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HOMEOMORPHISMS BETWEEN INVERSE LIMITS WITH N-SHAPED SET-VALUED FUNCTIONS

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Abstract. Suppose the graph of $g : [0,1] \to [0,1]$ consists of three straight line segments, one joining $(0,0)$ to $(1/3,1)$, one joining $(1/3,1)$ to $(2/3,0)$, and one joining $(2/3,0)$ to $(1,1)$, so that $\varprojlim \mathbf{g}$ is an indecomposable continuum. Using itineraries, we prove that various inverse limits with "N-shaped" upper semi-continuous bonding functions are homeomorphic to $\varprojlim \mathbf{g}$. Thus, we answer a question recently raised by W. T. Ingram in the affirmative.

1. Introduction

Let us consider the inverse limit with the single upper semi-continuous (u.s.c.) bonding function $f : [0,1] \to C([0,1])$ whose graph consists of three straight line segments, one joining $(0,0)$ to $(1/2,1)$, one joining $(1/2,1)$ to $(1/2,0)$, and one joining $(1/2,0)$ to $(1,1)$. This inverse limit, $\varprojlim \mathbf{f}$, has been studied closely and has turned out to be a fruitful example for the theory of u.s.c. inverse limits. $\varprojlim \mathbf{f}$ was first examined by W. T. Ingram in [2], where he proved it had the full projection property and that it was an indecomposable continuum. This same example later turned out to be useful in other ways, e.g., as an example of a u.s.c. inverse limit that is treelike [5], and then, chainable [4].

In [4, Remark 5.2], Ingram asks whether $\varprojlim \mathbf{f}$ is, in fact, homeomorphic to the two-endpoint Knaster continuum $\varprojlim \mathbf{g}$, where $g : [0,1] \to [0,1]$ is the mapping whose graph consists of three straight line segments, one joining $(0,0)$ to $(1/3,1)$, one joining $(1/3,1)$ to $(2/3,0)$, and one joining $(2/3,0)$ to $(1,1)$. Our main result in this paper, Theorem 3.1, answers this

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question in the affirmative. In fact, we prove something more general: If $0 < p < 1$ and the graph of $f : [0,1] \to C([0,1])$ consists of three straight line segments, one joining $(0,0)$ to $(p,1)$, one joining $(p,1)$ to $(p,0)$, and one joining $(p,0)$ to $(1,1)$, then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$. (If we let $p = 1/2$, then f is Ingram's original function as described above.) We then prove a similar theorem for inverse limits with another type of, roughly speaking, "N-shaped" bonding function. Finally, we end this paper by studying a few examples that do not fit the same mold. We hope that the techniques we illustrate here may be useful to others who, in the course of their research, need to prove certain u.s.c. inverse limits are (or are not) homeomorphic.

2. Definitions and Preliminary Remarks

Much of the background material we will need is the same as was given by the author in [10]. Suppose X and Y are compact Hausdorff spaces, and define 2^Y to be the set of all non-empty compact subsets of Y. A function $f : X \to 2^Y$ is called *upper semi-continuous* (*u.s.c.*) if, for any $x \in X$ and open V in Y containing $f(x)$, there exists an open U in X containing x so that $f(u) \subseteq V$ for all $u \in U$. A non-empty subset C of a Hausdorff space is a *continuum* if C is compact and connected. Thus, if $f : X \to 2^Y$ is u.s.c. and $f(x)$ is connected for each $x \in X$, then f is a *u.s.c. continuum-valued* function; in this case, for emphasis, we will usually write $f : X \to C(Y)$ instead, where $C(Y)$ is the set of all subcontinua of Y. If $f : X \to 2^Y$ is u.s.c. and $f(x) = \{y\}$ for some $x \in X$ and $y \in Y$, then, although f is a set-valued function, we use the convention of writing simply $f(x) = y$. Therefore, in the case where $f : X \to 2^Y$ is u.s.c., but $f(x)$ is degenerate for all $x \in X$, we may regard f as the corresponding continuous function $f : X \to Y$. A continuous function will also be referred to as a *mapping*. If $f : X \to 2^Y$ is u.s.c. and A is a subset of X, then $f|_A$ is the restriction of f to A.

Again, let X and Y be compact Hausdorff spaces and let $f : X \to 2^Y$ be a u.s.c. function. If $y \in Y$, then the *preimage of y via f* is $f^{-1}(y) = \{x \in X|\ y \in f(x)\}$. More generally, if $A \subseteq Y$, then the *preimage of A via f* is $f^{-1}(A) = \{x \in X|\ f(x) \cap A \neq \emptyset\}$. (If, on the other hand, f is simply a mapping, then the standard definition of preimage applies.) We say f is *surjective* if, for each $y \in Y$, $f^{-1}(y)$ is non-empty. Assuming that $f : X \to 2^Y$ is a surjective u.s.c. function, the *inverse of f*, i.e., the set-valued function $f^{-1} : Y \to 2^X$, is given by $f^{-1}(y) = \{x \in X|\ y \in f(x)\}$. Given compact Hausdorff spaces X and Y and a u.s.c. function $f : X \to 2^Y$, the *graph of f*, abbreviated $G(f)$, is the set $\{(x,y) \in X \times Y \mid y \in f(x)\}$.

Now suppose that, for each positive integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is a u.s.c. function. We define $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$

to be the set $\{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X_i \mid x_i \in f_i(x_{i+1})$ for all positive integers $i\}$. (For convenience, we will denote sequences by boldface letters: For example, we denote the sequence $(x_1, x_2, x_3, \ldots)$ by $\mathbf{x}$ and denote the sequence of functions $(f_1, f_2, f_3, \ldots)$ by $\mathbf{f}$. Thus, we may abbreviate $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ by $\varprojlim \mathbf{f}$.) Then we say $\varprojlim \mathbf{f}$ is an *inverse limit space with u.s.c. bonding functions*, and a basis for the topology on $\varprojlim \mathbf{f}$ is $\{O \cap \varprojlim \mathbf{f} \mid O$ is basic open in $\prod_{i=1}^{\infty} X_i\}$. For brevity's sake, we will sometimes call an inverse limit space with u.s.c. bonding functions simply a *u.s.c. inverse limit space*. Finally, in the special case where X is a compact Hausdorff space, $f : X \to 2^X$ is u.s.c., and $\mathbf{f} = (f, f, f, \ldots)$, we say $\varprojlim \mathbf{f}$ is the *inverse limit with the single bonding function* f. (If, in the description of a particular inverse limit, only the single bonding function $f : X \to 2^X$ is given, then it will be clear from context that $\varprojlim \mathbf{f}$ is the inverse limit with the single bonding function f.)

Let us note in advance that this paper will only deal with inverse limits whose factor spaces are all the unit interval $[0,1]$, i.e., $X_i = [0,1]$ for each positive integer i. However, as we will see, it is still helpful to use the notation $X_1, X_2, X_3, \ldots$ in order to distinguish between different factor spaces of an inverse limit.

If $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ is a basic open set in $\varprojlim \mathbf{f}$ with O_n a proper subset of X_n and $O_k = X_k$ for all $k > n$, then O is a *basic open set of order* n. We may also say that O has order n. Note that if $\mathbf{x} \in \varprojlim \mathbf{f}$ and V is any open subset of $\varprojlim \mathbf{f}$ containing $\mathbf{x}$, then there exists some positive integer n and some basic open set O of order n so that $\mathbf{x} \in O \subseteq V$.

Suppose, for each positive integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c. Suppose, for some positive integer n, U_n is a subset of X_n. Then $\overleftarrow{U_n} = \{\mathbf{x} \in \varprojlim \mathbf{f} \mid x_n \in U_n\}$. If U_n is an open subset of X_n, then, since $\overleftarrow{U_n} = (X_1 \times X_2 \times \ldots \times X_{n-1} \times U_n \times X_{n+1} \times \ldots) \cap \varprojlim \mathbf{f}$, $\overleftarrow{U_n}$ is open in $\varprojlim \mathbf{f}$. Thus, if U_n is an open subset of X_n, then we will call $\overleftarrow{U_n}$ an *open set rooted in the nth factor space* X_n.

The following lemma will be useful in the proof of our major theorem and will also help familiarize the reader with the "rooted" open sets of $\varprojlim \mathbf{f}$.

Lemma 2.1. *Suppose, for each positive integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to X_i$ is a mapping. Suppose that, for some positive integer n, $U_{n+1} \subseteq X_{n+1}$ and $f_n(U_{n+1}) \subseteq U_n \subseteq X_n$. Then $\overleftarrow{U_{n+1}} \subseteq \overleftarrow{U_n}$.*

Proof. Let $\mathbf{x} \in \overleftarrow{U_{n+1}}$. Then $x_{n+1} \in U_{n+1}$, so that $f_n(x_{n+1}) \in f_n(U_{n+1})$. Since $\mathbf{x} \in \varprojlim \mathbf{f}$, $f_n(x_{n+1}) = x_n$. By assumption, $f_n(U_{n+1}) \subseteq U_n$. Thus, $x_n \in U_n$, from which it follows that $\mathbf{x} \in \overleftarrow{U_n}$. □

For further background information on inverse limits, the reader should see [7] or [3].

3. Results and Examples

As we stated in the introduction, our main goal is to prove a general theorem which will imply that Ingram's example, $\varprojlim \mathbf{f}$, is homeomorphic to the two-endpoint Knaster continuum $\varprojlim \mathbf{g}$. We will then move on to study inverse limits whose graphs have a similar "italicized N" shape. Again, in the proofs of the following theorems, we will let $X_i = [0,1]$ for each positive integer i, so that the factor spaces of $\varprojlim \mathbf{f}$ (and of $\varprojlim \mathbf{g}$) may be denoted by $X_1, X_2, X_3, \ldots$. Also, we will use the notion of itineraries of an inverse limit to define the homeomorphisms we need; however, we define our itinerary spaces from scratch, so no prior experience with itineraries should be necessary for the reader to understand the proof. The itinerary technique we use here is reminiscent of the one the author used in [9], yet involves a much different sort of strategy. For a more detailed treatment of itineraries, see, e.g., Stewart Baldwin [1].

Theorem 3.1. *Suppose* $0 < p < 1$. *Let* $f : [0,1] \to C([0,1])$ *be the u.s.c. function whose graph consists of three straight line segments, one joining* $(0,0)$ *to* $(p,1)$, *one joining* $(p,1)$ *to* $(p,0)$, *and one joining* $(p,0)$ *to* $(1,1)$. *Let* $g : [0,1] \to [0,1]$ *be the mapping whose graph consists of three straight line segments, one joining* $(0,0)$ *to* $(1/3,1)$, *one joining* $(1/3,1)$ *to* $(2/3,0)$, *and one joining* $(2/3,0)$ *to* $(1,1)$. *(See Figure 1.) Then* $\varprojlim \boldsymbol{f}$ *and* $\varprojlim \boldsymbol{g}$ *are homeomorphic.*

Proof. We will denote $f|_{[0,p)}$ by f_a, $f|_{\{p\}}$ by f_b, and $f|_{(p,1]}$ by f_c. Thus, f_a, f_b, and f_c are mutually exclusive and $f = f_a \cup f_b \cup f_c$. Note that f_a is a homeomorphism from $[0,p)$ onto $[0,1)$, f_b is the u.s.c. function from $\{p\}$ into $C([0,1])$ given by $f_b(p) = [0,1]$, and f_c is a homeomorphism from $(p,1]$ onto $(0,1]$.

Next, let us denote $g|_{[0,1/3)}$ by g_a, $g|_{[1/3,2/3]}$ by g_b, and $g|_{(2/3,1]}$ by g_c. Thus, g_a, g_b, and g_c are mutually exclusive and $g = g_a \cup g_b \cup g_c$. Note that g_a is a homeomorphism from $[0,1/3)$ onto $[0,1)$, g_b is a homeomorphism from $[1/3,2/3]$ onto $[0,1]$, and g_c is a homeomorphism from $(2/3,1]$ onto $(0,1]$.

We will now define a map from one inverse limit to the other using itineraries. First, we create an itinerary space for $\varprojlim \mathbf{f}$. Let $\mathcal{H} =$

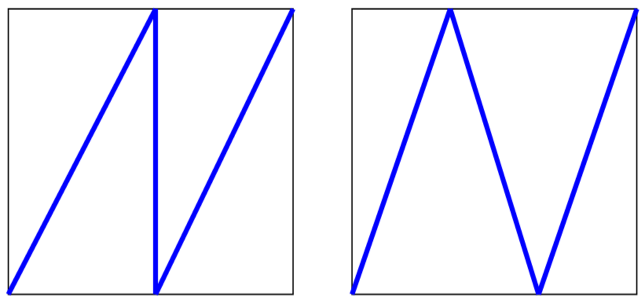

FIGURE 1. Left: The graph of the bonding function f in Theorem 3.1 with $p = 1/2$. Right: The graph of g.

$\{f_a, f_b, f_c\}$ and let $I_f = [0,1] \times \prod_{i=2}^{\infty} \mathcal{H}$. Define the function $\phi : \varprojlim \mathbf{f} \to I_f$ by $\phi(\mathbf{x}) = (x_1, \alpha_2, \alpha_3, \alpha_4, \ldots)$, where for each $i \geq 2$, α_i is the element of $\mathcal{H}$ such that $(x_i, x_{i-1}) \in G(\alpha_i)$. Then $\phi(\varprojlim \mathbf{f})$, which we denote by $\widetilde{I_f}$, is the itinerary space for $\varprojlim \mathbf{f}$.

To create an itinerary space for $\varprojlim \mathbf{g}$, we let $\mathcal{K} = \{g_a, g_b, g_c\}$ and let $I_g = [0,1] \times \prod_{i=2}^{\infty} \mathcal{K}$. Define $\psi : \varprojlim \mathbf{g} \to I_g$ by $\psi(\mathbf{x}) = (x_1, \beta_2, \beta_3, \beta_4, \ldots)$, where, for each $i \geq 2$, β_i is the element of $\mathcal{K}$ such that $(x_i, x_{i-1}) \in G(\beta_i)$. Then $\psi(\varprojlim \mathbf{g})$, which we denote by $\widetilde{I_g}$, is the itinerary space for $\varprojlim \mathbf{g}$.

Now, direct inspection shows that $\widetilde{I_f}$ consists exactly of all sequences of the following forms:

(1) $(x_1, \alpha_2, \alpha_3, \ldots)$, where $x_1 \in (0,1)$ and for each $i \geq 2$, $\alpha_i \in \mathcal{H}$,

(2) $(0, f_a, f_a, f_a, \ldots)$,

(3) $(0, f_a, f_a, \ldots, f_a, f_b, \alpha_k, \alpha_{k+1}, \ldots)$; i.e., for some $k \geq 4$, coordinates 2 through $k-2$ are f_a, coordinate $k-1$ is f_b, and $\alpha_i \in \mathcal{H}$ for each $i \geq k$,

(4) $(0, f_b, \alpha_3, \alpha_4, \ldots)$, where $\alpha_i \in \mathcal{H}$ for each $i \geq 3$,

(5) $(1, f_c, f_c, f_c, \ldots)$,

(6) $(1, f_c, f_c, \ldots, f_c, f_b, \alpha_k, \alpha_{k+1}, \ldots)$; i.e., for some $k \geq 4$, coordinates 2 through $k-2$ are f_c, coordinate $k-1$ is f_b, and $\alpha_i \in \mathcal{H}$ for each $i \geq k$,

(7) $(1, f_b, \alpha_3, \alpha_4, \ldots)$, where $\alpha_i \in \mathcal{H}$ for each $i \geq 3$.

Similarly, by direct inspection, we see that $\widetilde{I_g}$ consists exactly of all sequences of the following forms:

(1) $(x_1, \beta_2, \beta_3, \ldots)$, where $x_1 \in (0,1)$ and for each $i \geq 2$, $\beta_i \in \mathcal{K}$,
(2) $(0, g_a, g_a, g_a, \ldots)$,
(3) $(0, g_a, g_a, \ldots, g_a, g_b, \beta_k, \beta_{k+1}, \ldots)$; i.e., for some $k \geq 4$, coordinates 2 through $k-2$ are g_a, coordinate $k-1$ is g_b, and $\beta_i \in \mathcal{K}$ for each $i \geq k$,
(4) $(0, g_b, \beta_3, \beta_4, \ldots)$, where $\beta_i \in \mathcal{K}$ for each $i \geq 3$,
(5) $(1, g_c, g_c, g_c, \ldots)$,
(6) $(1, g_c, g_c, \ldots, g_c, g_b, \beta_k, \beta_{k+1}, \ldots)$; i.e., for some $k \geq 4$, coordinates 2 through $k-2$ are g_c, coordinate $k-1$ is g_b, and $\beta_i \in \mathcal{K}$ for each $i \geq k$,
(7) $(1, g_b, \beta_3, \beta_4, \ldots)$, where $\beta_i \in \mathcal{K}$ for each $i \geq 3$.

We note that whenever $\mathbf{x}, \mathbf{y} \in \varprojlim \mathbf{f}$ with $\mathbf{x} \neq \mathbf{y}$, $\phi(\mathbf{x}) \neq \phi(\mathbf{y})$. So ϕ is one-to-one, and ψ is one-to-one by the same reasoning. Of course, ϕ maps onto its image $\widetilde{I_f}$, just as ψ maps onto its image $\widetilde{I_g}$. There is also a natural one-to-one onto function from $\widetilde{I_g}$ to $\widetilde{I_f}$: let $\rho : \widetilde{I_g} \to \widetilde{I_f}$ be given by $\rho((x_1, \beta_2, \beta_3, \ldots)) = (x_1, \alpha_2, \alpha_3, \ldots)$, where, for each $i \geq 2$, $\beta_i = g_a$ iff $\alpha_i = f_a$, $\beta_i = g_b$ iff $\alpha_i = f_b$, and $\beta_i = g_c$ iff $\alpha_i = f_c$.

Now we define $h : \varprojlim \mathbf{g} \to \varprojlim \mathbf{f}$ by $h = \phi^{-1}|_{\widetilde{I_f}} \circ \rho \circ \psi$. We intend to show h is a homeomorphism.

Since h is a composition of one-to-one onto functions, h is one-to-one and onto. It remains to show that h and h^{-1} are continuous. Certainly, $\varprojlim \mathbf{g}$ is a compact space since it is an inverse limit on $[0,1]$ with a single continuous bonding function; also, $\varprojlim \mathbf{f}$ is a Hausdorff space. Thus, it suffices to show that h is continuous (for then h^{-1} is continuous automatically; see [10, Theorem 2.21]). So, let us assume $\mathbf{x} \in \varprojlim \mathbf{g}$. We will show that, for each positive integer n, if $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ is a basic open subset of $\varprojlim \mathbf{f}$ of order n containing $h(\mathbf{x})$, there is an open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ (rooted in X_n, the nth factor space of $\varprojlim \mathbf{g}$) containing $\mathbf{x}$ with $h(\overleftarrow{U_n}) \subseteq O$. We will prove this claim by induction.

Suppose $O = (O_1 \times X_2 \times X_3 \times \ldots) \cap \varprojlim \mathbf{f}$ is a basic open set in $\varprojlim \mathbf{f}$ that has order 1 and contains $h(\mathbf{x})$. Note that O consists of all elements $\mathbf{w} \in \varprojlim \mathbf{f}$ with $w_1 \in O_1$. By the way h is defined, the first coordinate of $h(\mathbf{x})$ is the same as the first coordinate of $\mathbf{x}$, namely x_1. So, since $h(\mathbf{x}) \in O$, $x_1 \in O_1$. Thus, the set $\overleftarrow{O_1} = \{\mathbf{y} \in \varprojlim \mathbf{g} \mid y_1 \in O_1\}$ is an open set in $\varprojlim \mathbf{g}$ rooted in X_1 that contains $\mathbf{x}$, and (by the way h was defined) $h(\overleftarrow{O_1}) \subseteq O$.

Now assume that for any basic open set $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ of order n that contains $h(\mathbf{x})$, there is an open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ rooted in X_n and containing $\mathbf{x}$ such that $h(\overleftarrow{U_n}) \subseteq O$. We need to show the claim is true for $n+1$. (For the rest of the proof, to avoid confusion, whenever we mention a rooted open set $\overleftarrow{U_n}$, it is understood that $\overleftarrow{U_n}$ is a subset of $\varprojlim \mathbf{g}$; the subscript n indicates that $\overleftarrow{U_n}$ is rooted in X_n. Also, for simplicity, let $h(\mathbf{x}) = \mathbf{z} = (z_1, z_2, z_3, \ldots)$.)

We begin by assuming $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ is basic open of order $n+1$ and $\mathbf{z} \in O$. We need to show that there exists an open set $\overleftarrow{U_{n+1}}$ rooted in X_{n+1} and containing $\mathbf{x}$ such that $h(\overleftarrow{U_{n+1}}) \subseteq O$.

Let us break this task into cases, based on characteristics of $\mathbf{z}$.

Case 1: $(z_{n+1}, z_n) \in G(f_a)$.

Since $\mathbf{z} \in O$, we know $z_n \in O_n$ and $z_{n+1} \in O_{n+1}$. Since $(z_{n+1}, z_n) \in G(f_a)$, it follows that $z_{n+1} \in [0,p)$. So, let $V_{n+1} = O_{n+1} \cap [0,p)$; note that V_{n+1} is an open subset of X_{n+1} containing z_{n+1}. Since $z_n = f_a(z_{n+1})$, we know $z_n \in f_a(V_{n+1})$. Since f_a is a homeomorphism from $[0,p)$ onto $[0,1)$, $f_a(V_{n+1})$ is an open subset of $[0,1) \subseteq X_n$. So, let $V_n = O_n \cap f_a(V_{n+1})$, so that V_n is an open proper subset of X_n containing z_n.

Thus, $\widetilde{O} = (O_1 \times O_2 \times \ldots \times O_{n-1} \times V_n \times X_{n+1} \times \ldots) \cap \varprojlim \mathbf{f}$ is a basic open set in $\varprojlim \mathbf{f}$ of order n containing $\mathbf{z}$. Therefore, the inductive hypothesis applies, and there exists some open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ rooted in X_n that contains $\mathbf{x}$ with $h(\overleftarrow{U_n}) \subseteq \widetilde{O}$. So $x_n \in U_n$. Since $(z_{n+1}, z_n) \in G(f_a)$, by the way h is defined, we also have $(x_{n+1}, x_n) \in G(g_a)$, which implies that $x_n \in [0,1)$. So, let $B_n = U_n \cap [0,1)$, so that B_n is open in X_n and contains x_n. Since $(x_{n+1}, x_n) \in G(g_a)$, we know $x_{n+1} = g_a^{-1}(x_n)$; thus, $x_{n+1} \in g_a^{-1}(B_n)$. Let $U_{n+1} = g_a^{-1}(B_n)$, and note that U_{n+1} is open in X_{n+1}. Moreover, note that $\mathbf{x} \in \overleftarrow{U_{n+1}}$, where $\overleftarrow{U_{n+1}}$ is rooted in X_{n+1}. Since $g(U_{n+1}) = B_n \subseteq U_n$, by Lemma 2.1, we have $\overleftarrow{U_{n+1}} \subseteq \overleftarrow{U_n}$, so that $h(\overleftarrow{U_{n+1}}) \subseteq h(\overleftarrow{U_n}) \subseteq \widetilde{O}$.

Now we claim $h(\overleftarrow{U_{n+1}}) \subseteq O$. Let $\mathbf{y} \in \overleftarrow{U_{n+1}}$ (and let $h(\mathbf{y}) = \mathbf{w}$). Then, since $h(\overleftarrow{U_{n+1}}) \subseteq \widetilde{O}$, we know $\mathbf{w} \in \widetilde{O}$, and thus, $w_n \in V_n$. Now since $\mathbf{y} \in \overleftarrow{U_{n+1}}$, we know $(y_{n+1}, y_n) \in G(g_a)$. Thus, $(w_{n+1}, w_n) \in G(f_a)$. That means $w_{n+1} = f_a^{-1}(w_n) \in f_a^{-1}(V_n)$. But f_a is a homeomorphism, so since $V_n \subseteq f_a(V_{n+1})$, we know $f_a^{-1}(V_n) \subseteq V_{n+1}$. Thus, $w_{n+1} \in V_{n+1} \subseteq O_{n+1}$. Since $\mathbf{w} \in \widetilde{O}$, we also know $w_1 \in O_1, w_2 \in O_2, \ldots, w_{n-1} \in O_{n-1}$, and $w_n \in V_n \subseteq O_n$. Thus, $\mathbf{w} \in O$.

Case 2: $(z_{n+1}, z_n) \in G(f_c)$.

This case is very similar to Case 1; we leave the details to the reader.

Case 3: $(z_{n+1}, z_n) \in G(f_b)$ and $0 < z_n < 1$.

Note that, in this case, $z_{n+1} = p$. Let $V_n = O_n \cap (0,1)$, so that V_n is an open subset of $(0,1) \subseteq X_n$ containing z_n. Since V_n is a proper subset of X_n that contains z_n, the set $\widetilde{O} = (O_1 \times O_2 \times \ldots \times O_{n-1} \times V_n \times X_{n+1} \times \ldots) \cap \varprojlim \mathbf{f}$ is basic open of order n and contains $\mathbf{z}$. Therefore, the inductive hypothesis applies: There exists some open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ rooted in X_n and containing $\mathbf{x}$ with $h(\overleftarrow{U_n}) \subseteq \widetilde{O}$. That means $x_n \in U_n$. Since $z_n \in (0,1)$, it follows from the definition of h that $x_n \in (0,1)$. Let $B_n = U_n \cap (0,1)$, so that B_n is an open subset of X_n containing x_n. Because $(z_{n+1}, z_n) \in G(f_b)$, we know $(x_{n+1}, x_n) \in G(g_b)$; thus, $g_b^{-1}(B_n)$ is an open subset of $(1/3, 2/3) \subseteq X_{n+1}$ containing x_{n+1}. Let $U_{n+1} = g_b^{-1}(B_n)$, and note that $\mathbf{x} \in \overleftarrow{U_{n+1}}$, where $\overleftarrow{U_{n+1}}$ is rooted in X_{n+1}. Since $g(U_{n+1}) = B_n \subseteq U_n$, $\overleftarrow{U_{n+1}} \subseteq \overleftarrow{U_n}$, so that $h(\overleftarrow{U_{n+1}}) \subseteq h(\overleftarrow{U_n}) \subseteq \widetilde{O}$.

Now we claim $h(\overleftarrow{U_{n+1}}) \subseteq O$. Let $\mathbf{y} \in \overleftarrow{U_{n+1}}$ (and let $h(\mathbf{y}) = \mathbf{w}$). Since $\mathbf{y} \in \overleftarrow{U_{n+1}}$, we know $(y_{n+1}, y_n) \in G(g_b)$. Thus, by the way h is defined, $(w_{n+1}, w_n) \in G(f_b)$. That means $w_{n+1} = p$. But $p = z_{n+1}$ and $z_{n+1} \in O_{n+1}$. Thus, $w_{n+1} \in O_{n+1}$. Since $\mathbf{w} \in \widetilde{O}$, we also know $w_1 \in O_1$, $w_2 \in O_2, \ldots, w_{n-1} \in O_{n-1}$, and $w_n \in V_n \subseteq O_n$. Thus, $\mathbf{w} \in O$.

Case 4: $z_n = 0$ and $z_{n+1} = p$.

We begin by noting that, in this case, $x_n = 0$ and $x_{n+1} = 2/3$. Since $p = z_{n+1} \in O_{n+1}$, there exists an interval of form $[p, r)$ that is a subset of $O_{n+1} \cap [p, 1)$. Let $V_{n+1} = (p, r)$. Note that, by the way f is defined, $f_c(V_{n+1}) = (0, s)$ for some s with $0 < s < 1$. Let $V_n = O_n \cap (f_c(V_{n+1}) \cup \{0\})$, so that V_n is an open subset of X_n containing $z_n = 0$; since $1 \notin f_c(V_{n+1})$, we know V_n is a proper subset of X_n.

Thus, $\widetilde{O} = (O_1 \times O_2 \times \ldots \times O_{n-1} \times V_n \times X_{n+1} \times \ldots) \cap \varprojlim \mathbf{f}$ is a basic open set of order n containing $\mathbf{z}$. The inductive hypothesis applies: We conclude that there exists some open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ rooted in X_n that contains $\mathbf{x}$ with $h(\overleftarrow{U_n}) \subseteq \widetilde{O}$. Let B_n be an open interval subset of $U_n \cap [0, 1/2)$ containing $x_n = 0$. Then $g_b^{-1}(B_n) \cup g_c^{-1}(B_n)$ is an open interval subset of X_{n+1} containing $x_{n+1} = 2/3$. Let $U_{n+1} = g_b^{-1}(B_n) \cup g_c^{-1}(B_n)$; note that $g(U_{n+1}) = B_n \subseteq U_n$, so that $\overleftarrow{U_{n+1}} \subseteq \overleftarrow{U_n}$ and $h(\overleftarrow{U_{n+1}}) \subseteq h(\overleftarrow{U_n}) \subseteq \widetilde{O}$.

It remains to show that $h(\overleftarrow{U_{n+1}}) \subseteq O$. Let $\mathbf{y} \in \overleftarrow{U_{n+1}}$ and let $\mathbf{w} = h(\mathbf{y})$. Since $y_{n+1} \in U_{n+1}$, either $(y_{n+1}, y_n) \in G(g_b)$ or $(y_{n+1}, y_n) \in G(g_c)$.

Suppose $(y_{n+1}, y_n) \in G(g_b)$; then $(w_{n+1}, w_n) \in G(f_b)$. Thus, we know $w_{n+1} = p$. But $p = z_{n+1} \in O_{n+1}$, so $w_{n+1} \in O_{n+1}$. Because $\mathbf{w} \in \widetilde{O}$, using the same reasoning as in the previous cases, we may conclude $\mathbf{w} \in O$.

Suppose $(y_{n+1}, y_n) \in G(g_c)$. Then $(w_{n+1}, w_n) \in G(f_c)$. Moreover, $\mathbf{w} \in \widetilde{O}$, so $w_n \in V_n$. Thus, $w_{n+1} = f_c^{-1}(w_n) \in f_c^{-1}(V_n)$. Since 0 is not in the range of f_c, we know $f_c^{-1}(V_n) = f_c^{-1}(V_n \setminus \{0\})$. Because $V_n \subseteq f_c(V_{n+1}) \cup \{0\}$, we may deduce that $V_n \setminus \{0\} \subseteq f_c(V_{n+1})$. Thus, since f_c is a homeomorphism, we have that $f_c^{-1}(V_n \setminus \{0\}) \subseteq V_{n+1}$. Since $V_{n+1} \subseteq O_{n+1}$, it follows that $w_{n+1} \in O_{n+1}$. Once again, since we know $\mathbf{w} \in \widetilde{O}$ already, we may now conclude that $\mathbf{w} \in O$.

Case 5: $z_n = 1$ and $z_{n+1} = p$.
This case is very similar to Case 4.

All cases have been accounted for, so we have completed the induction and shown that h is continuous. From this we conclude that h is in fact a homeomorphism, and the proof is complete. □

Having seen Theorem 3.1, it is a natural next step to investigate inverse limits with some different types of "N-shaped" u.s.c. bonding functions; we define one such function in Theorem 3.2. It turns out that the same sort of technique from the proof of Theorem 3.1 can be used to prove this inverse limit is also homeomorphic to $\varprojlim \mathbf{g}$. The inverse limit $\varprojlim \mathbf{f}$ in Theorem 3.2 was the first example known to the author of an inverse limit on $[0,1]$ with a single surjective, u.s.c., non-continuum-valued bonding function that gives rise to an indecomposable continuum. (However, other examples have been found independently by James P. Kelly and Jonathan Meddaugh [8].)

Theorem 3.2. *Suppose* $0 < p < q < 1$. *Let* $f : [0,1] \to 2^{[0,1]}$ *be the u.s.c. function whose graph is the union of three straight line segments, the first joining* $(0,0)$ *to* $(q,1)$, *the second joining* $(q,1)$ *to* $(p,0)$, *and the third joining* $(p,0)$ *to* $(1,1)$. *(See Figure 2.) Let* g *be defined as in Theorem 3.1. Then* $\varprojlim \boldsymbol{f}$ *and* $\varprojlim \boldsymbol{g}$ *are homeomorphic.*

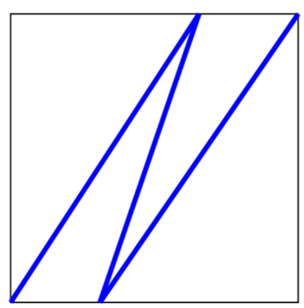

FIGURE 2. The graph of the bonding function f in Theorem 3.2 with $p = 1/3$ and $q = 2/3$.

Proof. Let $f_a : [0,q) \to [0,1)$ be given by $f_a(t) = \frac{1}{q}t$, let $f_b : [p,q] \to [0,1]$ be given by $f_b(t) = \frac{1}{q-p}(t-p)$, and let $f_c : (p,1] \to (0,1]$ be given by $f_c(t) = \frac{1}{1-p}(t-p)$. Note that f_a, f_b, and f_c are mutually exclusive and $f = f_a \cup f_b \cup f_c$; also note that f_a is a homeomorphism from $[0,q)$ onto $[0,1)$, f_b is a homeomorphism from $[p,q]$ onto $[0,1]$, and f_c is a homeomorphism from $(p,1]$ onto $(0,1]$. We define g_a, g_b, and g_c as in the proof of Theorem 3.1.

As before, we will define a map from one inverse limit to the other using itineraries. Let $\mathcal{H} = \{f_a, f_b, f_c\}$ and let $I_f = [0,1] \times \prod_{i=2}^{\infty} \mathcal{H}$. Define $\phi : \varprojlim \mathbf{f} \to I_f$ by $\phi(\mathbf{x}) = (x_1, \alpha_2, \alpha_3, \alpha_4, \ldots)$, where, for each $i \geq 2$, α_i is the element of $\mathcal{H}$ such that $(x_i, x_{i-1}) \in G(\alpha_i)$. Then $\phi(\varprojlim \mathbf{f})$, which we denote by $\widetilde{I_f}$, is the itinerary space for $\varprojlim \mathbf{f}$. The itinerary space for $\varprojlim \mathbf{g}$, namely $\psi(\varprojlim \mathbf{g}) = \widetilde{I_g}$, was already defined in Theorem 3.1.

Now, direct inspection shows that $\widetilde{I_f}$ consists of exactly the same sequences that $\widetilde{I_f}$ consisted of in the proof of Theorem 3.1. Of course, $\widetilde{I_g}$ is the same as in Theorem 3.1. We also define $\rho : \widetilde{I_g} \to \widetilde{I_f}$ as before, and conclude that $h = \phi^{-1}|_{\widetilde{I_f}} \circ \rho \circ \psi$ is a one-to-one onto function from $\varprojlim \mathbf{g}$ to $\varprojlim \mathbf{f}$. It remains to show that h and h^{-1} are continuous; as we argued in Theorem 3.1, it suffices to show that h is continuous. So, let us assume $\mathbf{x} \in \varprojlim \mathbf{g}$. We will show that, for each positive integer n, if $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ is a basic open subset of $\varprojlim \mathbf{f}$ of order n containing $h(\mathbf{x})$, there is an open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ (rooted in X_n, the nth factor space of $\varprojlim \mathbf{g}$) containing $\mathbf{x}$ with $h(\overleftarrow{U_n}) \subseteq O$. Again, we will prove this claim by induction.

Suppose $O = (O_1 \times X_2 \times X_3 \times \ldots) \cap \varprojlim \mathbf{f}$ is a basic open set in $\varprojlim \mathbf{f}$ that has order 1 and contains $h(\mathbf{x})$. Note that O consists of all elements $\mathbf{w} \in \varprojlim \mathbf{f}$ with $w_1 \in O_1$. Thus, as in Theorem 3.1, the set $\overleftarrow{O_1} = \{\mathbf{y} \in \varprojlim \mathbf{g} \mid y_1 \in O_1\}$ is an open set in $\varprojlim \mathbf{g}$ rooted in X_1 that contains $\mathbf{x}$, and (by the way h was defined) $h(\overleftarrow{O_1}) \subseteq O$.

Now assume that, for any basic open set $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ of order n that contains $h(\mathbf{x})$, there is an open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ rooted in X_n and containing $\mathbf{x}$ such that $h(\overleftarrow{U_n}) \subseteq O$. We need to show the claim is true for $n+1$. (Once more, as in the proof of the previous theorem, whenever we mention a rooted open set $\overleftarrow{U_n}$, it is understood that $\overleftarrow{U_n}$ is a subset of $\varprojlim \mathbf{g}$; the subscript n indicates that $\overleftarrow{U_n}$ is rooted in X_n. Also, for simplicity, let $h(\mathbf{x}) = \mathbf{z} = (z_1, z_2, z_3, \ldots)$.)

We begin by assuming $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ is basic open of order $n+1$ and $\mathbf{z} \in O$. We need to show that there exists an open set $\overleftarrow{U_{n+1}}$ rooted in X_{n+1} and containing $\mathbf{x}$ such that $h(\overleftarrow{U_{n+1}}) \subseteq O$.

Let us break this task into cases, based on characteristics of $\mathbf{z}$.

Case 1: $(z_{n+1}, z_n) \in G(f_a)$.

Since $\mathbf{z} \in O$, we know $z_n \in O_n$ and $z_{n+1} \in O_{n+1}$. Since $(z_{n+1}, z_n) \in G(f_a)$, it follows that $z_{n+1} \in [0, q)$. So, let $V_{n+1} = O_{n+1} \cap [0, q)$; note that V_{n+1} is an open subset of X_{n+1} containing z_{n+1}. Since $z_n = f_a(z_{n+1})$, we know $z_n \in f_a(V_{n+1})$. Since f_a is a homeomorphism from $[0, q)$ onto $[0, 1)$, $f_a(V_{n+1})$ is an open subset of $[0, 1) \subseteq X_n$. So, let $V_n = O_n \cap f_a(V_{n+1})$, so that V_n is an open proper subset of X_n containing z_n.

Thus, $\widetilde{O} = (O_1 \times O_2 \times \ldots \times O_{n-1} \times V_n \times X_{n+1} \times \ldots) \cap \varprojlim \mathbf{f}$ is a basic open set in $\varprojlim \mathbf{f}$ of order n containing $\mathbf{z}$. Therefore, the inductive hypothesis applies, and there exists some open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ rooted in X_n that contains $\mathbf{x}$ with $h(\overleftarrow{U_n}) \subseteq \widetilde{O}$. The rest of the proof in this case is identical to the corresponding argument for Case 1 of Theorem 3.1.

Case 2: $(z_{n+1}, z_n) \in G(f_c)$.

This case is very similar to Case 1; we leave the details to the reader.

Case 3: $(z_{n+1}, z_n) \in G(f_b)$ and $p < z_{n+1} < q$.

Let $V_{n+1} = O_{n+1} \cap (p, q)$, so that V_{n+1} is an open subset of $(p, q) \subseteq X_{n+1}$ containing z_{n+1}. Thus, $f_b(V_{n+1})$ is an open subset of X_n containing z_n. Let $V_n = O_n \cap f_b(V_{n+1})$, so that V_n is an open subset of X_n containing z_n. Since $V_{n+1} \subseteq (p, q)$, $f_b(V_{n+1}) \subseteq (0, 1)$, so V_n is a proper subset of X_n. Thus, the inductive hypothesis applies to $\widetilde{O} = (O_1 \times O_2 \times \ldots \times O_{n-1} \times V_n \times X_{n+1} \times \ldots) \cap \varprojlim \mathbf{f}$, and there exists some open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ rooted in X_n and containing $\mathbf{x}$ with $h(\overleftarrow{U_n}) \subseteq \widetilde{O}$. Thus, $x_n \in U_n$. Again, since $z_{n+1} \in (p, q)$, $z_n \in (0, 1)$, so it follows from the definition of h that $x_n \in (0, 1)$. Let $B_n = U_n \cap (0, 1)$, so that B_n is an open subset of X_n containing x_n. Thus, $g_b^{-1}(B_n)$ is an open subset of $(1/3, 2/3) \subseteq X_{n+1}$ containing x_{n+1}. Let $U_{n+1} = g_b^{-1}(B_n)$; the rest of the proof is analogous to the argument that finishes Case 1.

Case 4: $z_n = 0$ and $z_{n+1} = p$.

Since $z_{n+1} \in O_{n+1}$, there exists some r with $p < r < q$ such that the interval $[p, r)$ is a subset of $O_{n+1} \cap [p, q)$. Note that, by the way f is defined, $f_b([p, r)) = [0, s)$ where $s = \frac{r-p}{q-p}$, and $f_c((p, r)) = (0, t)$ where $t = \frac{r-p}{1-p}$ (so $t < s < 1$). Let V_n be an open interval subset of $O_n \cap [0, t)$ containing $z_n = 0$; note that V_n is a proper subset of X_n.

Thus, $\widetilde{O} = (O_1 \times O_2 \times \ldots \times O_{n-1} \times V_n \times X_{n+1} \times \ldots) \cap \varprojlim \mathbf{f}$ is a basic open set of order n containing $\mathbf{z}$. The inductive hypothesis applies: We conclude that there exists some open set $\overleftarrow{U_n} \subseteq \varprojlim \mathbf{g}$ rooted in X_n that contains $\mathbf{x}$ with $h(\overleftarrow{U_n}) \subseteq \widetilde{O}$. Let B_n be an open interval subset of $U_n \cap [0, 1/2)$ containing $x_n = 0$. Then $g_b^{-1}(B_n) \cup g_c^{-1}(B_n)$ is an open interval subset of

X_{n+1} containing $x_{n+1} = 2/3$. Let $U_{n+1} = g_b^{-1}(B_n) \cup g_c^{-1}(B_n)$; note that $g(U_{n+1}) = B_n \subseteq U_n$, so that $\overleftarrow{U_{n+1}} \subseteq \overleftarrow{U_n}$ and $h(\overleftarrow{U_{n+1}}) \subseteq h(\overleftarrow{U_n}) \subseteq \widetilde{O}$.

It remains to show that $h(\overleftarrow{U_{n+1}}) \subseteq O$. Let $\mathbf{y} \in \overleftarrow{U_{n+1}}$ and let $\mathbf{w} = h(\mathbf{y})$. Since $y_{n+1} \in U_{n+1}$, either $(y_{n+1}, y_n) \in G(g_b)$ or $(y_{n+1}, y_n) \in G(g_c)$.

Suppose $(y_{n+1}, y_n) \in G(g_b)$; then $(w_{n+1}, w_n) \in G(f_b)$. $\mathbf{w} \in \widetilde{O}$, so $w_n \in V_n$. That means that $w_{n+1} = f_b^{-1}(w_n) \in f_b^{-1}(V_n)$. But since $V_n \subseteq [0, t) \subseteq [0, s) = f_b([p, r))$ and f_b is a homeomorphism, $f_b^{-1}(V_n) \subseteq [p, r) \subseteq O_{n+1}$. So $w_{n+1} \in O_{n+1}$, and since $\mathbf{w} \in \widetilde{O}$, we conclude $\mathbf{w} \in O$.

Suppose $(y_{n+1}, y_n) \in G(g_c)$. Then $(w_{n+1}, w_n) \in G(f_c)$. Moreover, $\mathbf{w} \in \widetilde{O}$, so $w_n \in V_n$. Since 0 is not in the range of f_c, we know $w_n \neq 0$. Thus, $w_n \in V_n \setminus \{0\}$, and therefore $w_{n+1} = f_c^{-1}(w_n) \in f_c^{-1}(V_n \setminus \{0\})$. Since $V_n \subseteq [0, t)$, $V_n \setminus \{0\} \subseteq (0, t)$. However, $(0, t) = f_c((p, r))$. So, because f_c is a homeomorphism, we have $f_c^{-1}(V_n \setminus \{0\}) \subseteq (p, r)$. Since $(p, r) \subseteq O_{n+1}$, it follows that $w_{n+1} \in O_{n+1}$. Once again, since we know $\mathbf{w} \in \widetilde{O}$ already, we may conclude that $\mathbf{w} \in O$.

Case 5: $z_n = 1$ and $z_{n+1} = q$.

This case is very similar to Case 4.

All cases have been accounted for, so we have completed the induction and shown that h is continuous. From this we conclude that h is, in fact, a homeomorphism, and the proof is complete. □

The preceding two theorems motivate a definition that will make our intuitive notion of an "N-shaped" function more precise. Let us say that $f : [0, 1] \to 2^{[0,1]}$ is an *N-graph* if there exist p and q with $0 < p \leq 1$ and $0 \leq q < 1$ such that f is the union of three straight line segments, one from $(0, 0)$ to $(q, 1)$, one from $(q, 1)$ to $(p, 0)$, and one from $(p, 0)$ to $(1, 1)$. We will distinguish between the types of N-graphs we have already seen as follows: If $0 < q < p < 1$, let us refer to f as *α-type*; if $p = q$, we call f *β-type*; if $0 < p < q < 1$, f is *γ-type*.

It is a basic exercise using itineraries to show that if f is any α-type N-graph and g is defined as in the previous theorems, then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$. Theorem 3.1 (Theorem 3.2, respectively) says that if f is a β-type (γ-type, respectively) N-graph, $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$. Thus, putting all these results together, we have the following theorem.

Theorem 3.3. *Suppose $f : [0, 1] \to 2^{[0,1]}$ is an N-graph of α-, β-, or γ-type. Then $\varprojlim \boldsymbol{f}$ is homeomorphic to the two-endpoint Knaster continuum $\varprojlim \boldsymbol{g}$.*

The only remaining N-graphs that are not addressed by Theorem 3.3 are those such that either $p = 1$ or $q = 0$. Interestingly, if f is an N-graph that satisfies either $p = 1$ or $q = 0$, then $\varprojlim \mathbf{f}$ is not homeomorphic to the two-endpoint Knaster continuum $\varprojlim \mathbf{g}$. One simple way to prove this is to use the author's techniques from [9] and [10] to show that $\varprojlim \mathbf{f}$ would be a decomposable continuum.

Theorem 3.4. *Suppose $f : [0,1] \to 2^{[0,1]}$ is an N-graph with either $p = 1$ or $q = 0$. Then $\varprojlim \boldsymbol{f}$ is a decomposable continuum, and hence, it is not homeomorphic to the two-endpoint Knaster continuum $\varprojlim \boldsymbol{g}$.*

Proof. Suppose $p = 1$. Since $q < 1$, $p \neq q$. By the definition of N-graph, the graph of f is the union of three straight line segments, one joining $(0,0)$ to $(q,1)$, one joining $(q,1)$ to $(1,0)$, and one joining $(1,0)$ to $(1,1)$. f is a continuum-valued u.s.c. function; so, by [3, Theorem 2.7] (or [6, Theorem 4.7], the original source), $\varprojlim \mathbf{f}$ is a continuum. Let the graph of $j_1 : [0,1] \to C([0,1])$ be the union of two line segments, one joining $(0,0)$ to $(q,1)$ and one joining $(q,1)$ to $(1,0)$. Then j_1 is also continuum-valued, and $G(j_1) \subseteq G(f)$. For each positive integer $i \geq 2$, let $j_i = f$. Then $\varprojlim \mathbf{j}$ is a subcontinuum of $\varprojlim \mathbf{f}$. Since $(1,1,1,1,\ldots) \in \varprojlim \mathbf{f} \setminus \varprojlim \mathbf{j}$, we know $\varprojlim \mathbf{j}$ is a proper subcontinuum of $\varprojlim \mathbf{f}$. Moreover, if $O_2 = [0,1/2)$ and $O_i = [0,1]$ for each positive integer $i \neq 2$, then $(\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ is an open subset of $\varprojlim \mathbf{f}$ that is a subset of $\varprojlim \mathbf{j}$. Thus, $\varprojlim \mathbf{f}$ contains a proper subcontinuum that is not nowhere dense, and it follows that $\varprojlim \mathbf{f}$ is a decomposable continuum. Since $\varprojlim \mathbf{g}$ is indecomposable, $\varprojlim \mathbf{f}$ is not homeomorphic to $\varprojlim \mathbf{g}$. The proofs are similar in the cases where $q = 1$ or both $p = 0$ and $q = 1$. $\square$

It is worth mentioning, however, that inverse limits with N-graphs that satisfy either $p = 1$ or $q = 0$ are surprisingly complicated continua in their own right and deserve further study. To close this paper, we give two examples of such N-graphs and indicate a few properties of the resulting inverse limits that hint at their complexity.

Example 3.5. *Let the graph of $f : [0,1] \to C([0,1])$ consist of the straight line segments joining $(0,0)$ to $(1/2,1)$, joining $(1/2,1)$ to $(1,0)$ and joining $(1,0)$ to $(1,1)$. (This is an N-graph with $p = 1$ and $q = 1/2$; see Figure 3.) Then $\varprojlim \boldsymbol{f}$ contains a bucket handle continuum H that is the limiting set for uncountably many sequences of arcs, where (1) any arc A in any of these sequences satisfies $A \cap H = \emptyset$, and (2) if $\mathcal{A}$ and $\mathcal{B}$ are any two of these sequences of arcs, then $\bigcup \mathcal{A} \cap \bigcup \mathcal{B} = \emptyset$.*

Proof. Let $\Lambda : [0,1] \to [0,1]$ be the function whose graph consists of the straight line segments joining $(0,0)$ to $(1/2,1)$, and joining $(1/2,1)$ to $(1,0)$. Then $\varprojlim \boldsymbol{\Lambda}$ is well known to be the bucket handle continuum H,

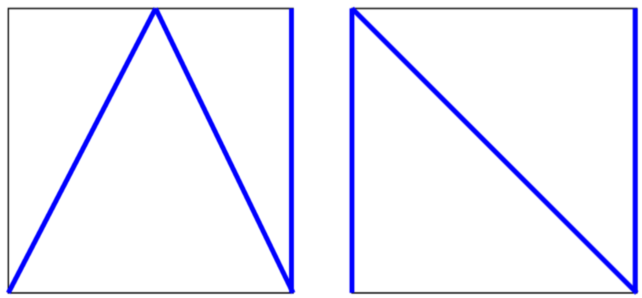

FIGURE 3. Left: The graph of the bonding function f in Example 3.5. Right: The graph of the bonding function f in Example 3.6.

and, since $G(\Lambda) \subseteq G(f)$, clearly $\varprojlim \mathbf{\Lambda} \subseteq \varprojlim \mathbf{f}$. We will now show how to construct the aforementioned sequences of arcs.

Let $\mathbf{y}$ be a particular sequence in $\varprojlim \mathbf{f}$ with $y_1 = 1$, $y_2 = 1$, and $(y_{d+1}, y_d) \in G(\Lambda)$ for each integer $d \geq 2$. Then we define the sequence $\mathcal{A}^{\mathbf{y}} = \{A_1^{\mathbf{y}}, A_2^{\mathbf{y}}, A_3^{\mathbf{y}}, \ldots\}$ as follows: For each positive integer n, let $A_n^{\mathbf{y}} = \{\mathbf{x} \in \varprojlim \mathbf{f} \mid (x_{i+1}, x_i) \in G(\Lambda)$ for all i with $1 \leq i \leq n$; $x_i = y_{i-(n+1)}$ for each positive integer $i \geq n+2\}$. (Thus, $A_n^{\mathbf{y}}$ consists of all $\mathbf{x}$ in $\varprojlim \mathbf{f}$ with $(x_{i+1}, x_i) \in G(\Lambda)$ for $1 \leq i \leq n$ and with the "tail" of $\mathbf{x}$ from coordinate $n+2$ onward being $\mathbf{y}$, i.e., $(x_{n+2}, x_{n+3}, x_{n+4}, \ldots) = (y_1, y_2, y_3, \ldots)$). It is straightforward to prove that for each positive integer n, $A_n^{\mathbf{y}}$ is a continuum that contains exactly two non-cut points (namely, the element of $A_n^{\mathbf{y}}$ with $x_{n+1} = 0$ and the element of $A_n^{\mathbf{y}}$ with $x_{n+1} = 1$), so that each $A_n^{\mathbf{y}}$ is an arc.

To see that $\varprojlim \mathbf{\Lambda}$ is the limiting set for $\mathcal{A}^{\mathbf{y}}$, first let $\mathbf{z} \in \varprojlim \mathbf{\Lambda}$ and let $O = (\prod_{i=1}^{\infty} O_i) \cap \varprojlim \mathbf{f}$ be a basic open set of order k containing $\mathbf{z}$. Thus, in particular, $z_i \in O_i$ for each $i \leq k$. We also know that for each positive integer n, $(z_{n+1}, z_n) \in G(\Lambda)$. Therefore, we know that for each positive integer $n \geq k$, $(z_1, z_2, \ldots, z_n, z_{n+1}, y_1, y_2, y_3, \ldots) \in A_n^{\mathbf{y}} \cap O$. On the other hand, suppose by way of contradiction that $\mathbf{z} \in \varprojlim \mathbf{f}$ and every open set in $\varprojlim \mathbf{f}$ containing $\mathbf{z}$ intersects infinitely many arcs in $\mathcal{A}^{\mathbf{y}}$, but $\mathbf{z} \notin \varprojlim \mathbf{\Lambda}$. Then we know for some positive integer n, $(z_{n+1}, z_n) \in G(f) \setminus G(\Lambda)$. Let

$O_{n+1} \times O_n$ be a basic open subset of $X_{n+1} \times X_n$ containing (z_{n+1}, z_n) and missing $G(\Lambda)$. Then $(X_1 \times X_2 \times \ldots \times X_{n-1} \times O_n \times O_{n+1} \times X_{n+2} \times \ldots) \cap \varprojlim \mathbf{f}$ is an open subset of $\varprojlim \mathbf{f}$ containing $\mathbf{z}$ that can intersect only finitely many arcs in $\mathcal{A}^{\mathbf{y}}$; this is a contradiction. Thus, we have proven that $\varprojlim \mathbf{\Lambda}$ is the limiting set for $\mathcal{A}^{\mathbf{y}}$.

Finally, we turn to conditions (1) and (2). To show (1), note that $y_1 = y_2 = 1$ and $(1,1) \notin G(\mathbf{\Lambda})$, so that every arc in the sequence $\mathcal{A}^{\mathbf{y}}$ does not intersect $\varprojlim \mathbf{\Lambda}$. For (2), let $\mathbf{y}, \mathbf{w} \in \varprojlim \mathbf{f}$ with $y_1 = y_2 = w_1 = w_2 = 1$, $(y_{d+1}, y_d) \in G(\Lambda)$ for all $d \geq 2$, and $(w_{d+1}, w_d) \in G(\Lambda)$ for all $d \geq 2$. We wish to show that if $\mathbf{y} \neq \mathbf{w}$, then $\bigcup \mathcal{A}^{\mathbf{y}} \cap \bigcup \mathcal{A}^{\mathbf{w}} = \emptyset$. Suppose $\mathbf{z} \in \bigcup \mathcal{A}^{\mathbf{y}} \cap \bigcup \mathcal{A}^{\mathbf{w}}$. Then $\mathbf{z} \in A_n^{\mathbf{y}}$ for some positive integer n and $\mathbf{z} \in A_m^{\mathbf{w}}$ for some positive integer m. If $n = m$, then by the way $A_n^{\mathbf{y}}$ and $A_n^{\mathbf{w}}$ are defined, $\mathbf{y}$ and $\mathbf{w}$ are both the same as the sequence $(z_{n+2}, z_{n+3}, z_{n+4}, \ldots)$. This implies $\mathbf{y} = \mathbf{w}$, which is a contradiction. So, suppose $n < m$. Then, since $\mathbf{z} \in A_n^{\mathbf{y}}$, we know $z_{n+2} = 1$, $z_{n+3} = 1$, and $(z_{d+1}, z_d) \in G(\Lambda)$ for all $d \geq n+3$. However, since $\mathbf{z} \in A_m^{\mathbf{w}}$ and $n < m$, we know that for some $k \geq n+3$, $z_k = 1$ and $z_{k+1} = 1$. That means $(z_{k+1}, z_k) \notin G(\Lambda)$, which is a contradiction. A similar contradiction is reached if $m < n$. Thus, we conclude that $\bigcup \mathcal{A}^{\mathbf{y}} \cap \bigcup \mathcal{A}^{\mathbf{w}} = \emptyset$.

Since there are uncountably many different elements $\mathbf{y}$ in $\varprojlim \mathbf{f}$ with $y_1 = y_2 = 1$ and $(y_{d+1}, y_d) \in G(\Lambda)$ for all $d \geq 2$, there are uncountably many different sequences of arcs with the desired properties. Thus, the proof is complete. $\square$

Example 3.6. Let the graph of $f : [0,1] \to C([0,1])$ consist of the straight line segments joining $(0,0)$ to $(0,1)$, joining $(0,1)$ to $(1,0)$, and joining $(1,0)$ to $(1,1)$. (This is an N-graph with $p = 1$ and $q = 0$; see Figure 3.)

Letting $\Omega : [0,1] \to C([0,1])$ be the u.s.c. function whose graph is the union of two straight line segments, one joining $(0,0)$ to $(0,1)$ and one joining $(0,1)$ to $(1,0)$, we see that $\varprojlim \mathbf{\Omega} \subseteq \varprojlim \mathbf{f}$. The complicated inverse limit $\varprojlim \mathbf{\Omega}$ has been studied thoroughly by Ingram (see [3, Example 2.15]), who discovered that it is a non-planar continuum that contains a topologist's sine curve, fans, and n-ods for each n. The addition of the line segment from $(1,0)$ to $(1,1)$ in the graph of f will add even more complexity to this space in a manner similar to the way the line segment from $(1,0)$ to $(1,1)$ added significant complexity to $\varprojlim \mathbf{\Lambda}$ in Example 3.5.

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