TOPOLOGY PROCEEDINGS
Volume 44, 2014
Pages 223–231
http://topology.auburn.edu/tp/

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Electronically published on September 26, 2013

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 44 (2014)
Pages 223-231
E-Published on September 26, 2013

CONNECTED OPEN NEIGHBORHOODS OF SUBCONTINUA OF PRODUCT CONTINUA WITH INDECOMPOSABLE FACTORS

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Abstract. In a product either of Knaster continua or of pseudo-arcs, a continuum W has arbitrarily small connected open neighborhoods if and only if the projection of W to every factor is onto. This is not true for all products of indecomposable continua.

1. Introduction

The general question considered here is the following: Suppose M is a subcontinuum of a product X of at least two nondegenerate indecomposable continua. Under what conditions does it follow that M has arbitrarily small connected open neighborhoods in X? Precisely, when is it true that given any open set $\mathcal{O}$ with $M \subseteq \mathcal{O} \subseteq X$, there exists a connected open set U satisfying $M \subseteq U \subseteq \mathcal{O}$? The particular cases to which we present solutions involve products of Knaster-type indecomposable continua, of solenoids, and of pseudo-arcs. Both positive and negative results are included, along with some open questions.

We did most of this research a number of years ago, but we held off on publishing it in the hope of getting improved results. The only result of this delay was the strengthening of Theorem 4.4 from the case of a product of two pseudo-arcs to all finite products (and, in light of the observation in §5, to all products of pseudo-arcs). Since the research in this paper was done, there has been a lot more progress in this area. Important work, especially, is contained in the papers of Janusz R. Prajs and Keith Whittington [13], [14], and to some extent, in the papers of Karen

2010 *Mathematics Subject Classification.* Primary 54B10, 54F15 ; Secondary 54H11.
Key words and phrases. Knaster continua, product continua, pseudo-arc, solenoid.

Villarreal [15], [16]. The referee has stated that the property of having arbitrarily small connected open neighborhoods, for a subcontinuum of a Kelly continuum, is equivalent to the property of being an ample subcontinuum. The notion of ample subcontinua was introduced by Prajs and Whittington in [13].

2. Definitions and Notation

A *continuum* is a compact connected metric space. A continuum X is *locally connected at* $p \in X$ provided that for every open subset $\mathcal{O}$ of X with $p \in \mathcal{O}$, there exists a connected open set U with $p \in U \subseteq \mathcal{O}$; X is *locally connected* if it is locally connected at every point of itself. The continuum X is *indecomposable* provided that whenever A and B are proper subcontinua of X, $A \cup B$ is a proper subset of X; X is *hereditarily indecomposable* provided that every subcontinuum of X is indecomposable.

A reader unfamiliar with inverse limits may wish to consult W. T. Ingram's book [8]. If $\{X_k; f_k\}_{k\in\mathbb{N}}$ is an inverse sequence of continua and continuous maps, the *inverse limit* of $\{X_k; f_k\}_{k\in\mathbb{N}}$ is the set

$$\{\langle x_k\rangle \in \prod_{k=1}^{\infty} X_k \,|\, \text{for each } k \geq 2, f_k(x_k) = x_{k-1}\}.$$

A *basic open set* in an inverse limit is a set obtained by restricting a single coordinate to an open set in a single factor. (It is not difficult to verify that the collection of such open sets is a base, not merely a subbase.)

Since the theorems herein sometimes involve products of inverse limits, it is convenient to use notation that is not quite standard. When $\{X_k; f_k\}_{k\in\mathbb{N}}$ is an inverse sequence of continua, the multi-step bonding maps will be denoted $f[m,n] : X_m \to X_n$, where $f_k = f[k, k-1]$ for each $k \geq 2$.

When there are finitely many inverse limits under consideration, the $i-$th inverse limit will be denoted using superscripts, $\{X^i_k; f^i_k\}_{k\in\mathbb{N}}$, with the multi-step bonding maps being $f^i[m,n] : X^i_m \to X^i_n$.

A continuous map $f : [0,1] \to [0,1]$ is a *folding map* if and only if there is a subdivision (in the sense of Riemann integration) $\langle x_j\rangle_{j=0}^n$ of $[0,1]$ such that for every j, $0 < j \leq n$, $f|[x_{j-1}, x_j]$ is a homeomorphism of $[x_{j-1}, x_j]$ onto $[0,1]$. A continuum which is an inverse limit of a sequence of arcs $[0,1]$ and folding maps is called a *continuum of Knaster-type*, or simply a *Knaster continuum.*

Any nondegenerate continuum which is homeomorphic to an inverse limit of a sequence of arcs $[0,1]$ and continuous maps is called *chainable* or *arc-like* (an older equivalent term is *snake-like.*) A *pseudo-arc* is a continuum which is both arc-like and hereditarily indecomposable (see [1], [4], [5], [9], [10], [11]).

Suppose that for each i, $1 \leq i \leq n$, $f^i : X^i \to Y^i$ is a continuous map. Then the product mapping $\prod_{i=1}^{n} f^i : \prod_{i=1}^{n} X^i \to \prod_{i=1}^{n} Y^i$ is defined by

$$\big(\prod_{i=1}^{n} f^i\big)\big(\langle x^i\rangle_{i=1}^{n}\big) = \langle f^i(x^i)\rangle_{i=1}^{n},$$

where we are indexing with superscripts instead of subscripts in keeping with our notational convention introduced above.

The unit circle of complex numbers is denoted by S; S is a topological group under complex multiplication. A map which sends each $z \in S$ to z^k for some integer $k \notin \{-1, 0, 1\}$ is a *power mapping.* An inverse limit of a sequence of copies of S and power mappings is a *solenoid.* A solenoid is both an indecomposable continuum and a topological group under term-by-term complex multiplication.

A *face* of the n-dimensional cube $\prod_{i=1}^{n}[0,1]$ is the subset obtained by setting a single coordinate to either 0 of 1; thus $\prod_{i=1}^{n}[0,1]$ has $2n$ faces.

3. Introductory and Background Information

The following results are mostly either well known or easy to prove. They are included here for ease of reference.

Lemma 3.1. *An indecomposable continuum with more than one point is not locally connected at any point.*

Lemma 3.2. *Suppose that X and Y are continua and $p \in Y$. Then $X \times \{p\}$ has arbitrarily small connected open neighborhoods in $X \times Y$ if and only if Y is locally connected at p.*

Lemma 3.3. *Suppose $f = \prod_{i=1}^{n} f^i$ is a product mapping of $\prod_{i=1}^{n}[0,1]$ to itself, where each f^i is a folding map. Suppose A is a connected subset of $\prod_{i=1}^{n}[0,1]$ and suppose A intersects every face of this cube. Then $f^{-1}(A)$ is connected and also intersects every face of $\prod_{i=1}^{n}[0,1]$.*

To make the notation less cumbersome, the letter I, with or without subscripts or superscripts, will henceforth denote the real interval $[0,1]$.

The next lemma is a special case of the well-known, but somewhat vague, statement that inverse limits commute with products.

Lemma 3.4. *If K^i is a Knaster continuum for $1 \leq i \leq n$ and K^i is the inverse limit of $\{I^i_k; f^i_k\}_{k\in\mathbb{N}}$, then $\prod_{i=1}^{n} K^i$ is the inverse limit of $\{\prod_{i=1}^{n} I^i_k; \prod_{i=1}^{n} f^i_k\}_{k\in\mathbb{N}}$.*

This result has the effect of making it possible to represent a point of a product of n inverse limit continua as either an n-tuple of infinite sequences or as an infinite sequence of n-tuples, interchangeably. It is important for our purposes in the case when each f^i_k is a folding map.

The next lemma is more technical, since without compactness an inverse limit of connected sets need not be connected.

Lemma 3.5. *Suppose each K^i is a Knaster continuum, $1 \leq i \leq n$, and suppose $\mathcal{O}$ is a connected open subset of $\prod_{i=1}^{n} I$ and that $\mathcal{O}$ intersects every face of the cube. Let $K^i = \varprojlim\{I^i_k; f^i_k\}$ where the f^i_k are folding maps, and suppose that m is a positive integer. Let $U = \{\langle x_j\rangle_{j=1}^{\infty} \in \prod_{i=1}^{n} K^i | x_m \in \mathcal{O}\}$. Then U is a connected open subset of $\prod_{i=1}^{n} K^i$. (Here, each x_j is a point of the n-cube, and hence is an n-tuple of points of I, making use of Lemma 3.4.)*

Proof. The set U is, by definition, a basic open set in the inverse limit of n-cubes, so only the connectedness of U needs to be established. Suppose $\langle p_k\rangle_{k=1}^{\infty}$ and $\langle q_k\rangle_{k=1}^{\infty}$ are points of U. Then $p_m, q_m \in \mathcal{O}$. Since $\mathcal{O}$ is a connected open subset of a cube, it is arcwise connected, so there is a continuum W_m such that $p_m, q_m \in W_m \subseteq \mathcal{O}$ and W_m intersects every face of $\prod_{i=1}^{n} I^i$. For $k < m$, define $W_k = f[m,k](W_m)$, and for $k > m$, $W_k = (f[k,m])^{-1}(W_m)$. Then each W_k is a continuum by continuity and Lemma 3.3, and W, the inverse limit of the W_k's, is a subcontinuum of $\prod_{i=1}^{n} K^i$. The points $\langle p_k\rangle_{k=1}^{\infty}$ and $\langle q_k\rangle_{k=1}^{\infty}$ belong to W and $W \subseteq U$.

The two points of U chosen were arbitrary, so U is a connected subset of $\prod_{i=1}^{n} K^i$. □

Lemma 3.6. *Let X^i be an indecomposable continuum for each i, $1 \leq i \leq n$, and let W be a subcontinuum of $\prod_{i=1}^{n} X^i$. If for some j, the $j-th$ projection $\pi_j(W)$ is a proper subcontinuum of X_j, then W cannot have arbitrarily small connected open neighborhoods in $\prod_{i=1}^{n} X^i$.*

Proof. Assuming that W and $\prod_{i=1}^{n} X^i$ are as in the hypotheses, note that every connected open subset of the indecomposable continuum X^j is dense. There exists an open set $\mathcal{O} \subseteq X^j$ such that $\pi_j(W) \subseteq \mathcal{O}$ and $\mathcal{O}$ is not dense in X^j. Then the component K of $\mathcal{O}$ containing $\pi_j(W)$ has empty interior in X^j. Thus, $\pi_j^{-1}(K)$ has empty interior in $\prod_{i=1}^{n} X^i$, since π_j is an open map. Therefore, there is no connected open neighborhood U of W in $\prod_{i=1}^{n} X^i$ contained in the open set $\pi_j^{-1}(\mathcal{O})$. □

Lemma 3.7. *Let $\{X_k; f_k\}_{k\in\mathbb{N}}$ be an inverse sequence of continua and continuous mappings. Let X denote the inverse limit and suppose $M \subseteq U \subseteq X$, with M closed in X and U open in X. Then there exists V open in X such that $M \subseteq V \subseteq U$, and V is a basic open set.*

Proof. Let $\{X_k; f_k\}$ be an inverse system with inverse limit X as given. Let $M \subset U \subset X$ and assume M is closed in X and U is open in X. For each $x = \langle x_k\rangle_{k=1}^{\infty} \in M$, let V_x be a basic open set given by $V_x = \{\langle y_i\rangle_{i=1}^{\infty} \,|\, y_{i(x)} \in V_{i(x)}\}$ where $V_{i(x)}$ is an open subset of $X_{i(x)}$, satisfying $x \in V_x \subset U$. The collection $\{V_x\}$ is an open cover of M and so has a finite subcover $\{V_{x_j}\}$. Let $k = \max\{i(x_j)\}_{j=1}^{n}$, and let $f[k, i(x_j)]^{-1}(V_{i(x_j)})$ be called simply $V_{j(k)}$. Let $U_j = \{\langle x_i\rangle_{i=1}^{\infty} \in X \,|\, x_k \in V_{j(k)}\}$, and let $V = \bigcup_{j=1}^{n} U_j$. Then $V = \{\langle x_i\rangle_{i=1}^{\infty} \,|\, x_j \in \bigcup_{k=1}^{n} V_{j(k)}\}$ which is a basic open set, as required. □

Suppose X and Y are continua and $f : X \to Y$ is a continuous map. Then f is called *universal* provided that, for every map $g : X \to Y$, there exists $x \in X$ such that $g(x) = f(x)$.

Lemma 3.8 ([7])**.** *Every map of a continuum onto a chainable continuum is universal.*

4. Principal Results

Theorem 4.1. *Let* $\prod_{i=1}^{n} K^i$ *be a product of Knaster continua, and suppose* W *is a subcontinuum of* $\prod_{i=1}^{n} K^i$. *Then* W *has arbitrarily small connected open neighborhoods in* $\prod_{i=1}^{n} K^i$ *if and only if for every* i, *the projection of* W *to* K^i *is all of* K^i.

Proof. Let K^i and W be as in the hypotheses. If for some j, $\pi_j(W) \neq K^j$, it follows from Lemma 3.6 that W cannot have arbitrary small connected open neighborhoods in $\prod_{i=1}^{n} K^i$.

To prove the converse, let $X = \prod_{i=1}^{n} K^i$, and suppose $\pi_j(W) = K^j$ for each j. Let U be an open subset of X with $W \subseteq U$. By Lemma 3.7, there is a basic open set $\tilde{V}$, such that $W \subseteq \tilde{V} \subseteq U$. Let $\tilde{V}$ be determined by the $m-$th term in the inverse limit. Then let V_m denote the component of U containing W_m, the $m-$th projection of W. By local connectedness of this cube, V_m is open. Let V be the set of points of $\prod_{i=1}^{n} K^i$ with the $m-$th coordinate (in the inverse limit, using Lemma 3.4) belonging to V_m. By Lemma 3.5, V is a connected open subset of $\prod_{i=1}^{n} K^i$. Since $W \subseteq V \subseteq \tilde{V} \subseteq U$, the proof is complete. $\square$

Theorem 4.2. *If* G *is both a topological group and a continuum, the diagonal* Δ *of* $G \times G$ *has arbitrarily small connected open neighborhoods in* $G \times G$ *if and only if* G *is locally connected.*

Proof. Let G be a topological group with identity element e. The function $f : G \times G \to G \times G$ defined by $f(x, y) = (y^{-1}x, y^{-1})$ is a homeomorphism which interchanges the $\{e\} \times G$ and Δ. Therefore, Δ has the same kinds of neighborhoods as $\{e\} \times G$. By Lemma 3.2, $\{e\} \times G$ has arbitrarily small connected open neighborhoods if and only if G is locally connected at e. Of course, G is locally connected at e if and only if it is locally connected at every point. $\square$

Corollary 4.3. *The diagonal in the product of a solenoid with itself does not have arbitrarily small connected open neighborhoods.*

Theorem 4.4. *A subcontinuum W of a finite product of pseudo-arcs has arbitrarily small connected open neighborhoods if and only if each projection of W is onto the pseudo-arc factor.*

Proof. This proof is adapted from an approach suggested to the authors by Prajs [12]. Without his idea, we had only been able to prove the result for the product of two pseudo-arcs. We appreciate his help.

Let P denote a pseudo-arc and let W be a subcontinuum of $\displaystyle\prod_{i=1}^{n} P$ satisfying $\pi_j(W) = P$ for each j, $1 \leq j \leq n$. Let U be an arbitrary open subset of $\displaystyle\prod_{i=1}^{n} P$ with $W \subseteq U$. Let ρ be the max metric on $\displaystyle\prod_{i=1}^{n} P$, that is $\rho(\langle x_i\rangle_{i=1}^n, \langle y_i\rangle_{i=1}^n) = \max_{1\leq i\leq n} d(x_i, y_i)$, where d is a metric on P. Let $\epsilon > 0$ be such that the ϵ-neighborhood of W in $\displaystyle\prod_{i=1}^{n} P$ is a subset of U, and such that any homeomorphism, within ϵ of the identity, of $\displaystyle\prod_{i=1}^{n} P$ to itself is a product homeomorphism $h(\langle x_i\rangle_{i=1}^n) = \langle h_i(x_i)\rangle_{i=1}^n$. Using [2, Theorem 3], it is easy to see that, for ϵ less than half the diameter of P, any self homeomorphism of $\prod_{i=1}^n P$ within ϵ of the identity is a product homeomorphism. Now, by choice of ρ, h is within ϵ of the identity if and only if each h_i is. Let $\mathcal{O} = H(W)$, where H is the ϵ-neighborhood of the identity in the full homeomorphism group of $\displaystyle\prod_{i=1}^{n} P$. Then by the Effros theorem, $\mathcal{O}$ is open in $\displaystyle\prod_{i=1}^{n} P$. It remains to be shown that $\mathcal{O}$ is connected. Let $\langle y_i\rangle_{i=1}^n \in \mathcal{O}$. Then for some $\langle x_i\rangle_{i=1}^n \in W$ and some homeomorphism $h \in H$, $h(\langle x_i\rangle_{i=1}^n) = \langle y_i\rangle_{i=1}^n$. Since h is a product homeomorphism, there exist homeomorphisms $h_i : P \to P$ such that $h = \displaystyle\prod_{i=1}^{n} h_i$ and, in particular, $h_i(x_i) = y_i$ for each i. Furthermore, h_i is at a distance less than ϵ from the identity on P.

For $0 \leq j \leq n$, let $p(j)$ be the point $\langle p_i(j)\rangle_{i=1}^n$ given by $p_i(j) = y_i$ for $i \leq j$ and $p_i(j) = x_i$ for $i > j$. Then $p(0) = \langle x_i\rangle_{i=1}^n$ and $p(n) = \langle y_i\rangle_{i=1}^n$. Let $H_j : P^n \to P^n$ be the homeomorphism which is h_j in the $j-$th factor of P^n and the identity in every other factor. Then h is equal to the composition of the H_j's, $1 \leq j \leq n$ (in any order) and $H_j(p(j-1)) = p(j)$ for each j, $1 \leq j \leq n$.

Define $W_0 = W$ and, for each j, define $W_j = H_j(W_{j-1})$. In particular, $W_n = h(W)$ while $W_0 = W$. Let $M = \bigcup_{j=0}^{n} W_j$. Then $W \subseteq M \subseteq \mathcal{O}$ and $\langle y_i \rangle_{i=1}^n \in M$.

Now since P is chainable, the map $\pi_j | W_{j-1} : W_{j-1} \to P$ is universal, so there exists $p = \langle p_i \rangle_{i=1}^n \in W_{j-1}$ such that $\pi_j(p) = \pi_j \circ H_j(p)$. Since $\pi_j \circ H_j(p) = h_j(p_j)$, it follows that $p_j = h_j(p_j)$. Therefore, $p \in W_{j-1} \cap H_j(W_{j-1})$, so that $W_{j-1} \cap W_j = W_j \cap H_j(W_{j-1}) \neq \emptyset$. Thus, M is a continuum, and since $\langle y_i \rangle_{i=1}^n$ was an arbitrary point of $\mathcal{O}$, the set $\mathcal{O}$ is connected also. $\square$

5. Final Observations

The referee has pointed out to us that Theorem 4.1 and Theorem 4.4 are true for arbitrary products, not just finite ones. To see this, notice that given any compact set $W \subseteq \prod_{\alpha \in A} X_\alpha$ and any open U in this product with $W \subseteq U$, there is a finite $F \subseteq A$ and an open $V \subseteq \prod_{\alpha \in F} X_\alpha$ satisfying the condition that $W_F \subseteq V$ (where W_F is the projection of W into $\prod_{\alpha \in F} X_\alpha$) and $W \subseteq V \times \prod_{\alpha \in A \setminus F} X_\alpha \subseteq U$. By the respective theorems, where W is a continuum projecting onto each factor, the same is true for W_F, and V can then be chosen to be connected. Since V is connected, so is $V \times \prod_{\alpha \in A \setminus F} X_\alpha$.

6. Questions

In the following products, does the result hold that a continuum W has arbitrarily small connected open neighborhoods if its projection to each factor is onto:

(1) the product of a Knaster continuum and a pseudo-arc,
(2) the product of a Knaster continuum and a solenoid, either its associated one or a different one,
(3) any product of chainable indecomposable continua, or more generally, any product of chainable continua,
(4) the product of two non-homeomorphic solenoid groups.

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