5–3.8, we apply Theorem 3.3 with the following choice of Hodel operator: $\Phi(V) = \overline{V}$ for every open set V.

Corollary 3.5 (Bella [2]). *If X is a T_2-space, then $|X| \leq 2^{L(X)F_c(X)\psi(X)}$.*

Proof. Let $\kappa = L(X)F_c(X)\psi(X)$. Since $L(X)F_c(X) \leq \kappa$ implies $aL_\kappa(X) \leq \kappa$ (see [2] or [5, Proposition 2(g)]), condition (3) in Theorem 3.3 holds. Hence, by Theorem 3.3, $|X| \leq 2^{L(X)F_c(X)\psi(X)}$. □

Corollary 3.6 (Cammaroto-Catalioto-Porter [5])**.** *If X is a Hausdorff space, then* $|X| \leq 2^{aL_\kappa(X)\psi_c(X)}$.

Proof. Let $\kappa = aL_\kappa(X)\psi_c(X)$. It is not difficult to check that the hypotheses in Theorem 3.3 are satisfied. □

Corollary 3.7 (Bella-Cammaroto [3])**.** *If X is a Hausdorff space, then* $|X| \leq 2^{aL_c(X)\psi_c(X)t(X)}$.

Proof. Let $\kappa = aL_c(X)\psi_c(X)t(X)$. Clearly, we need only to check that condition (3) in Theorem 3.3 holds, but this follows from [5, Proposition 2(e)]. □

The following corollary follows easily from Theorem 3.3.

Corollary 3.8. *Let X be a Hausdorff space with $\psi_c(X) \leq \kappa$. If $L(A) \leq \kappa$ for every κ-closed $A \subseteq X$, then $|X| \leq 2^\kappa$.*

Finally, in [7], Hodel asks the following question.

Question 3.9. Let X be a Lindelöf first countable T_1-space. Can we prove that $|X| \leq 2^\omega$?

This question has been considered by Raushan Z. Buzyakova in [4].

We will use Theorem 3.3 to obtain an affirmative partial answer to Question 3.9.

Corollary 3.10. *Let X be a T_1-space. If X is a monolithic space (for every $A \subseteq X$, $nw(\overline{A}) \leq |A|$), then $|X| \leq 2^{L(X)\chi(X)}$. In particular, every monolithic Lindelöf first countable T_1-space has cardinality $\leq 2^\omega$.*

Proof. Let $\kappa = L(X)\chi(X)$ and consider the Hodel operator $\Phi(V) = V$ for every open set V.

Let $A \in [X]^{2^\kappa}$. Then $|Cl_\kappa(A)| = |\bigcup_{B\in[A]^\kappa} \overline{B}| \leq \sum_{B\in[A]^\kappa} |\overline{B}| \leq 2^\kappa \cdot 2^\kappa = 2^\kappa$, since $|\overline{B}| \leq 2^{nw(\overline{B})} \leq 2^{|B|} \leq 2^\kappa$ where the second inequality follows from the fact that X is monolithic and the first inequality follows from $|Y| \leq 2^{nw(Y)}$, which is true for every T_0 space Y (see [6]). Thus, condition (2) in the Theorem 3.3 holds.

Moreover, $t(X) \leq \kappa$ implies that every κ-closed subset of X is closed in X; hence, as $L(X) \leq \kappa$, we have that for every κ-closed subset A of X, $L(A) \leq \kappa$. Thus, condition (3) in Theorem 3.3 holds. Therefore, $|X| \leq 2^{L(X)\chi(X)}$. □

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Facultad de Ciencias de la Electrónica; Universidad Autónoma de Puebla; Av. San Claudio y 18 sur s.n, Col. Jardines de San Manuel; c.p. 72570, Puebla, Pue.,México

E-mail address: aparamo@ece.buap.mx