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# Semicoverings, Coverings, Overlays, and Open Subgroups of the Quasitopological Fundamental Group

by

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# SEMICOVERINGS, COVERINGS, OVERLAYS, AND OPEN SUBGROUPS OF THE QUASITOPOLOGICAL FUNDAMENTAL GROUP

JEREMY BRAZAS

Abstract. In this paper, we study the classification of semicovering maps, classical covering maps, and Ralph H. Fox's overlays in the context of open subgroups of (quasi)topological fundamental groups. For a given space $X$, we say a subgroup $H \subseteq \pi_1(X, x_0)$ is a semicovering (covering, overlay) subgroup if there is a semicovering (covering, overlay) $p : Y \to X$, $p(y_0) = x_0$ such that $H$ is the image of the monomorphism induced on fundamental groups. Using a new type of Spanier group, we show that every overlay subgroup has open core (i.e., contains an open normal subgroup). We also use semicoverings to show that if $X$ is a so-called locally wep-connected space, then every subgroup of $\pi_1(X, x_0)$ with open core is a covering subgroup. The converse holds for locally path connected spaces but not for general locally wep-connected spaces. We find application to the general theory of topological groups by identifying a large class of spaces $Z$ whose free Graev topological group $F_G(Z, z)$ admits an open subgroup $H$ with non-open core. This is achieved by constructing a covering map $p : E \to B$, which is not an overlay, similar to a well-known example of Fox.

## 1. Introduction

The covering spaces of a path connected, locally path connected, and semilocally simply connected topological space $X$ are classified by the subgroups of the fundamental group $\pi_1(X, x_0)$. This classification is often stated in categorical terms: The categories of the coverings of $X$,

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$\pi_1(X, x_0)$-sets, functors $\pi X \to \mathbf{Set}$ on the fundamental groupoid, and groupoid covering morphisms $\mathscr{G} \to \pi X$ are all equivalent. The covering theoretic notion of classifying maps by algebraic structures is useful in numerous areas of mathematics, including many areas of topology and geometry, group theory, topological group theory, topos theory, Galois categories, etc.

When a topological space $X$ has more complicated local structure, there need not be a universal covering corresponding to the trivial subgroup. Since the mid 1900s, many authors have made an effort to extend covering theoretic methods to these more complicated spaces. A common approach is to designate those properties of a covering map which are deemed important for the intended application and attempt to classify the resulting covering-like maps (or maps with additional structure) in terms of some algebraic or topological algebraic structure. For instance, generalized coverings of locally path connected [12] [14] [19] and one-dimensional [20] spaces have been defined to directly retain choice properties of coverings. Covering theories for objects in other categories, such as uniform spaces [4] [11], their generalizations [29], and topological groups [3], have also appeared.

In many instances, it is appropriate to replace the usual fundamental group with another algebraic object, such as the fundamental progroup [1] [26], fundamental profinite group, first shape homotopy group, Brown-Grossman fundamental group, first steenrod homotopy group, or fundamental localic group [33]. (See [27] for comparisons of many of these objects and how they can be used to classify certain classical covering maps.)

A well-known case is the notion of overlay due to Ralph H. Fox [22] [23]. Overlays are defined to be special types of covering maps (in the classical sense) but can be classified by representations of the fundamental trope in symmetric groups for any path connected separable metric space. Overlays also admit alternative classifications for general connected spaces [28] [30].

Many of the classifying algebraic structures mentioned above naturally inherit topological structure(s) closely related to covering maps and their generalizations [6] [9] [12] [21] [27] [37]. The *quasitopological fundamental group* $\pi_1^{qtop}(X, x_0)$ of a based topological space is the usual fundamental group $\pi_1(X, x_0)$ equipped with the quotient topology with respect to the function $\pi : \Omega(X, x_0) \to \pi_1(X, x_0)$ identifying homotopy class in the based loop space (with the compact-open topology), i.e., the finest topology on $\pi_1(X, x_0)$ such that $\pi$ is continuous. The resulting object $\pi_1^{qtop}(X, x_0)$ need not be a topological group [5] [16] [17] but is a quasitopological group in the sense that inversion is continuous and group multiplication is continuous in each variable. Despite the complication that multiplication can fail to be continuous, $\pi_1^{qtop}(X, x_0)$ retains the notions of homotopically path-Hausdorff [18] and $\pi_1$-shape injective [9] as separation properties and can distinguish shape equivalent spaces. See [10] for more on the theory of quasitopological fundamental groups.

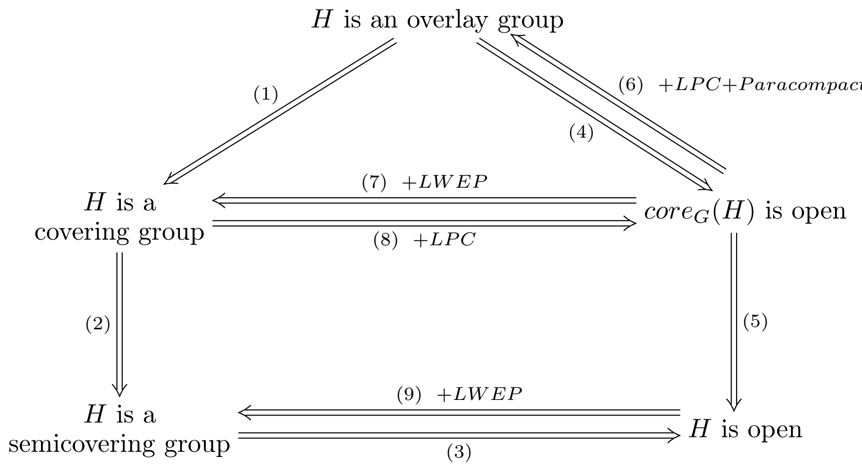
Another natural topology on $\pi_1(X, x_0)$, introduced in [7], is the finest topology on $\pi_1(X, x_0)$ such that $\pi : \Omega(X, x_0) \to \pi_1(X, x_0)$ is continuous *and* such that $\pi_1(X, x_0)$ is a topological group. We call the resulting topological group $\pi_1^\tau(X, x_0)$ the *topological fundamental group.* Though the quotient topology may be strictly finer than the topology of $\pi_1^\tau(X, x_0)$, the two share the same open subgroups. Thus, every result on open subgroups which holds for one topology also holds for the other.

The semicovering maps defined by the author in [6] generalize the classical notion of covering and are classified by the open subgroups of $\pi_1^{qtop}(X, x_0)$ (equivalently, $\pi_1^\tau(X, x_0)$) for all locally path connected and many non-locally path connected spaces (so-called *locally wep-connected spaces*). A natural relationship between $\pi_1^\tau(X, x_0)$ and universal constructions of topological groups has led to new results on free topological groups using semicoverings [8].

Since every overlay is a covering and every covering is a semicovering, the open subgroups of the above topologized fundamental groups provide a natural context for relating these three types of maps. For a given space $X$ and subgroup $H \subseteq \pi_1(X, x_0)$, we say $H$ is a *semicovering (covering, overlay*, respectively) *subgroup* if there is a connected semicovering (covering, overlay, respectively) $p : Y \to X$, $p(y_0) = x_0$ such that $H = p_*(\pi_1(Y, y_0))$. Recall the *core* of a subgroup $H \subseteq G$ is the largest normal subgroup $core_G(H) = \bigcap_{g\in G} gHg^{-1}$ of $G$ contained in $H$. If $G$ is a quasitopological group, then $core_G(H)$ is open in $G$ if and only if there is an open normal subgroup $N$ of $G$ such that $N \subseteq H$.

The main purpose of this paper is to use the characterization of open subgroups as semicovering subgroups to compare covering and overlay subgroups with those subgroups having an open core.

The results of the current paper and some previously known results are summarized in the following diagram. Here $H$ is a subgroup of $G = \pi_1^{qtop}(X, x_0)$ (or $G = \pi_1^\tau(X, x_0)$). A "$+LPC$" means $X$ is assumed to be locally path connected, "$+Paracompact$" means $X$ is assumed to be paracompact, and "$+LWEP$" means $X$ is assumed to be locally wep-connected (a natural generalization of local path connectivity defined in section 2.4). The original contributions of the current paper to this diagram (found in sections 3 and 4) are showing that the arrows (4) and (7) hold and that (8) cannot be extended to locally wep-connected spaces.




Note that (1)–(5) hold for arbitrary $X$.

(1) follows from the fact that every overlay is a covering by definition.
(2) is the known fact that every covering map is a semicovering map [6, Proposition 3.7].
(3) is proved in [6, Theorem 5.5]. See Corollary 3.4 for a more direct proof.
(4) is proved in Theorem 3.9 using a new type of Spanier group which we call the "path-Spanier group."
(5) follows from the elementary fact that if $N \subseteq H$ where $N$ is an open normal subgroup of $G$, then $H$ is the union of cosets of $N$ and is therefore open.
(6) follows from the known fact that a covering of a paracompact locally path connected space is an overlay [30].
(7) is Corollary 4.3 below. See [21] and [32] for very nice proofs of the locally path connected case.
(8) follows directly from (2), (3), and Spanier's approach to covering theory [36] (See Theorem 4.8). In section 4.3, we show there is a compact, locally wep-connected planar set $X$ for which (8) does not hold.
(9) is proved in [6] as part of the classification of semicoverings.

Finally, in section 5 we find application to topological algebra. Free topological groups are important objects in the general theory of topological groups [2] and have an extensive literature dating back to their introduction by A. A. Markov [31] and M. I. Graev [24]. These groups are typically defined in terms of their universal property, which is the topological analogue of the universal property of free groups; however, explicit descriptions of their topological structure are often quite complicated.

It is shown in [8] that every free Graev topological group $F_G(Z, z)$ on a based topological space $(Z, z)$ is isomorphic to the topological fundamental group $\pi_1^\tau(|\Gamma|, v)$ of a so-called **Top**-graph $|\Gamma|$, i.e., a graph where the edge space between fixed pairs of vertices need not be discrete. A **Top**-graph $|\Gamma|$ is locally wep-connected so the semicoverings of $|\Gamma|$ are classified by the open subgroups $H \subset F_G(Z, z)$. Thus, one can extend the study of free groups via coverings of graphs to the study of open subgroups of free topological groups via semicoverings of **Top**-graphs.

In Theorem 5.8, we show that if $Z$ has the first compact ordinal as a quotient, then $F_G(Z, z)$ has an open subgroup $H$ with non-open core. We avoid calling upon an explicit set of generators of $H$ by constructing a covering $p : E \to B$ of a **Top**-graph $B$ which fails to be an overlay. In the effort to further understand the open subgroup structure of free topological groups, we leave the reader with the open problem of whether or not the arrows (2) and (4) in the above diagram can be reversed in the case that $X$ is a **Top**-graph (Problem 5.11).

## 2. Preliminaries

In this section, we recall a number of definitions, examples, and known results to be used in the following sections. Throughout this paper $X$ will be a path connected topological space with basepoint $x_0 \in X$.

### 2.1. Semicovering maps, covering maps, and overlays.

Let $p : Y \to X$ be an open map of topological spaces. An open set $V \subset Y$ is a *p-slice* if $p^{-1}(p(V))$ decomposes as a disjoint union $\coprod_{\lambda\in\Lambda} V_\lambda$ of open sets $V_\lambda \subset Y$ such that $V = V_\lambda$ for some $\lambda$ and $p|V_\lambda : V_\lambda \to p(V)$ is a homeomorphism onto $p(V)$ for each $\lambda \in \Lambda$. An open set $U \subset X$ is *evenly covered* by $p$ if there is a $p$-slice $V \subset Y$ such that $U = p(V)$. Sometimes we say $V$ is a *p-slice over* $U = p(V)$.

A *covering structure* for $p : Y \to X$ is a pair $(\mathscr{S}, \mathscr{E})$ where $\mathscr{E}$ is an open cover of $X$ consisting of sets $U$ which are evenly covered by $p$ and $\mathscr{S}$ is an open cover of $Y$ consisting of every $p$-slice over $U$ for each $U \in \mathscr{E}$.

**Definition 2.1.** A map $p : Y \to X$ is a *covering map* if it admits a covering structure $(\mathscr{S}, \mathscr{E})$.

The definition of overlay is more restrictive than that of covering; however, the payoff is that the classification of overlays applies well beyond the scope of classical covering theory [22] [23] [28] [30] [34]. We use the more recent characterization of overlays in [15] which agrees with Fox's original definition in the connected case and is reminiscent of the definition of paracompactness via open star refinement.

An *overlay structure* for a map $p : Y \to X$ is a covering structure $(\mathscr{S}, \mathscr{E})$ such that for every $y \in Y$, the open star $St(y, \mathscr{S}) = \bigcup \{S \in \mathscr{S} | y \in S\}$ is a $p$-slice.

**Definition 2.2.** A map $p : Y \to X$ is an *overlay* if it admits an overlay structure $(\mathscr{S}, \mathscr{E})$.

In [15], Jerzy Dydak uses methods of Valera Berestovskii and Conrad Plaut [4] to present a chain-lifting condition, which is equivalent to the open-star condition in the definition of overlay structure. Given an open cover $\mathscr{U}$ of a space $X$, a $\mathscr{U}$-chain is a finite sequence $\{x_0, ..., x_n\} \subset X$ such that for each $i = 1, ..., n$, there is a $U_i \in \mathscr{U}$ such that $\{x_{i-1}, x_i\} \subset U_i$. Given a surjective map $p : Y \to X$, a lift of a sequence $\{x_0, ..., x_n\} \subset X$ is a sequence $\{y_0, ..., y_n\} \subset Y$ such that $p(y_i) = x_i$ for each $i$.

**Lemma 2.3** ([15])**.** *Suppose $(\mathscr{S}, \mathscr{E})$ is a covering structure for $p : Y \to X$. Then $(\mathscr{S}, \mathscr{E})$ is an overlay structure for $p$ if and only if for every $\mathscr{E}$-chain $\{x_0, ..., x_n\}$ and $y_0 \in p^{-1}(x_0)$, there is a unique $\mathscr{S}$-chain $\{y_0, ..., y_n\}$ which is a lift of $\{x_0, ..., x_n\}$.*

While coverings and overlays are defined via local triviality, semicoverings are defined via lifting of paths. We recall a few basic constructions.

Let $\mathcal{P}X$ be the space of paths $[0, 1] \to X$ with the compact-open topology. A subbasis for this topology consists of sets $\langle K, U \rangle = \{\alpha \in \mathcal{P}X | \alpha(K) \subseteq U\}$ where $K \subseteq [0, 1]$ is compact and $U \subseteq X$ is open. A convenient basis for this topology consists of sets of the form $\bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], U_j \rangle$ where $0 = t_0 < t_1 < \cdots < t_n = 1$ and $U_j$ is open in $X$. Such sets satisfying $t_j = \frac{j}{n}$ also form a basis.

Let $(\mathcal{P}X)_{x_0} = \{\alpha \in \mathcal{P}X | \alpha(0) = x_0\}$ be the subspace of paths starting at $x_0$ and $\Omega(X, x_0) = \{\alpha \in \mathcal{P}X | \alpha(0) = x_0 = \alpha(1)\}$ be the subspace of loops based at $x_0$. Sometimes it is convenient to view $\Omega(X, x_0)$ as the space of based maps $(S^1, (1, 0)) \to (X, x_0)$.

A map $p : Y \to X$ is said to have *continuous lifting of paths* if the induced map $\mathcal{P}p : (\mathcal{P}Y)_y \to (\mathcal{P}X)_{p(y)}$, $\alpha \mapsto p \circ \alpha$ is a homeomorphism for each $y \in Y$. When $p$ has continuous lifting of paths, we let $\widetilde{\alpha}_y$ denote the unique lift of a path $\alpha : [0, 1] \to X$ starting at $y \in p^{-1}(\alpha(0))$.

**Definition 2.4** ([6])**.** A map $p : Y \to X$ is a *semicovering map* if it is a local homeomorphism which has continuous lifting of paths.

**Remark 2.5.** The definition of semicovering given in Definition 2.4 is simpler than, but equivalent to, the original definition of semicovering given in [6]. We justify this simplification with the following observation. Let $D^2 \subset \mathbb{R}^2$ be the closed unit disk with basepoint $d_0 = (1,0)$ and $(\Phi X)_{x_0}$ be the space of based maps $f : (D^2, d_0) \to (X, x_0)$ with the compact-open topology. In [6] a semicovering is defined to be a local homeomorphism $p : Y \to X$ such that for each $y \in Y$ the induced maps $\mathcal{P}p : (\mathcal{P}Y)_y \to (\mathcal{P}X)_{p(y)}$ and $\Phi p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ are both homeomorphisms. However, observe that $(\Phi X)_{x_0}$ is naturally homeomorphic to the space $\Omega((\mathcal{P}X)_{x_0}, c_{x_0})$ of loops $S^1 \to (\mathcal{P}X)_{x_0}$ based at the constant loop $c_{x_0}$ at $x_0$: If $\ell_t : [0,1] \to D^2$ is the linear path from $d_0$ to a point $t \in S^1$, then $\Psi : (\Phi X)_{x_0} \to \Omega((\mathcal{P}X)_{x_0}, c_{x_0})$, $\Psi(f)(t) = f \circ \ell_t$ is a natural homeomorphism. Hence, the functorality of based loop spaces implies that $\Phi p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ is a homeomorphism whenever $\mathcal{P}p : (\mathcal{P}Y)_y \to (\mathcal{P}X)_{p(y)}$ is a homeomorphism.

Since every path and homotopy of paths in $X$ has a unique lift with respect to a based semicovering map $p : Y \to X$, $p(y_0) = x_0$, we have the following familiar characterization of loops which lift to loops.

**Proposition 2.6.** *If $p : Y \to X$, $p(y_0) = x_0$ is a semicovering, and $\alpha \in \Omega(X, x_0)$ is a loop, then $\widetilde{\alpha}_{y_0}$ is a loop (i.e., $\widetilde{\alpha}_{y_0}(1) = y_0$) if and only if $[\alpha] \in p_*(\pi_1(Y, y_0))$.*

Though semicoverings, coverings, and overlays are not defined to be surjective, it is evident from the path connectivity of $X$ and the existence of path lifts that a semicovering $p : Y \to X$ is surjective whenever $Y \neq \emptyset$. A semicovering (covering, overlay) $p : Y \to X$ is said to be *connected* if $Y$ is non-empty and path connected.

## 2.2. Known results comparing semicoverings, coverings, and overlays.

The following lemma compares the three types of maps under consideration.

**Lemma 2.7.** *For a given map $p : Y \to X$,*

*$p$ is an overlay $\Rightarrow$ $p$ is a covering map $\Rightarrow$ $p$ is a semicovering map.*

*All three types of maps are Serre fibrations with discrete fibers.*

*Proof.* The first implication is by definition. The second is proven in [6, Proposition 3.7]; we sketch the idea here. Clearly, $p$ is a local homeomorphism. Given $y \in Y$, $\mathcal{P}p : (\mathcal{P}Y)_y \to (\mathcal{P}X)_{p(y)}$ is continuous by functorality and is bijective based on classical lifting arguments. Let $(\mathscr{S}, \mathscr{E})$ be a covering structure for $p$ such that if $U \in \mathscr{E}$ and $V \subseteq U$ is open, then

$V \in \mathscr{E}$. Take a non-empty neighborhood $\mathcal{U} = \bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], S_j \rangle \cap (\mathcal{P}Y)_y$ in $(\mathcal{P}Y)_y$ where $S_j \in \mathscr{S}$, $t_0 = 0$, and $t_n = 1$. Observe that

$$\mathcal{P}p(\mathcal{U}) = \bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], p(S_j) \rangle \cap \bigcap_{j=1}^{n-1} \langle \{t_j\}, p(S_j \cap S_{j+1}) \rangle \cap (\mathcal{P}X)_{p(y)}$$

is open in $(\mathcal{P}X)_{p(y)}$. It follows that $\mathcal{P}p$ is a continuous open bijection and thus a homeomorphism.

Semicoverings are Serre fibrations since their lifting properties are essentially the same as those for covering maps [6, Remark 3.3]. Every fiber of a local homeomorphism is discrete. □

Neither arrow of Lemma 2.7 is reversible for compact metric spaces; however, the following case was proven by Sibe Mardesic and Vlasta Matijevic [30].

**Lemma 2.8.** *Every covering of a locally path connected, paracompact Hausdorff space is an overlay.*

In particular, every covering of a Peano continuum (a path connected, locally path connected, compact metric space) is an overlay. A covering of a non-locally path connected compact planar set which is not an overlay was first constructed by Fox [23]; a similar construction appears in section 4.3 below.

The notion of semicovering map coincides with that of covering map in the case that $X$ satisfies the conditions required in classical covering space theory.

**Proposition 2.9** ([6, Corollary 7.2])**.** *If $X$ is path connected, locally path connected, and semilocally 1-connected, then every semicovering of $X$ is a covering.*

Observe that Proposition 2.9 does not require the semicovering in question be connected. Thus, to construct a (connected or non-connected) semicovering which is not a covering, one must take the base space to be either non-locally path connected or non-semilocally simply connected. There are known examples of semicoverings of the Hawaiian earring $\mathbb{H} = \bigcup_{n \geq 1} \{(x, y) \in \mathbb{R}^2 | (x - (1/n))^2 + y^2 = (1/n)^2\}$ which are not coverings due to the failure of local triviality [6] (See [25, ch. 1.3, Exercise 6]). A particularly extreme semicovering of $\mathbb{H}$ is constructed in [21].

One approach to constructing semicoverings which are not coverings is to exploit operations on coverings which fail to yield a covering. For instance, coverings and overlays fail to have the following general property which semicoverings do enjoy.

**Proposition 2.10** ([6]). *The composition of semicoverings is a semicovering.*

*Proof.* Suppose $p : Y \to X$ and $q : Z \to Y$ are semicoverings. Since the composition of local homeomorphisms is a local homeomorphism, $p \circ q$ is a local homeomorphism. If $z \in Z$, then $\mathcal{P}q : (\mathcal{P}Z)_z \to (\mathcal{P}Y)_{q(z)}$ and $\mathcal{P}p : (\mathcal{P}Y)_{q(z)} \to (\mathcal{P}X)_{p(q(z))}$ are homeomorphisms by assumption. Thus, $\mathcal{P}(p \circ q) = \mathcal{P}p \circ \mathcal{P}q : (\mathcal{P}Z)_z \to (\mathcal{P}X)_{p(q(z))}$ is a homeomorphism. □

Recall that the coproduct of a family of maps $f_j : Y_j \to X$, $j \in J$ over $X$ is the map $f : \coprod_{j \in J} Y_j \to X$ defined as $f_j$ on the summand $Y_j$ for each $j \in J$. The following corollary shows that the category of semicoverings over a given space is closed under arbitrary coproducts.

**Corollary 2.11.** *The coproduct of a family of semicoverings over $X$ is a semicovering.*

*Proof.* Suppose $p_j : Y_j \to X$ $j \in J$ is a family of semicoverings. Note $\coprod_{j \in J} Y_j \to \coprod_{j \in J} X$ is a semicovering. Additionally, the map $\coprod_{j \in J} X \to X$ defined as $id_X$ on each summand is a semicovering (since it is a covering map). Since the coproduct $p : \coprod_{j \in J} Y_j \to X$ over $X$ is the composition of these two maps, $p$ is a semicovering by Proposition 2.10. □

The following example, derived from Spanier's text [36, ch. 2.2, Example 8], provides an example of a non-connected semicovering which is not a covering. This example also illustrates that the categories of coverings and overlays over a Peano continuum $X$ need not be closed under infinite coproducts.

**Example 2.12.** Let $A$ be a path connected but non-simply connected space with a simply connected covering $q : \widetilde{A} \to A$ (e.g., $A = S^1$ and $\widetilde{A} = \mathbb{R}$). Let $X = \prod_{n \geq 1} A$ be the countable product. Since $X$ has the product topology and $\pi_1(A, a_0) \neq 1$, $X$ is not semi-locally simply connected. If $A$ is a Peano continuum, then $X$ is a Peano continuum and $q$ is an overlay.

Let $Y_n = \left(\widetilde{A}\right)^n \times \prod_{m > n} A$ and let $r_n : Y_n \to X$ be the covering map $r_n = q^n \times \prod_{m > n} id_A$. Let $Y = \coprod_{n \geq 1} Y_n$ and $p : Y \to X$ be the coproduct of the covering maps $r_n$ over $X$. By Corollary 2.11, $p$ is a semicovering. On the other hand, $p$ cannot be a covering map. Indeed, if $U = \prod_{1 \leq m \leq k} U_m \times \prod_{m > k} A$ is a basic neighborhood of $X$ (i.e., $U_k \neq A$ for finitely many $k$) and $n > k + 1$, then

$$r_n^{-1}(U) = \prod_{1 \leq m \leq k} q^{-1}(U_m) \times \prod_{k < m \leq n} \widetilde{A} \times \prod_{m > n} A \subset Y_n \subset Y.$$

But $q : \widetilde{A} \to A$ is a connected covering with fibers of cardinality $> 1$. Thus, there are at least two points, $a$ and $b$, in the same path component of $r_n^{-1}(U)$ such that $r_n(a) = r_n(b)$.

### 2.3. Topologized fundamental groups.

**Definition 2.13.** The *quasitopological fundamental group* $\pi_1^{qtop}(X, x_0)$ of a based space $(X, x_0)$ is the usual fundamental group $\pi_1(X, x_0)$ equipped with the quotient topology with respect to the canonical map $\pi : \Omega(X, x_0) \to \pi_1(X, x_0)$, $\pi(\alpha) = [\alpha]$ identifying based homotopy classes.

Equivalently, $\pi_1^{qtop}(X, x_0)$ has the finest topology on $\pi_1(X, x_0)$ such that $\pi : \Omega(X, x_0) \to \pi_1(X, x_0)$ is continuous.

A *quasitopological group* is a group $G$ equipped with a topology such that inversion $g \mapsto g^{-1}$ is continuous and multiplication $G \times G \to G$ is continuous in each variable (i.e., equivalently all translations are homeomorphisms) [2]. It is known that $\pi_1^{qtop}$ is a functor from the category of based topological spaces to the category of quasitopological groups and continuous homomorphisms. Though $\pi_1^{qtop}$ is an invariant of homotopy type, $\pi_1^{qtop}(X, x_0)$ can fail to be a topological group [5][16][17]. It is typically non-trivial to determine if $\pi_1^{qtop}(X, x_0)$ is a topological group or not. See [10] for more on quasitopological fundamental groups.

The following topology on $\pi_1(X, x_0)$, introduced in [7], gives $\pi_1(X, x_0)$ the structure of a topological group by definition.

**Definition 2.14.** The *topological fundamental group* $\pi_1^{\tau}(X, x_0)$ of a based space $(X, x_0)$ is the usual fundamental group $\pi_1(X, x_0)$ equipped with the finest *group topology* such that $\pi : \Omega(X, x_0) \to \pi_1(X, x_0)$ is continuous.

The topology of $\pi_1^{\tau}(X, x_0)$ is guaranteed to exist since the category of quasitopological groups is a reflective subcategory of the category of topological groups. We refer to [7] for explicit constructions of $\pi_1^{\tau}(X, x_0)$ and proof of the following lemma.

**Lemma 2.15** ([7])**.** *The quotient topology of $\pi_1^{qtop}(X, x_0)$ is finer than the topology of $\pi_1^{\tau}(X, x_0)$ though the two have the same open subgroups. The two topologies are equal if and only if $\pi_1^{qtop}(X, x_0)$ is a topological group.*

It is shown in [13] that if $X$ is locally path connected and semilocally simply connected, then $\pi_1^{qtop}(X, x_0)$ is a discrete group.

**Corollary 2.16.** *If $X$ is locally path connected and semilocally simply connected, then $\pi_1^{qtop}(X, x_0)$ and $\pi_1^{\tau}(X, x_0)$ are both discrete groups.*

Since $\pi_1^{qtop}(X, x_0)$ and $\pi_1^{\tau}(X, x_0)$ share the same open subgroups, any result on open subgroups which holds for one topology must also hold for the other. We prove most results using $\pi_1^{qtop}(X,x_0)$ since it is more direct to study the quotient topology. The topological fundamental group $\pi_1^{\tau}(X,x_0)$ is used in the last section for application to free topological groups.

The main connection between semicoverings and these two topologized fundamental groups is the following.

**Lemma 2.17.** *If $p:Y\to X$, $p(y_0)=x_0$ is a semicovering, covering, or overlay, then the induced continuous homomorphisms $p_*:\pi_1^{qtop}(Y,y_0)\to \pi_1^{qtop}(X,x_0)$ and $p_*:\pi_1^{\tau}(Y,y_0)\to\pi_1^{\tau}(X,x_0)$ are open embeddings of quasitopological and topological groups, respectively. In particular, the image $p_*(\pi_1(Y,y_0))$ is open in both $\pi_1^{qtop}(X,x_0)$ and $\pi_1^{\tau}(X,x_0)$.*

*Proof.* The lemma is proven for semicoverings in [6, Theorem 5.5] in the more general context of fundamental groupoids. Since every overlay is a covering and every covering is a semicovering, the other cases follow. □

**Remark 2.18.** It is worth noting that there are many other natural topologies one can place on the fundamental group. We remark on two alternatives here. The so-called *shape topology* on $\pi_1(X,x_0)$ is the initial (or pullback) topology with respect to the canonical homomorphism $\Psi:\pi_1(X,x_0)\to\check{\pi}_1(X,x_0)$ to the first shape group (where the shape group is naturally prodiscrete). The shape topology is coarser than the topology of $\pi_1^{\tau}(X,x_0)$ for any $X$ [7, §3.24].

One can also generate a *Spanier topology* on $\pi_1(X,x_0)$ by taking a neighborhood base at the identity to consist of the family of normal *Spanier* subgroups $\pi^s(\mathscr{U},x_0)\subset\pi_1(X,x_0)$ over all open covers $\mathscr{U}$ of $X$ [27] [37] (see Definition 3.1 below). If $X$ is locally path connected, then the Spanier topology is coarser than the topology of $\pi_1^{\tau}(X,x_0)$ and finer than the shape topology. In fact, Theorem 4.8 below implies that the open normal subgroups of $\pi_1^{\tau}(X,x_0)$ generate the Spanier topology. If, in addition, $X$ is paracompact Hausdorff, then the shape and Spanier topologies are equal [9]. Known cases where all four topologies agree include when $X$ is locally path connected and semilocally simply connected (all groups are discrete) and when $X$ is an infinite product of CW-complexes.

### 2.4. Locally wep-connected spaces.

We recall the following recursive definition which naturally generalizes local path connectivity.

**Definition 2.19.** A path $\alpha:[0,1]\to X$ is (*locally*) *well-targeted* if, for every open neighborhood $\mathcal{U}$ of $\alpha$ in $(\mathcal{P}X)_{\alpha(0)}$, there is an open neighborhood $V$ of $\alpha(1)$ such that, for each $v\in V$, there is a (well-targeted) path $\beta\in\mathcal{U}$ with $\beta(1)=v$. Given a basepoint $x_0\in X$, the space $X$ is (*locally*) *wep-connected* if, for any point $x \in X$, there is a (locally) well-targeted path from $x_0$ to $x$.

The definition of a (locally) wep-connected space does not depend on the choice of starting point $x_0 \in X$ since if $\alpha$ and $\beta$ are paths where $\beta$ is (locally) well-targeted and $\alpha(1) = \beta(0)$, then $\alpha * \beta$ is (locally) well-targeted.

It is easy to see that if $X$ is locally path connected at $\alpha(1)$, then $\alpha$ is locally well-targeted. Thus, for any path connected space,

$$\text{local path connectivity} \Rightarrow \text{local wep-connectivity} \Rightarrow \text{wep-connectivity}$$

follows easily from the definitions. An example of a locally wep-connected space which is not locally path connected is the space $B$ in section 4.3.

**Remark 2.20.** The notion of a (locally) well-targeted path was introduced in [7] and [6] alongside the notion of a (*locally*) *well-ended path* (a two-sided version of the (locally) well-targeted property) from which the letters "wep" in "wep-connected" are derived. We retain the nomenclature of these previous papers for consistency. This is justified since the definitions of (locally) wep-connected space obtained using (locally) well-targeted and (locally) well-ended paths are equivalent (See [6, §6]). Our choice to use well-targeted paths is due to the fact that we typically require the use of a basepoint $x_0 \in X$.

To study locally wep-connected spaces we require special notation for certain operations on paths and neighborhoods of paths. For any subinterval $[s,t] \subset [0,1]$, there is a linear homeomorphism $L_{[s,t]} : [0,1] \to [s,t]$, $L_{[s,t]}(r) = (t-s)r + s$. Given a path $\alpha : [0,1] \to X$, let $\alpha_{[s,t]}$ be the path $\alpha|[s,t] \circ L_{[s,t]}$. All of the following neighborhoods are taken in $\mathcal{P}X$. If $\mathcal{U} = \bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], U_j \rangle$ is an open neighborhood of $\alpha$, then

$$\mathcal{U}_{[s,t]} = \bigcap_{[t_{j-1},t_j] \cap [s,t] \neq \emptyset} \left\langle L_{[s,t]}^{-1}\left([t_{j-1}, t_j]\right), U_j \right\rangle$$

is an open neighborhood of $\alpha_{[s,t]}$. If $\mathcal{U} = \bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], U_j \rangle$ and $\mathcal{V} = \bigcap_{i=1}^{m} \langle [s_{i-1}, s_i], V_i \rangle$ are basic neighborhoods of paths $\alpha$ and $\beta$, respectively, and $\alpha(1) = \beta(0)$, then $\mathcal{U}^{-1} = \{\gamma^{-1} | \gamma \in \mathcal{U}\}$ is an open neighborhood of $\alpha^{-1}$ and

$$\mathcal{U}\mathcal{V} = \bigcap_{j=1}^{n} \left\langle L_{[0,1/2]}\left([t_{j-1}, t_j]\right), U_j \right\rangle \cap \bigcap_{i=1}^{m} \left\langle L_{[1/2,1]}\left([s_{i-1}, s_i]\right), V_i \right\rangle$$

is a neighborhood of the concatenation $\alpha * \beta$. For instance, if $\alpha, \beta \in \mathcal{U}$ with $\alpha(1) = \beta(1)$, then $\mathcal{U}\mathcal{U}^{-1}$ is a neighborhood of $\alpha * \beta^{-1}$.

Locally wep-connected spaces are significant because they are the spaces to which the classification of semicovering applies. See [6] for the proof of the following theorem.

**Theorem 2.21.** *Suppose $X$ is locally wep-connected and $x_0 \in X$. A subgroup $H \subset \pi_1(X, x_0)$ is open in $\pi_1^{qtop}(X, x_0)$ if and only if $H$ is a semicovering subgroup of $\pi_1(X, x_0)$.*

## 3. A New Type of Spanier Group

**Definition 3.1.** Given an open cover $\mathscr{V}$ of $X$, the *Spanier group of $X$ with respect to $\mathscr{V}$* is the subgroup $\pi^s(\mathscr{V}, x_0)$ of $\pi_1(X, x_0)$ generated by classes of the form $[\alpha * \gamma * \alpha^{-1}]$ where $\alpha \in (\mathcal{P}X)_{x_0}$ and $\gamma$ is a loop based at $\alpha(1)$ with image in some $V \in \mathscr{V}$.

By construction, $\pi^s(\mathscr{V}, x_0)$ is a normal subgroup of $\pi_1(X, x_0)$. These subgroups originally appeared in Spanier's textbook [36] where they are used to detect the existence of coverings of locally path connected spaces. Recently, Spanier groups have received a great deal of attention due to their utility in studying the first shape group, generalized covering maps, and topologized fundamental groups [9] [12] [18] [19] [27] [32].

To study semicoverings, coverings, and overlays of spaces which are not necessarily locally path connected, we require a more general notion of Spanier group. In particular, we define the so-called *path-Spanier groups* in terms of open covers of the path space $(\mathcal{P}X)_{x_0}$. Our approach is related to that in [32]; however, our definition of path-Spanier group is somewhat less cumbersome and need not be a normal subgroup.

If $\mathcal{U}$ is an open neighborhood in $(\mathcal{P}X)_{x_0}$, let

$$\pi(\mathcal{U}, x_0) = \left\{[\alpha * \beta^{-1}] \in \pi_1(X, x_0) | \alpha, \beta \in \mathcal{U} \text{ and } \alpha(1) = \beta(1)\right\}.$$

If $\mathscr{U}$ is an open cover of $(\mathcal{P}X)_{x_0}$, let $\pi^{ps}(\mathscr{U}, x_0)$ be the subgroup of $\pi_1(X, x_0)$ generated by the sets $\pi(\mathcal{U}, x_0)$ and $\mathcal{U} \in \mathscr{U}$. We call $\pi^{ps}(\mathscr{U}, x_0)$ the *path-Spanier group of $X$ with respect to $\mathscr{U}$*.

**Proposition 3.2.** *For any open cover $\mathscr{U}$ of $(\mathcal{P}X)_{x_0}$, $\pi^{ps}(\mathscr{U}, x_0)$ is open in $\pi_1^{qtop}(X, x_0)$.*

*Proof.* Since $\pi : \Omega(X, x_0) \to \pi_1^{qtop}(X, x_0)$ is quotient, it suffices to show $\pi^{-1}(\pi^{ps}(\mathscr{U}, x_0))$ is open in $\Omega(X, x_0)$. Suppose $\alpha \in \pi^{-1}(\pi^{ps}(\mathscr{U}, x_0))$. Since $\alpha \in (\mathcal{P}X)_{x_0}$, there is a $\mathcal{U} \in \mathscr{U}$ such that $\alpha \in \mathcal{U}$. It suffices to check that $\mathcal{U} \cap \Omega(X, x_0) \subseteq \pi^{-1}(\pi^{ps}(\mathscr{U}, x_0))$. If $\beta \in \mathcal{U}$ is a loop based at $x_0$, then clearly, $[\alpha * \beta^{-1}] \in \pi(\mathcal{U}, x_0) \subseteq \pi^{ps}(\mathscr{U}, x_0)$. Since $[\alpha] \in \pi^{ps}(\mathscr{U}, x_0)$ and $\pi^{ps}(\mathscr{U}, x_0)$ is a subgroup, it follows that $[\beta] \in \pi^{ps}(\mathscr{U}, x_0)$. Thus, $\beta \in \pi^{-1}(\pi^{ps}(\mathscr{U}, x_0))$. ☐

For a given semicovering map $p : Y \to X$, $p(y_0) = x_0$, let $\mathscr{S}$ be an open cover of $Y$ consisting of neighborhoods $S$ such that $p|S : S \cong p(S)$. Define the open cover $\mathscr{L}$ of $(\mathcal{P}X)_{x_0}$ which consists of neighborhoods of the form

$$\mathcal{L} = \bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], p(S_j)\rangle \cap \bigcap_{j=1}^{n-1} \langle \{t_j\}, p(S_j \cap S_{j+1})\rangle$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$ and $S_j \in \mathscr{S}$. If $\mathcal{S} = \bigcap_{j=1}^{n}\langle [t_{j-1}, t_j], S_j\rangle$ where $y_0 \in S_1$, then $\mathcal{P}p : (\mathcal{P}Y)_{y_0} \to (\mathcal{P}X)_{x_0}$ maps $\mathcal{S}$ bijectively onto $\mathcal{L}$ (recall the proof of Lemma 2.7). In particular, if $\alpha, \beta \in \mathcal{L}$, then the unique lifts $\widetilde{\alpha}_{y_0}$ and $\widetilde{\beta}_{y_0}$ both lie in $\mathcal{S}$ and if $\alpha(1) = \beta(1)$, then $\widetilde{\alpha}_{y_0}(1) = \widetilde{\beta}_{y_0}(1)$.

**Lemma 3.3.** $\pi^{ps}(\mathscr{L}, x_0) \subseteq p_*(\pi_1(Y, y_0))$.

*Proof.* Suppose $[\alpha * \beta^{-1}] \in \pi(\mathcal{L}, x_0)$ where $\mathcal{L} \in \mathscr{L}$ as above. Thus, $\widetilde{\alpha}_{y_0}$ and $\widetilde{\beta}_{y_0}$ both lie in $\mathcal{S}$. In particular, $\widetilde{\alpha}_{y_0}(1) = \widetilde{\beta}_{y_0}(1)$ in $S_n$. By Proposition 2.6, we have $[\alpha * \beta^{-1}] \in p_*(\pi_1(Y, y_0))$. Since $p_*(\pi_1(Y, y_0))$ contains all generators of $\pi^{ps}(\mathscr{L}, x_0)$, the inclusion $\pi^{ps}(\mathscr{L}, x_0) \subseteq p_*(\pi_1(Y, y_0))$ holds. □

**Corollary 3.4.** *Every semicovering subgroup of* $\pi_1(X, x_0)$ *is open in* $\pi_1^{qtop}(X, x_0)$.

*Proof.* Since $\pi^{ps}(\mathscr{L}, x_0) \subseteq p_*(\pi_1(Y, y_0))$ by the previous lemma, and $\pi^{ps}(\mathscr{L}, x_0)$ is open, it follows that $p_*(\pi_1(Y, y_0))$ is open in $\pi_1^{qtop}(X, x_0)$ for any semicovering $p : Y \to X$ with $p(y_0) = x_0$. □

Compare the following theorem with the characterization of coverings (Theorem 4.8) and overlays (Theorem 4.9) of locally path connected spaces in terms of ordinary Spanier groups.

**Theorem 3.5.** *Suppose* $X$ *is locally wep-connected and* $H$ *is a subgroup of* $\pi_1(X, x_0)$. *The following are equivalent:*

(1) $H$ *is open in* $\pi_1^{qtop}(X, x_0)$*;*
(2) $H$ *is a semicovering subgroup of* $\pi_1(X, x_0)$*;*
(3) *there is an open cover* $\mathscr{U}$ *of* $(\mathcal{P}X)_{x_0}$ *such that* $\pi^{ps}(\mathscr{U}, x_0) \subseteq H$.

*Proof.* (1) ⇔ (2) is Theorem 2.21.

(2) ⇒ (3). If $p : Y \to X$, $p(y_0) = x_0$ is a semicovering such that $p_*(\pi_1(Y, y_0)) = H$, then, by Lemma 3.3, there is an open cover $\mathscr{L}$ of $(\mathcal{P}X)_{x_0}$ such that $\pi^{ps}(\mathscr{L}, x_0) \subseteq p_*(\pi_1(Y, y_0)) = H$.

(3) ⇒ (1). Suppose there is an open cover $\mathscr{U}$ of $(\mathcal{P}X)_{x_0}$ such that $\pi^{ps}(\mathscr{U}, x_0) \subseteq H$. By Proposition 3.2, $\pi^{ps}(\mathscr{U}, x_0)$ is open in $\pi_1(X, x_0)$. Since $H$ is the disjoint union of cosets of $\pi^{ps}(\mathscr{U}, x_0)$, it follows that $H$ is open in $\pi_1(X, x_0)$. □

If $\mathscr{V}$ is an open cover of $X$, there is a canonical way to construct an open cover of $(\mathcal{P}X)_{x_0}$: Let $\mathcal{P}\mathscr{V}$ be the open cover of $(\mathcal{P}X)_{x_0}$ consisting of all neighborhoods of the form $\bigcap_{j=1}^{n}\langle[t_{j-1},t_j],V_j\rangle\cap(\mathcal{P}X)_{x_0}$ where $0\leq t_0<t_1<\cdots<t_n=1$ and $V_j\in\mathscr{V}$.

**Proposition 3.6.** *If $\mathscr{V}$ is an open cover of $X$, then $\pi^{ps}(\mathcal{P}\mathscr{V},x_0)$ is an open normal subgroup of $\pi_1^{qtop}(X,x_0)$.*

*Proof.* Note $\pi^{ps}(\mathcal{P}\mathscr{V},x_0)$ is open by Lemma 3.2. Since the groups $\pi(\mathcal{V},x_0)$, $\mathcal{V}\in\mathcal{P}\mathscr{V}$, generate $\pi^{ps}(\mathcal{P}\mathscr{V},x_0)$, it suffices to show that

$$[\gamma]\pi(\mathcal{V},x_0)[\gamma^{-1}]\subseteq\pi^{ps}(\mathcal{P}\mathscr{V},x_0)$$

for each loop $\gamma$ based at $x_0$.

Suppose $[\alpha*\beta^{-1}]\in\pi(\mathcal{V},x_0)$ for $\mathcal{V}\in\mathcal{P}\mathscr{V}$. In particular, suppose

$$\alpha,\beta\in\mathcal{V}=\bigcap_{j=1}^{n}\langle[t_{j-1},t_j],V_j\rangle\cap(\mathcal{P}X)_{x_0}$$

where $0\leq t_0<t_1<\cdots<t_n=1$ and $V_j\in\mathscr{V}$. Let $\gamma$ be any loop in $X$ based at $x_0$ and find $0\leq s_0<s_1<\cdots<s_m=1$ and $W_i\in\mathscr{V}$ such that $\gamma\in\mathcal{W}=\bigcap_{i=1}^{m}\langle[s_{i-1},s_i],W_i\rangle$.

There is a subdivision $0<r_0<r_1<...<r_{m+n}=1$ and neighborhoods $A_k\in\mathscr{V}$ such that

$$\gamma*\alpha\in\mathcal{V}'=\bigcap_{k=1}^{m+n}\langle[r_{k-1},r_k],A_k\rangle\cap(\mathcal{P}X)_{x_0}.$$

In particular, $A_k=W_k$ for $0\leq k\leq m$ and $A_k=V_{k-m}$ for $m+1\leq k\leq m+n$. It is clear that $\gamma*\beta\in\mathcal{V}'$ as well. The construction of $\mathcal{V}'$ guarantees $\mathcal{V}'\in\mathcal{P}\mathscr{V}$. Thus,

$$[\gamma][\alpha*\beta^{-1}][\gamma^{-1}]=[(\gamma*\alpha)*(\gamma*\beta)^{-1}]\in\pi(\mathcal{V}',x_0)\subseteq\pi^{ps}(\mathcal{P}\mathscr{V},x_0).\quad\square$$

It is worthwhile to compare path-Spanier groups and ordinary Spanier groups.

**Proposition 3.7.** *If $\mathscr{V}$ is an open cover of $X$, then $\pi^{s}(\mathscr{V},x_0)\subseteq\pi^{ps}(\mathcal{P}\mathscr{V},x_0)$.*

*Proof.* Let $[\alpha*\gamma*\alpha^{-1}]$ be a generator of $\pi^{s}(\mathscr{V},x_0)$ where $\gamma$ has image in $V\in\mathscr{V}$. Pick a subdivision $0\leq t_0<t_1<\cdots<t_n=1$ such that $\alpha([t_{i-1},t_i])\subset V_i$ for sets $V_i\in\mathscr{V}$. Let $V_{n+1}=V$, $[s_{i-1},s_i]=L_{[0,1/2]}([t_{i-1},t_i])$, and $s_{n+1}=1$. Now $\alpha*\gamma_{[0,1/2]}$ and $\alpha*\gamma_{[1/2,1]}^{-1}$ are elements of $\mathcal{V}=\bigcap_{i=1}^{n+1}\langle[s_{i-1},s_i],V_i\rangle\cap(\mathcal{P}X)_{x_0}$ where $V_i\in\mathscr{V}$ for each $i=1,...,n+1$. This shows that

$$[\alpha * \gamma * \alpha^{-1}] = \left[\alpha * \gamma_{[0,1/2]} * \left(\alpha * \gamma^{-1}_{[1/2,1]}\right)^{-1}\right] \in \pi(\mathcal{V}, x_0) \subset \pi^{ps}(\mathcal{P}\mathscr{V}, x_0). \quad \square$$

**Lemma 3.8.** *If $(\mathscr{S}, \mathscr{E})$ is an overlay structure for $p : Y \to X$, then*

$$p_*(\pi^{ps}(\mathcal{P}\mathscr{S}, y_0)) = \pi^{ps}(\mathcal{P}\mathscr{E}, x_0).$$

*Proof.* Let $\eta$ and $\zeta$ be paths in $Y$ based at $y_0$ such that $\eta(1) = \zeta(1)$. Suppose there is a partition $0 = t_0 < t_1 < \dots t_n = 1$ and $S_j \in \mathscr{S}$ such that $\mathcal{U} = \bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], S_j \rangle \cap (\mathcal{P}Y)_{y_0}$ is a neighborhood of both $\eta$ and $\zeta$. Thus, $[\eta * \zeta^{-1}]$ is a generator of $\pi^{ps}(\mathcal{P}\mathscr{S}, y_0)$. If $U_j = p(S_j) \in \mathscr{E}$, then $\bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], U_j \rangle \cap (\mathcal{P}X)_{x_0}$ is a neighborhood of $p \circ \eta$ and $p \circ \zeta$ in $(\mathcal{P}X)_{x_0}$. Thus, $p_*([\eta * \zeta^{-1}]) = [(p \circ \eta) * (p \circ \zeta)^{-1}] \in \pi^{ps}(\mathcal{P}\mathscr{E}, x_0)$, and the inclusion $p_*(\pi^{ps}(\mathcal{P}\mathscr{S}, y_0)) \subseteq \pi^{ps}(\mathcal{P}\mathscr{E}, x_0)$ follows.

For the other inclusion, suppose $\alpha, \beta \in (\mathcal{P}X)_{x_0}$ are paths such that $\alpha(1) = \beta(1)$ and such that there is a partition $0 = t_0 < t_1 < \dots < t_n = 1$ and $U_j \in \mathscr{E}$ making $\mathcal{U} = \bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], U_j \rangle \cap (\mathcal{P}X)_{x_0}$ a neighborhood of both $\alpha$ and $\beta$. Thus, $[\alpha * \beta^{-1}]$ is a generator of $\pi^{ps}(\mathcal{P}\mathscr{E}, y_0)$. Note that $\{x_0, \alpha(t_1), ..., \alpha(t_n)\}$ and $\{x_0, \beta(t_1), ..., \beta(t_n)\}$ are $\mathscr{E}$-chains. By the characterization of overlay structures in Lemma 2.3, these $\mathscr{E}$-chains have unique lifts $\{y_0 = a_0, a_1, ..., a_n\}$ and $\{y_0 = b_0, b_1, ..., b_n\}$ in $Y$, respectively, which are $\mathscr{S}$-chains. In particular, there is a sequence $S_1, ..., S_n \in \mathscr{S}$ where $S_j$ is a $p$-slice over $U_j$ and $\{a_{j-1}, a_j, b_{j-1}, b_j\} \subseteq S_j$ for $j = 1, ..., n$. Since $p|_{S_j} : S_j \to U_j$ is a homeomorphism, the lifts $\widetilde{\alpha}_{y_0}$ and $\widetilde{\beta}_{y_0}$ must lie in the neighborhood $\mathcal{V} = \bigcap_{j=1}^{n} \langle [t_{j-1}, t_j], S_j \rangle \cap (\mathcal{P}Y)_{y_0}$. Note $p$ maps $S_n$ homeomorphically onto $U_n$, and thus $\widetilde{\alpha}_{y_0}(1) = \widetilde{\beta}_{y_0}(1)$. It follows that $\left[\widetilde{\alpha}_{y_0} * \widetilde{\beta}^{-1}_{y_0}\right] \in \pi^{ps}(\mathcal{P}\mathscr{S}, y_0)$ where $p_*\left(\left[\widetilde{\alpha}_{y_0} * \widetilde{\beta}^{-1}_{y_0}\right]\right) = \left[\alpha * \beta^{-1}\right]$. $\square$

**Theorem 3.9.** *If $X$ is path connected and $H \subseteq \pi_1(X, x_0)$ is an overlay subgroup, then there is an open normal subgroup $N \subseteq \pi_1^{qtop}(X, x_0)$ such that $N \subseteq H$.*

*Proof.* Suppose $p : Y \to X$, $p(y_0) = x_0$ is an overlay with $p_*(\pi_1(Y, y_0)) = H$. Let $(\mathscr{S}, \mathscr{E})$ be an overlay structure for $p$. Then, by Lemma 3.8,

$$\pi^{ps}(\mathcal{P}\mathscr{E}, x_0) = p_*(\pi^{ps}(\mathcal{P}\mathscr{S}, y_0)) \subseteq p_*(\pi_1(Y, y_0)) = H.$$

According to Proposition 3.6, $N = \pi^{ps}(\mathcal{P}\mathscr{E}, x_0)$ is an open normal subgroup of $\pi_1^{qtop}(X, x_0)$. $\square$

## 4. Open Normal Subgroups and Covering Subgroups

### 4.1. The locally-wep connected case.

The following lemma provides a topological condition on $\pi_1^{qtop}(X, x_0)$ sufficient to know that a semicovering is a covering.

**Theorem 4.1.** *Suppose $X$ is locally wep-connected and $p : Y \to X$, $p(y_0) = x_0$ is a connected semicovering. If there is an open normal subgroup $N \subseteq \pi_1^{qtop}(X, x_0)$ such that $N \subseteq p_*(\pi_1(Y, y_0))$, then $p$ is a covering map.*

*Proof.* Given $x \in X$, we check that $x$ has a neighborhood evenly covered by $p$. Since $N$ is open in $\pi_1^{qtop}(X, x_0)$ and $\pi : \Omega(X, x_0) \to \pi_1^{qtop}(X, x_0)$ is continuous, $\pi^{-1}(N)$ is open in $\Omega(X, x_0)$. Let $\alpha : [0, 1] \to X$ be a locally well-targeted path from $x_0$ to $x$. Since the concatenation $\alpha * \alpha^{-1}$ is null-homotopic, it lies in $\pi^{-1}(N)$. Find a basic open neighborhood $\mathcal{U} = \bigcap_{j=1}^{n} \left\langle \left[\frac{j-1}{n}, \frac{j}{n}\right], U_j \right\rangle$ of $\alpha$ such that $\alpha * \alpha^{-1} \in \mathcal{U}\mathcal{U}^{-1} \cap \Omega(X, x_0) \subset \pi^{-1}(N)$. Since $\alpha$ is locally well-targeted, there is an open neighborhood $U$ of $x$ such that for each $z \in U$, there is a well-targeted path $\gamma_z : [0, 1] \to X$ in $\mathcal{U}$ from $x_0$ to $z$. When $z = x$, take $\gamma_x = \alpha$. We claim that $U$ is evenly covered by $p$.

For each loop $\beta \in \Omega(X, x_0)$, let $V_\beta = \left\{ \widetilde{(\beta * \gamma_z)}_{y_0}(1) | z \in U \right\}$. By definition, we have $p(V_\beta) \subset U$ for every $\beta$. It is clear from the uniqueness of lifts that $p$ maps $V_\beta$ bijectively onto $U$. On the other hand, if $y \in p^{-1}(U)$, then $\gamma_{p(y)} \in \mathcal{U}$. There is a point $y_1 \in p^{-1}(x_0)$ such that $\widetilde{(\gamma_{p(y)})}_{y_1}(1) = y$. Since $Y$ is path connected, there is a path $\delta : [0, 1] \to Y$ from $y_0$ to $y_1$. If $\beta = p \circ \delta \in \Omega(X, x_0)$, then we have $y \in V_\beta$. It follows that $p^{-1}(U) = \bigcup_\beta V_\beta$.

Let $H = p_*(\pi_1(Y, y_0))$. Note that if $[\beta_1 * \beta_2^{-1}] \in H$ for loops $\beta_1, \beta_2 \in \Omega(X, x_0)$, then $\widetilde{(\beta_1)}_{y_0}(1) = \widetilde{(\beta_2)}_{y_0}(1)$, and thus $V_{\beta_1} = V_{\beta_2}$. Conversely, suppose $y \in V_{\beta_1} \cap V_{\beta_2}$. Then $\widetilde{(\beta_1 * \gamma_a)}_{y_0}(1) = y = \widetilde{(\beta_2 * \gamma_b)}_{y_0}(1)$ for $a, b \in U$. But $a = p(y) = b$, and therefore $[\beta_1 * \gamma_a * (\beta_2 * \gamma_b)^{-1}] = [\beta_1 * \beta_2^{-1}] \in H$. It follows that $V_{\beta_1} = V_{\beta_2}$. We conclude that $p^{-1}(U)$ is the set theoretic disjoint union $\coprod_{H[\beta]} V_\beta$ where the union ranges over the right cosets of $H$.

The proof that each $V_\beta$ is open requires the assumption that $N$ is normal and $X$ is locally wep-connected and is given in the following proposition. □

**Proposition 4.2.** *For any $\beta \in \Omega(X, x_0)$, $V_\beta$ is open in $Y$.*

*Proof.* Since $X$ is locally wep-connected, $Y$ is locally wep-connected [6, Corollary 6.12]. Thus, evaluation $ev_1 : (\mathcal{P}Y)_{y_0} \to Y$, $ev_1(\zeta) = \zeta(1)$ is quotient [6, Proposition 6.2]. Additionally, $\mathcal{P}p : (\mathcal{P}Y)_{y_0} \to (\mathcal{P}X)_{x_0}$ has a continuous inverse $L : (\mathcal{P}X)_{x_0} \to (\mathcal{P}Y)_{y_0}$. Thus, the composition $ev_1 \circ L : (\mathcal{P}X)_{x_0} \to Y$, $\delta \mapsto \widetilde{\delta}_y(1)$ is quotient. To show $V_\beta$ is open in $Y$, it suffices to show $L^{-1}(ev_1^{-1}(V_\beta))$ is open in $(\mathcal{P}X)_{x_0}$.

Suppose $\delta \in L^{-1}(ev_1^{-1}(V_\beta))$. Then $\widetilde{\delta}_{y_0}(1) = \widetilde{(\beta * \gamma_z)}_{y_0}(1)$ for some $z \in U$. Thus, $[\beta * \gamma_z * \delta^{-1}] \in H$. Note that $\pi^{-1}(H)$ is an open neighborhood of $\beta * \gamma_z * \delta^{-1}$ in $\Omega(X, x_0)$. Find a basic open neighborhood $\mathcal{W} = \bigcap_{i=1}^{m} \left\langle \left[\frac{i-1}{m}, \frac{i}{m}\right], W_j \right\rangle$ of $\beta * \gamma_z * \delta^{-1}$ such that

- $\mathcal{W} \cap \Omega(X, x_0) \subseteq \pi^{-1}(H)$,
- $m$ is divisible by 3, and
- $\mathcal{W}_{[1/3,2/3]} \cap (\mathcal{P}X)_{x_0}$ is an open neighborhood of $\gamma_z$ contained in $\mathcal{U} \cap (\mathcal{P}X)_{x_0}$.

Since $\gamma_z$ is well-targeted, there is an open neighborhood $U'$ of $\delta(1) = z$ such that $U' \subseteq U$ and such that for each $w \in U'$, there is a path $\zeta_w \in \mathcal{W}_{[1/3,2/3]} \cap (\mathcal{P}X)_{x_0}$ from $x_0$ to $w$.

Note that $\left(\mathcal{W}_{[2/3,1]}\right)^{-1}$ is an open neighborhood of $\delta$ in $\mathcal{P}X$. We claim the open neighborhood $\mathcal{V} = \left(\mathcal{W}_{[2/3,1]}\right)^{-1} \cap \langle \{1\}, U' \rangle \cap (\mathcal{P}X)_{x_0}$ of $\delta$ is contained in $L^{-1}(ev_1^{-1}(V_\beta))$. Let $\epsilon \in \mathcal{V}$ and $w = \epsilon(1) \in U'$. Now the concatenation $\beta * \zeta_w * \epsilon^{-1}$ is a loop contained in $\mathcal{W}$. Thus, $[\beta * \zeta_w * \epsilon^{-1}] \in H$. This gives $ev_1 \circ L(\epsilon) = \widetilde{\epsilon}_{y_0}(1) = \widetilde{(\beta * \zeta_w)}_{y_0}(1)$.

Since $\zeta_w \in \mathcal{U} \cap (\mathcal{P}X)_{x_0}$ and $U' \subseteq U$, we have $\zeta_w(1) = w = \gamma_w(1)$ and $\zeta_w * (\gamma_w)^{-1} \in \mathcal{U}\mathcal{U}^{-1} \cap \Omega(X, x_0) \subseteq \pi^{-1}(N)$. Therefore, $[\zeta_w * (\gamma_w)^{-1}] \in N$. Our assumption that $N$ is normal gives

$$[\beta * \zeta_w * (\beta * \gamma_w)^{-1}] = [\beta][\zeta_w * (\gamma_w)^{-1}][\beta^{-1}] \in [\beta]N[\beta^{-1}] = N \subseteq H.$$

Thus,

$$ev_1 L(\epsilon) = \widetilde{(\beta * \zeta_w)}_{y_0}(1) = \widetilde{(\beta * \gamma_w)}_{y_0}(1) \in V_\beta\,,$$

and the proof is complete. □

**Corollary 4.3.** *Suppose $X$ is locally wep-connected and $H$ is a subgroup of $\pi_1(X, x_0)$. If there is an open normal subgroup $N \subseteq \pi_1^{qtop}(X, x_0)$ such that $N \subseteq H$ (i.e., $H$ has open core), then $H$ is a covering subgroup of $\pi_1(X, x_0)$.*

*Proof.* Since $N$ is open in $\pi_1^{qtop}(X, x_0)$, $H$ is the union of cosets of $N$ and is therefore open in $\pi_1^{qtop}(X, x_0)$. By Theorem 3.5, there is a semicovering $p : Y \to X$, $p(y_0) = x_0$ such that $p_*(\pi_1(Y, y_0)) = H$. Since $N$ is normal, we may apply Theorem 4.1 to see that $p$ is a covering. □

**Corollary 4.4.** *If $X$ is locally wep-connected and every open subgroup of $\pi_1^{qtop}(X, x_0)$ has an open core, then every connected semicovering of $X$ is a covering.*

**Example 4.5.** An infinite Cartesian product of manifolds or CW-complexes $\prod_\lambda X_\lambda$ is locally path connected but fails to be semilocally simply connected if infinitely many factors are not simply connected. Since $G_\lambda = \pi_1^{qtop}(X_\lambda, x_\lambda)$ is discrete for each $\lambda$, $G = \pi_1^{qtop}(X, x_0)$ is isomorphic to the product $\prod_\lambda G_\lambda$. If $H$ is an open subgroup of $G$, take a neighborhood

$$N = \prod_{\lambda \in F} e_\lambda \times \prod_{\lambda \notin F} G_\lambda$$

of the identity contained in $H$ where $e_\lambda$ is the (open) trivial subgroup of $G_\lambda$ and $F$ is a finite set. Since $N$ is an open normal subgroup of $G$ and $H$ is arbitrary, we can apply Corollary 4.4 to conclude that every connected semicovering of $X$ is a covering. This observation is in contrast with Example 2.12 where it is shown that a *non-connected* semicovering of an infinite product of CW-complexes need not be a covering.

As an application, we generalize the characterization of discreteness of fundamental groups in [13] to locally wep-connected spaces.

**Corollary 4.6.** *Suppose $X$ is locally wep-connected. Then $\pi_1^{qtop}(X, x_0)$ is discrete if and only if $X$ admits a simply connected covering space. Moreover, if $\pi_1^{qtop}(X, x_0)$ is discrete, then every semicovering of $X$ is a covering.*

*Proof.* If $p : Y \to X$ is a covering map where $Y$ is simply connected, then the trivial subgroup $\pi_*(\pi_1(Y, y_0))$ is open in $\pi_1^{qtop}(X, x_0)$ by Lemma 2.17. Thus, $\pi_1^{qtop}(X, x_0)$ is a discrete group. Conversely, if $\pi_1^{qtop}(X, x_0)$ is discrete, then the trivial subgroup is a normal open subgroup of $\pi_1^{qtop}(X, x_0)$. Theorem 3.5 gives the existence of a semicovering $p : Y \to X$ such that $p_*(\pi_1(Y, y_0))$ is the trivial subgroup. Since $p_*(\pi_1(Y, y_0))$ is normal, $p$ is a covering by Theorem 4.1.

The last statement of the corollary is immediate from Corollary 4.4. □

## 4.2. **The locally path connected case**.

We now observe that, for locally path connected spaces, the converse of Corollary 4.3 holds. The following theorem, which is a combination of Lemma 2.5.11 and Theorem 2.5.13 in [36], detects the existence of coverings of a locally path connected space via Spanier groups.

**Theorem 4.7.** *Suppose $X$ is path connected, locally path connected and $H$ is a subgroup of $\pi_1(X, x_0)$. There is a covering map $p : Y \to X$,*

*$p(y_0) = x_0$ such that $p_*(\pi_1(Y, y_0)) = H$ if and only if there is an open cover $\mathscr{V}$ of $X$ such that $\pi^s(\mathscr{V}, x_0) \subseteq H$.*

For a path connected, locally path connected space $X$, the following theorem classifies the covering subgroups of $\pi_1(X, x_0)$ as precisely those subgroups with an open core. This result has recently appeared in [21] and [32]. The authors of [27] provide another classification of the coverings of a locally path connected space in terms of continuous representations on the fundamental group with the Spanier topology (recall Remark 2.18).

**Theorem 4.8** ([21] [32])**.** *Suppose $X$ is path connected, locally path connected and $H$ is a subgroup of $\pi_1(X, x_0)$. The following are equivalent:*

1. *$H$ contains an open normal subgroup $N$ of $\pi_1^{qtop}(X, x_0)$;*
2. *$H$ is a covering subgroup of $\pi_1(X, x_0)$;*
3. *there is an open cover $\mathscr{V}$ of $X$ such that $\pi^s(\mathscr{V}, x_0) \subseteq H$.*

*Proof.* First, note that (2) $\Leftrightarrow$ (3) is precisely Theorem 4.7.

(1) $\Rightarrow$ (2). Since $H$ is open, Theorem 3.5 gives a connected semicovering $p : Y \to X$, $p(y_0) = x_0$ such that $p_*(\pi_1(Y, y_0)) = H$. By Theorem 4.1, $p$ is a covering map.

(3) $\Rightarrow$ (1). By Theorem 4.7, the normal subgroup $N = \pi^s(\mathscr{V}, x_0)$ is a covering group. Consequently, $N$ is open in $\pi_1^{qtop}(X, x_0)$ by Theorem 3.5. □

By Lemma 2.8, every covering of a locally path connected, paracompact Hausdorff space is an overlay. We combine this fact with Theorem 4.8 to classify overlays in a similar fashion.

**Theorem 4.9.** *Suppose $X$ is path connected, paracompact Hausdorff, and locally path connected, and $H$ is a subgroup of $\pi_1(X, x_0)$. The following are equivalent:*

1. *$H$ contains an open normal subgroup $N$ of $\pi_1^{qtop}(X, x_0)$;*
2. *$H$ is an overlay subgroup of $\pi_1(X, x_0)$;*
3. *there is an open cover $\mathscr{V}$ of $X$ such that $\pi^s(\mathscr{V}, x_0) \subseteq H$.*

### 4.3. **A covering subgroup with non-open core**.

In this section, we show that the converse of Corollary 4.3 does not hold for locally wep-connected metric spaces. In particular, we prove the following theorem.

**Theorem 4.10.** *There is a compact, locally wep-connected space $B \subset \mathbb{R}^2$ and a covering map $p : E \to B$, $p(e_0) = b_0$ such that $p_*(\pi_1(E, e_0))$ contains no open normal subgroup of $\pi_1^{qtop}(B, b_0)$.*

We prove Theorem 4.10 by constructing a covering map which is similar to an example of a covering map which is not an overlay due to Fox [23] (see also [34]). Though Fox's example can also be used to prove Theorem 4.10, we use a slightly altered version to provide direct application to free topological groups in the following section.

For each $n \geq 1$, let $Q_n \subset \mathbb{R}^2$ be the boundary of the quadrilateral which has vertices $(0,0)$, $(1/n, 1/8)$, $(1/n, 7/8)$, and $(0,1)$ and let $B = \bigcup_{n\geq 1} Q_n$. (See Figure 1.) Let $b_0 = (0,0)$ be the basepoint of $B$.

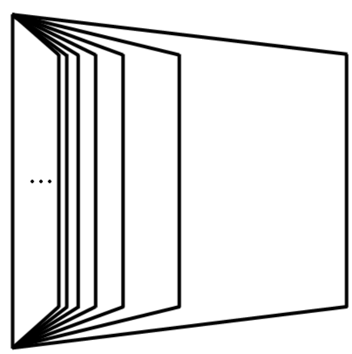


FIGURE 1. The planar set $B$.

**Proposition 4.11.** *$B$ is locally wep-connected.*

*Proof.* Observe that $B$ is locally path connected at every point $b$ in $B\backslash(\{0\}\times(0,1))$. Thus, every path from $b_0$ to $b \in B\backslash(\{0\}\times(0,1))$ is locally well-targeted. Suppose $b = (0,t)$ for $0 < t < 1$ and let $\alpha$ be any path from $b_0$ to $b$ in $\{0\}\times[0,1)$ such that $\alpha^{-1}(b_0) = \{0\}$.

For each $n \geq 1$, let $r_n : Q_n \to \{0\}\times[0,1]$ be the projection onto the y-axis. Let $\mathcal{U} = \bigcap_{i=1}^{m}\langle[t_{i-1}, t_i], U_i\rangle$ be a basic neighborhood of $\alpha$ in $(\mathcal{P}B)_{b_0}$. We may assume $U_1$ is of the form $\bigcup_{n\geq 1} r_n^{-1}(\{0\}\times[0,d_1))$ and $U_i$, where $2 \leq i \leq m$, is of the form $\bigcup_{n\geq N} r_n^{-1}(\{0\}\times(c_i,d_i))$ for some fixed $N \geq 1$. Consider any point $v = (s,t') \in U_m = V$.

Let $\delta : [0,1] \to \{0\}\times[0,1)$ be the concatenation of $\alpha$ followed by a linear path from $(0,t)$ to $(0,t')$ in $\{0\}\times(c_m,d_m)$. Take $\alpha'$ to be a suitable reparameterization of $\delta$ so that $\alpha' \in \mathcal{U}$. For each $n \geq N$, let $\alpha'_n$ be the unique path in $Q_n\backslash(\{0\}\times(0,1))$ such that $r_n \circ \alpha'_n = \alpha'$. Clearly, $\alpha'_n \in \mathcal{U}$ for each $n \geq N$. Notice that either $v = \alpha'(1)$ or $v = \alpha'_k(1)$ for some $k \geq N$. Thus, $\alpha$ is well-targeted.

Finally, observe that if $s \neq 0$, then $B$ is locally path connected at $v$, and thus $\alpha'_k$ is a well-targeted path from $b_0$ to $v$. If $s = 0$, then $\alpha'$ is a well-targeted path from $b_0$ to $v$ since the argument for $\alpha$ also applies to $\alpha'$. Consequently, $B$ is locally wep-connected. □

Let $\alpha_n : [0,1] \to B$ be a piecewise linear loop based at $b_0$ which traverses $Q_n$ once in the counterclockwise direction. Let $\alpha_0$ be a piecewise linear, null-homotopic loop which travels from $b_0$ to $(0,1)$ and back to $b_0$ on $\{0\} \times [0,1]$. Choose parameterizations of these loops so that $\alpha_n \to \alpha_0$ uniformly in the loop space $\Omega(B, b_0)$. Since $\pi\colon \Omega(B, b_0) \to \pi_1^{qtop}(X, x_0)$ is continuous, $[\alpha_n] \to [\alpha_0]$ in $\pi_1(B, b_0)$.

**Proposition 4.12.** *$\pi_1(B, b_0)$ is freely generated by the set $\{[\alpha_n] | n \geq 1\}$.*

*Proof.* Let $P(B)$ be the so-called universal Peano space of $B$ with the same underlying set as $B$ but whose topology is generated by the path components of open sets in $B$ (and is therefore finer than the topology of $B$). It is known that if $Z$ is locally path connected, then a function $f : Z \to B$ is continuous if and only if $f : Z \to P(B)$ is continuous [12] [14]. In particular, this correspondence applies to all paths and homotopies of paths. Therefore, the continuous identity $P(B) \to B$ induces an isomorphism $\pi_1(P(B), b_0) \to \pi_1(B, b_0)$ of fundamental groups. Since each loop $\alpha_n : [0,1] \to P(B)$ is continuous, it suffices to show $\pi_1(P(B), b_0)$ is freely generated by the set $\{[\alpha_n] | n \geq 1\}$. Observe that $P(B)$ is a graph with two vertices, $(0,0)$ and $(0,1)$, and infinitely many edges, $d_0 = \{0\} \times [0,1]$ and $d_n = Q_n \backslash (\{0\} \times (0,1))$. Since $d_0$ is a maximal tree in $P(B)$, application of classical graph theory shows the homotopy classes $[\alpha_n]$ freely generate $\pi_1(P(B), b_0)$. □

We define the covering $p : E \to B$ by splitting $B$ into two pieces, $C_0 = B \cap ([0,1] \times [0,1/2])$ and $C_0 = B \cap ([0,1] \times [1/2,1])$, so that $B = C_0 \cup C_1$. Let $\mathbb{N} = \{1, 2, ...\}$ be the natural numbers and $S = \{1, 1/2, ..., 1/n, ..., 0\}$. Define $E = (C_0 \times \mathbb{N} \times \{0\}) \cup (C_1 \times \mathbb{N} \times \{1\}) / \sim$ where

- $(1/n, 1/2, n, i) \sim (1/n, 1/2, n+1, 1-i)$ for all $n \geq 1$, $i \in \{0,1\}$,
- $(a, 1/2, 1, 0) \sim (a, 1/2, 1, 1)$ for $a \in S \backslash \{1\}$, and
- $(a, 1/2, n, 0) \sim (a, 1/2, n, 1)$ for all $a \in S \backslash \{1/(n-1), 1/n\}$.

The covering map $p : E \to B$ is given by $p[a, b, n, i] = (a, b)$. (See Figure 2.) We say a point of the form $[a, b, n, i] \in E$ lies on the n$^{\text{th}}$ level of $E$ if $(a, b) \in B \backslash (\{1/(n-1), 1/n\} \times (1/7, 7/8))$. In particular, $e_n = [0, 0, n, 0]$ lies on the n$^{\text{th}}$ level for $n \geq 1$. We take $e_0 = e_1$ on the first level to be the basepoint of $E$.

To complete the proof of Theorem 4.10, we observe some loops in $B$ which do not lift to loops in $E$ by keeping track of the level on which we can find the endpoint of the lift.

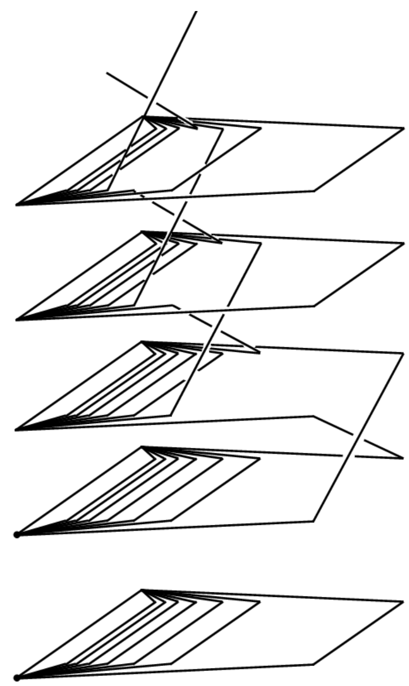


FIGURE 2. The covering map $p : E \to B$.

**Lemma 4.13.** *Suppose $\beta$ is a loop in $B$ such that $\widetilde{\beta}_{y_0}(1) = e_n$ lies on the $n^{th}$ level and $\epsilon \in \{\pm 1\}$.*

(1) *If $n = 1$ and $m > 1$, then $(\widetilde{\beta * \alpha_m^\epsilon})_{e_0}(1) = e_0$ remains on the $1^{st}$ level.*
(2) *If $n = 1$ and $m = 1$, then $(\widetilde{\beta * \alpha_m^\epsilon})_{e_0}(1) = e_2$ is on the $2^{nd}$ level.*
(3) *If $n > 1$ and $m \notin \{n-1, n\}$, then $(\widetilde{\beta * \alpha_m^\epsilon})_{e_0}(1) = e_n$ remains on the $n^{th}$ level.*
(4) *If $n > 1$ and $m = n$, then $(\widetilde{\beta * \alpha_m^\epsilon})_{e_0}(1) = e_{n+1}$ is on the $(n+1)^{th}$ level.*
(5) *If $n > 1$ and $m = n - 1$, then $(\widetilde{\beta * \alpha_m^\epsilon})_{e_0}(1) = e_{n-1}$ is on the $(n-1)^{th}$ level.*

Let $K = p_*(\pi_1(E, e_0))$. Since $p$ is a covering map, $K$ is a covering subgroup of $\pi_1(B, b_0)$ and is therefore open in $\pi_1^{qtop}(B, b_0)$. Theorem 4.10 follows immediately from the following lemma.

**Lemma 4.14.** *The open subgroup $K = p_*(\pi_1(E, e_0))$ contains no open normal subgroups of $\pi_1^{qtop}(B, b_0)$.*

*Proof.* Suppose $N$ is an open normal subgroup of $\pi_1^{qtop}(B, b_0)$ which is a subgroup of $K$. Thus, if $[\beta] \in N$, the lift $\widetilde{\beta}_{e_0}$ is a loop in $E$. Let $g_n = [\alpha_n]$. Since $N$ is open and $g_n \to [\alpha_0]$ where $[\alpha_0]$ is the identity element, we have $g_n \in N$ for all $n \geq n_0$. Thus, for all $p > q \geq n_0$, the product $g_q g_{q+1} \dots g_p \in N$. Fix any such element $g_q g_{q+1} \dots g_p$ of $N$.

Since $N$ is normal, the conjugate

$$(g_1 g_2 \dots g_p)(g_q g_{q+1} \dots g_p)(g_1 g_2 \dots g_p)^{-1} =$$
$$g_1 g_2 \dots g_p g_{q-1}^{-1} g_{q-2}^{-1} \dots g_2^{-1} g_1^{-1} \in N.$$

By assumption, $N \subseteq p_*(\pi_1(Y, y_0))$. Therefore, the lift of the loop $\alpha_1 * \alpha_2 * \cdots * \alpha_p * \alpha_{q-1}^{-1} * \alpha_{q-2}^{-1} * \cdots * \alpha_2^{-1} * \alpha_1^{-1}$ starting at $e_0$ must be a loop (i.e., end on the $1^{\text{st}}$ level). But the lift of $\alpha_1 * \alpha_2 * \cdots * \alpha_p$ ends at $e_{p+1}$ on the $(p+1)^{\text{th}}$ level. Note that $\{1, ..., q-1\} \cap \{p, p+1\} = \emptyset$. By inductively applying (3) of Lemma 4.13, we see that the lift of

$$\alpha_1 * \alpha_2 * \cdots * \alpha_p * \alpha_{q-1}^{-1} * \alpha_{q-2}^{-1} \dots \alpha_{q-k}^{-1}$$

remains on the $(p+1)^{\text{th}}$ level for each $k = 1, ..., q-1$. Thus, the lift of

$$\alpha_1 * \alpha_2 * \cdots * \alpha_p * \alpha_{q-1}^{-1} * \alpha_{q-2}^{-1} * \cdots * \alpha_2^{-1} * \alpha_1^{-1}$$

has endpoint $e_{p+1}$ on the $(p+1)^{\text{th}}$ level and cannot be a loop, which is a contradiction. □

By Theorem 3.9, every overlay subgroup of $\pi_1(B, b_0)$ has a core which is open in $\pi_1^{qtop}(B, b_0)$. Thus, we obtain the following corollary.

**Corollary 4.15.** *The covering map $p : E \to B$ is not an overlay.*

## 5. Open Subgroups of Free Topological Groups with Non-Open Core

Free topological groups are the topological analogues of free groups and are important objects in the general study of topological groups [2]. As an application of the distinction between coverings and overlays, we identify certain open subgroups of free topological groups (in the sense of Graev [24]) which have non-open core.

**Definition 5.1.** The *free Graev topological group* on a topological space $Z$ with basepoint $z \in Z$ is the unique (up to topological isomorphism) topological group $F_G(Z, z)$ equipped with a map $\sigma_Z : Z \to F_G(Z, z)$ taking $z$ to the identity element, which is universal in the sense that any continuous function $f : Z \to G$ to a topological group $G$ which takes $z$

to the identity element of $G$ induces a unique continuous homomorphism $\widetilde{f} : F_G(Z, z) \to G$ such that $\widetilde{f} \circ \sigma_Z = f$.

The choice of basepoint of $Z$ does not affect the isomorphism class of $F_G(Z, z)$; i.e, for any other $z' \in Z$, there is an isomorphism $F_G(Z, z) \to F_G(Z, z')$ of topological groups [24].

Algebraically, $F_G(Z, z)$ is the free group on the underlying set $Z\backslash\{z\}$. The underlying group is also the quotient group $F(Z)/N$ of the free group on $Z$ where $N$ is the conjugate closure of $\{z\}$. In categorical terms, $F_G : \mathbf{Top}_* \to \mathbf{TopGrp}$ is a functor from the category of based topological spaces to the category of topological groups and continuous homomorphisms which is left adjoint to the forgetful functor $\mathbf{TopGrp} \to \mathbf{Top}_*$. Explicit descriptions of the topological structure of free topological groups (in the sense of Graev [24] or Markov [31]) can be quite complicated [35]; however, they arise naturally as topological fundamental groups of the following class of spaces [8].

**Definition 5.2.** A **Top**-*graph* consists of a discrete space of vertices $\Gamma_0$, an edge space $\Gamma$, and continuous structure maps $\partial_0, \partial_1 : \Gamma \to \Gamma_0$. We denote a **Top**-graph simply by its edge space $\Gamma$. The *geometric realization* of $\Gamma$ is the topological space

$$|\Gamma| = \Gamma_0 \sqcup (\Gamma \times [0, 1])/\sim \text{ where } \partial_i(e) \sim (e, i) \text{ for } i = 0, 1.$$

We topologize $|\Gamma|$ so that $|\Gamma|\backslash\Gamma_0 \cong \Gamma \times (0, 1)$ and so the sets

$$B(v, r) = \left( \bigcup_{\partial_0(e)=v} \{e\} \times [0, r) \right) \cup \left( \bigcup_{\partial_1(e)=v} \{e\} \times (1 - r, 1] \right)$$

where $r > 0$ form a neighborhood base at a vertex $v \in \Gamma_0$.

**Example 5.3.** Recall from the previous section that

$$S = \{1, 1/2, 1/3, ..., 0\} \subset \mathbb{R}.$$

Let $\Gamma$ be the **Top**-graph with two vertices $\Gamma_0 = \{x_0, x_1\}$, edge space $\Gamma = S$, and constant structure maps $\partial_i(\Gamma) = x_i$. Then $|\Gamma|$ is homeomorphic to the planar set $B = \bigcup_{n \geq 1} Q_n$ constructed in the previous section (recall Figure 1).

The following theorem is a special case of a more general result in [8].

**Theorem 5.4** ([8]). *Suppose* $\Gamma$ *is a* **Top**-*graph with two vertices* $\Gamma_0 = \{v_0, v_1\}$, *a totally path disconnected edge space* $\Gamma$, *and constant structure maps* $\partial_i(\Gamma) = v_i$. *Then* $\pi_1^\tau(|\Gamma|, v_0)$ *is isomorphic to the free Graev topological group* $F_G(\Gamma, e)$ *for any edge* $e \in \Gamma$.

We apply this result to Example 5.3.

**Theorem 5.5.** *If $B$ is the planar set constructed in the previous section, then $\pi_1^{\tau}(B, b_0)$ is isomorphic to the free topological group $F_G(S, 0)$.*

Applying Lemma 4.14 and the fact that $\pi_1^{qtop}(B, b_0)$ and $\pi_1^{\tau}(B, b_0)$ have the same open subgroups (Lemma 2.15), we see that $F_G(S, 0)$ has an open subgroup with non-open core.

**Corollary 5.6.** *There is an open subgroup $K$ of the free topological group $F_G(S, 0)$ which contains no open normal subgroup of $F_G(S, 0)$.*

Clearly, the subgroup $K$ of $F_G(S, 0)$ is infinitely generated. An explicit set of generators for $K$ can be written down, but $K$ is probably best understood in terms of the fundamental group of the covering space $E$ of $B$.

We generalize Corollary 5.6 to other free topological groups using the following proposition.

**Proposition 5.7.** *If $q : (A, a) \to (B, b)$ is a based quotient map, then the induced continuous epimorphism $F_G(q) : F_G(A, a) \to F_G(B, b)$ is an open map.*

*Proof.* It is well known that any epimorphism of topological groups which is also a quotient map of spaces is an open map. Therefore, it suffices to check that $F_G(f)$ is quotient. Suppose $G$ is the underlying group of $F_G(B, b)$ equipped with the quotient topology of $F_G(A, a)$ with respect to $F_G(q)$. In general, if $r : H \to K$ is a group epimorphism and $H$ is a topological group, then $K$ becomes a topological group when endowed with the quotient topology with respect to $r$. It follows that $G$ is a topological group. Since $F_G(q) : F_G(A, a) \to F_G(B, b)$ is continuous, the topology of $G$ is finer than that of $F_G(B, b)$. It now suffices to show the identity homomorphism $f : F_G(B, b) \to G$ is continuous.

Let $\sigma_A : A \to F_G(A, a)$ and $\sigma_B : B \to F_G(B, b)$ be the continuous injection of generators and consider the following commutative diagram.

$$\begin{array}{ccccc} A & \xrightarrow{\sigma_A} & F_G(A, a) & & \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle F_G(q)} & \searrow^{F_G(q)} & \\ B & \xrightarrow[\sigma_B]{} & F_G(B, b) & \xrightarrow[f]{} & G \end{array}$$

Since $F_G(q) \circ \sigma_A = f \circ \sigma_B \circ q$ is continuous and $q$ is quotient, $f \circ \sigma_B$ is continuous. By the universal property of $F_G(B, b)$, $f$ is continuous. □

**Theorem 5.8.** *If a topological space $Z$ has $S = \{1, 1/2, 1/3, ..., 0\}$ as a quotient, then there is an open subgroup $L$ of the free Graev topological group $F_G(Z, z)$ which contains no open normal subgroup.*

*Proof.* Suppose $q : Z \to S$ is a quotient map. Since the isomorphism class of $F_G(Z, z)$ does not depend on the choice of the basepoint $z$, we may assume $q(z) = 0$. By Corollary 5.6, there is an open subgroup $K \subset F_G(S, 0)$ which does not contain any open subgroups which are normal in $F_G(S, 0)$. Let $f = F_G(q) : F_G(Z, z) \to F_G(S, 0)$ be the induced continuous epimorphism which is open by Proposition 5.7. By continuity, the subgroup $L = f^{-1}(K)$ is open in $F_G(Z, z)$. Suppose $N \subset L$ is an open subgroup which is normal in $F_G(Z, z)$. Since $f$ is an open surjection, $f(N)$ is an open normal subgroup of $F_G(S, 0)$. But $f(N) \subseteq f(L) = K$, contradicting our choice of $K$. □

**Example 5.9.** Let $\mathbb{Q}$ be the rational numbers with the standard topology and pick a convergence sequence of decreasing irrational numbers $a_n \to 0$. Note that $\mathbb{Q}$ is the union of the disjoint rational intervals $(-\infty, 0]$, $(a_n, a_{n-1})$, $n \geq 2$, and $(a_1, \infty)$. Identifying each of these intervals to a point yields the quotient space $S$, and thus, by Theorem 5.8, there is an open subgroup of $F_G(\mathbb{Q}, 0)$ with non-open core. One can make a similar argument for the middle-third cantor set $C \subset [0, 1]$ or any infinite compact ordinal.

**Remark 5.10.** If $Z$ is discrete, then $F_G(Z, z)$ is discrete and clearly every open subgroup has open core. At the other extreme, it is known that if $Z$ is connected, then so is $F_G(Z, z)$ [8] and therefore contains no proper open subgroups (otherwise, $F_G(Z, z)$ would decompose as a disjoint union of open cosets). Thus, the conclusion of Theorem 5.8 does not follow if $Z$ is connected or discrete. The author is unaware of any general characterization of $Z$ such that $F_G(Z, z)$ admits an open subgroup with non-open core.

**Problem 5.11.** Suppose $\Gamma$ is a **Top**-graph and $H \subseteq \pi_1^{\tau}(|\Gamma|, v)$ is a subgroup.

(1) Is $H$ an overlay group whenever $H$ has open core in $\pi_1^{\tau}(|\Gamma|, v)$?
(2) Is $H$ a covering group whenever $H$ is open (or equivalently, is a semicovering group)?

Since every free topological group is isomorphic to $\pi_1^{\tau}(|\Gamma|, v)$ for some **Top**-graph $\Gamma$ [8], a positive answer to (1) would imply that, within the classification of semicoverings, the subgroups of a free topological group $F_G(Z, z)$ with open core are classified by the overlays of some **Top**-graph $|\Gamma|$.

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