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CONCERNING CONTINUA IRREDUCIBLE ABOUT FINITELY MANY POINTS II

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Abstract. The purpose of this paper is to provide new characterizations of continua that are irreducible about finitely many points. One such characterization is that a continuum M is irreducible about finitely many points if and only if each monotonic collection of subcontinua of M such that the difference between any two members contains a non-separating open subset of M is finite.

1. Introduction

The purpose of this paper is to introduce several new characterizations of continua that are irreducible about finitely many points. The denials of these are given in parts (4) through (7) of the following theorem.

Theorem 1.1. *Suppose M is a continuum. The following are equivalent.*

(1) *M fails to be irreducible about a finite set.*
(2) *M contains infinitely many pair-wise disjoint non-separating open sets.*
(3) *M has infinitely many weakly non-separating subcontinua each of which has an interior point that fails to lie in the closure of the union of the others.*
(4) *M has infinitely many subcontinua each of which contains a non-separating open subset of M that fails to intersect any of the other subcontinua.*
(5) *There is an infinite monotonic collection of subcontinua of M such that the difference between any two members contains a non-separating open subset of M.*

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(6) *There is an infinite increasing sequence* $M_1, M_2, M_3, \dots$ *of subcontinua of* M *such that* $M_{n+1} - M_n$ *contains a non-separating open subset of* M *for each* $n \in \mathbb{N}$.
(7) *Either* M *is an* ∞*-od or there is an infinite increasing sequence* $M_1, M_2, M_3, \dots$ *of non-separating subcontinua of* M *such that* $M_{n+1} - M_n$ *contains a non-separating open subset of* M *for each* $n \in \mathbb{N}$.
(8) *(Maćkowiak)* M *is the union of an infinite monotonic collection of its proper subcontinua.*

The point of departure for the proof is the equivalence of (1) with (2), which, together with their equivalence to (3), was among the main results in this paper's namesake [3]. In that paper, R. H. Sorgenfrey's theorems [4], [5] on irreducibility were shown to follow readily from those results as testimony to their utility.

Another characterization of finite irreducibility, equal in beauty to Sorgenfrey's classic theorems, was suggested by Fugate in the Houston Problem Book (see [1, Problem 113] and later proved by T. Maćkowiak [2]. It asserts the equivalence of (the denials of) parts (1) and (8) in Theorem 1.1. The new results in this paper lead naturally to an alternate proof of Maćkowiak's theorem, which is thus included as testimony to their utility.

A *continuum* is a compact connected subset of a metric space. A continuum is said to be *irreducible* about a closed set H if and only if it contains H but has no proper subcontinuum that contains H.

If A is a subset of a continuum M, then A is said to be a subset of M *with interior* if and only if it contains a nonempty open subset of M.

A *non-separating* subset of a continuum M is a nonempty subset of M whose complement is connected. A *weakly non-separating* subset of M is a subset of M that contains a nonempty non-separating open subset of M.

Two sets are said to be *mutually separated* if and only if they are mutually exclusive and neither contains a limit point of the other.

If $\mathcal{A}$ is a collection of sets, then the notation $\mathcal{A}^*$ denotes the union of the sets belonging to $\mathcal{A}$.

2. **Proof of Theorem 1.1**

2.1. **A preliminary lemma**.

Lemma 2.1. *Suppose* M *is a continuum.*

(1) *If a non-separating open subset of* M *is the union of finitely many mutually separated sets, then each of them is a non-separating open set.*

(2) *If M has a subcontinuum whose complement has infinitely many components, then there is a subcontinuum K of M and a sequence $D_1, D_2, D_3, \ldots$ of non-separating open subsets of M such that $K \cup D_n$ is a continuum for each positive integer n and $M = K \cup D_1 \cup D_2 \cup D_3 \cup \ldots$.*

Proof. Suppose $D_1, D_2, \ldots, D_n$ are mutually separated sets whose union is a non-separating open subset D of M. Note that each of $D_1, D_2, \ldots, D_n$ is the union of a collection of components of D and that each component of D lies wholly within one of $D_1, D_2, \ldots, D_n$. Since each such component bumps the boundary of $M - D$, the union of $M - D$ with any collection of components of D is connected. Hence, the complement of each of $D_1, D_2, \ldots, D_n$ is connected.

Suppose L is a subcontinuum of M whose complement has infinitely many components. Then $M - L$ is the union of two mutually exclusive non-separating open sets A_1 and D_1, one of which, say A_1, contains infinitely many components of $M - L$. Similarly, A_1 is the union of two mutually exclusive non-separating open sets A_2 and D_2, one of which, say A_2, contains infinitely many components. Proceeding inductively yields a sequence $D_1, D_2, D_3, \ldots$ of pair-wise disjoint non-separating open subsets of M. Define K to be the union of L with each of the components of $M - L$ that fails to belong to one of $D_1, D_2, D_3, \ldots$. Then K is a continuum as is $K \cup D_n$ for each positive integer n, and $M = K \cup D_1 \cup D_2 \cup D_3 \cup \ldots$. $\square$

2.2. **Proof of Theorem 1.1.**

The equivalence of (1), (2), and (3) was proved in [3]. The remainder of the proof goes (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (1). The last implication is trivial, and the remaining implications are proved below.

Notice that any continuum satisfying the conclusion of Lemma 2.1(2) satisfies each of the conditions listed in Theorem 1.1. Thus, by Lemma 2.1, if a continuum M has a subcontinuum K such that $M - K$ has infinitely many components or, equivalently, if M contains a non-separating open set with infinitely many components, then each of the conditions in Theorem 1.1 holds. In the forthcoming proofs, this fact will be used regularly.

Proof of (2) $\Rightarrow$ (4). If M contains a non-separating open set with infinitely many components, then (4) follows from Lemma 2.1(2). Otherwise, since each component of a non-separating open set with finitely many components is itself a non-separating open set, M contains an infinite pair-wise disjoint collection of connected, non-separating, open subsets of M. The collection of closures of all such sets satisfies (4).

Proof of (4) $\Rightarrow$ (5). Suppose $\mathcal{A}$ is an infinite collection of subcontinua of M, each member of which contains a non-separating open subset of

M that fails to intersect any other term of $\mathcal{A}$. Denote the corresponding collection of non-separating open subsets by $\mathcal{A}'$. Since M is connected, it contains a point p that fails to belong to any member of $\mathcal{A}'$. Denote by $\mathcal{P}$ the collection to which a subcontinuum of M belongs if and only if it is the component of $M - \mathcal{B}^*$ containing p for some subcollection $\mathcal{B}$ of $\mathcal{A}'$.

We wish to show that if $P \in \mathcal{P}$ and $A' \in \mathcal{A}'$, then P either contains A' or fails to intersect A'. To that end suppose $P \in \mathcal{P}$, and suppose A' is a member of $\mathcal{A}'$ that shares a common point with P. Denote by $\mathcal{B}_P$ a subcollection of $\mathcal{A}'$ for which P is the component of $M - \mathcal{B}_P^*$ containing p, and denote by A the member of $\mathcal{A}$ that corresponds to A'. Recall that A fails to intersect any member of $\mathcal{A}' - \{A'\}$. In particular, A fails to intersect any member of $\mathcal{B}_P$. Hence, $P \cup A$ is a continuum that fails to intersect $\mathcal{B}_P^*$. As P is a component of $M - \mathcal{B}_P^*$, it follows that $P \cup A \subset P$. Consequently, P contains A'.

By the Hausdorff Maximality Principle, the collection $\mathcal{P}$, partially ordered by set containment, has a maximal totally ordered subcollection $\mathcal{T}$. To complete the proof it suffices to show that $\mathcal{T}$ is infinite or, equivalently, that $\mathcal{T}$ has more than n terms for each n. To that end suppose n is given, and suppose $T_1 < T_2 < \ldots < T_n$ is a finite subcollection of $\mathcal{T}$. Notice that M and the component of $M - \mathcal{A}'^*$ containing p are the unique maximum and minimum elements of $\mathcal{P}$, respectively. Hence, both belong to $\mathcal{T}$. If either T_n is not M or T_1 is not the component of $M - \mathcal{A}'^*$ that contains p, then $\mathcal{T}$ has more than n terms.

Suppose T_1 is the component of $M - \mathcal{A}'^*$ that contains p, and $T_n = M$. Then, for some index j, $T_j - T_{j-1}$ contains infinitely many of the non-separating open sets in $\mathcal{A}'$. Define $\mathcal{B}_j$ to be the subcollection of $\mathcal{A}'$ consisting of all terms that fail to intersect T_j, and define $\mathcal{B}_{j-1}$ similarly. Notice that T_j and T_{j-1} are the respective components of $M - \mathcal{B}_j^*$ and $M - \mathcal{B}_{j-1}^*$ that contain p. Choose $B' \in \mathcal{B}_{j-1} - \mathcal{B}_j$ and denote its corresponding term in $\mathcal{A}$ by B.

If B' and T_{j-1} share a common boundary point, then $T_{j-1} \cup B$ is a continuum that contains p, properly contains T_{j-1}, and fails to intersect any term of $\mathcal{B}_{j-1} - \{B'\}$. Thus, for T equal to the component of $M - (\mathcal{B}_{j-1} - \{B'\})^*$ that contains p, we have $T \in \mathcal{T}$ and $T_{j-1} < T < T_j$. On the other hand, if B' and T_{j-1} fail to share a boundary point, then the component of $T_j - B'$ that contains p properly contains T_{j-1}. Thus, for T equal to the component of $M - (\mathcal{B}_j \cup \{B'\})^*$ that contains p, we have $T \in \mathcal{T}$ and $T_{j-1} < T < T_j$. It follows that $\mathcal{T}$ has more than n terms.

Proof of (5) $\Rightarrow$ (6). The hypothesis guarantees that either (6) holds or there is a decreasing sequence $C_1, C_2, C_3, \ldots$ of continua such that, for each n, $C_n - C_{n+1}$ contains a non-separating open set D_n. Suppose the latter. Denote $\cap C_n$ by C, and note that C is a continuum.

For each $x \in M - C$ and each n for which $x \in M - C_n$, define $L_n(x)$ to be the component of $M - C_n$ that contains x and define $L(x)$ to be $\cup L_n(x)$. Notice that, for each $x \in M - C$, $L(x)$ is the nondecreasing union of connected sets and is, therefore, connected.

We show that each two sets of the form $L(x)$ are either identical or mutually exclusive. Suppose $x, y \in M - C$ such that $L(x)$ and $L(y)$ have a common point z. Then $L_n(x)$ and $L_n(y)$ both contain z for each of cofinitely many positive integers n. Hence, $L_n(x) = L_n(z) = L_n(y)$ for each such n, from which it follows that $L(x) = L(y)$.

If $M - C_n$ has infinitely many components for some $n \in \mathbb{N}$, then (6) follows from Lemma 2.1(2). Suppose $M - C_n$ has finitely many components for each $n \in \mathbb{N}$. It follows that $L_n(x)$ is open for each $x \in M - C$ and each n for which $x \in M - C_n$. Consequently, $L(x)$ is open for each $x \in M - C$.

In summary, $L(x)$ is connected and open for each $x \in M - C$ and the collection of all such sets forms a partition of $M - C$. Consequently, $L(x)$ is a component of $M - C$ for each x in $M - C$.

If $M - C$ has infinitely many components, then (6) follows from Lemma 2.1(2) . If $M - C$ has finitely many components, then there is a point p of $M - C$ such that $L(p)$ intersects infinitely many terms of $D_1, D_2, D_3, \ldots$. In other words, $\cup_{n \in \mathbb{N}} L_n(p) \cap D_m$ is nonempty for infinitely many $m \in \mathbb{N}$.

Recall that $L_n(p) \subset M - C_n$. Since $C_1, C_2, C_3, \ldots$ is a decreasing sequence, $L_n(p) \subset M - C_m$ for all $m \geq n$. It follows that $L_n(p)$ fails to intersect C_m and, therefore, D_m for $m \geq n$. Consequently, $m < n$ if either $L_n(p) \cap C_m$ or $L_n(p) \cap D_m$ is nonempty.

Choose m_1 and n_1 such that $L_{n_1}(p) \cap D_{m_1}$ is nonempty. Then $m_1 < n_1$. Choose m_2 and n_2 such that $n_1 < m_2$ and $L_{n_2}(p) \cap D_{m_2}$ is nonempty. Then $m_2 < n_2$. Proceeding inductively, we may define a pair of sequences $n_1, n_2, n_3, \ldots$ and $m_1, m_2, m_3, \ldots$ such that $m_1 < n_1 < m_2 < n_2 < m_3 < n_3 < \ldots$ and such that $L_{n_k}(p) \cap D_{m_k}$ is nonempty for all $k \in \mathbb{N}$.

Since, for each k, $D_{m_k} \subset M - C_{n_k}$ and $M - C_{n_k}$ is the union of finitely many disjoint open sets, of which $L_{n_k}(p)$ is one, it follows from Lemma 2.1(1) that $L_{n_k}(p) \cap D_{m_k}$ is a non-separating open set. Furthermore, since $n_{k-1} < m_k$, $L_{n_{k-1}}(p) \cap D_{m_k}$ is empty. It follows that $L_{n_1}(p), L_{n_2}(p), L_{n_3}(p), \ldots$ is an increasing sequence of subcontinua of M such that, for each k, $L_{n_{k+1}}(p) - L_{n_k}(p)$ contains the non-separating open set $L_{n_{k+1}}(p) \cap D_{m_{k+1}}$.

Proof of (6) $\Rightarrow$ (7). Suppose M is not an ∞-od, and denote by $D_2, D_3, D_4, \ldots$ a sequence of non-separating open sets such that $D_n \subset M_n - M_{n-1}$

for each n. Since M is not an ∞-od, D_n has finitely many components for each n. Furthermore, each component of D_n is itself a non-separating open subset of M that is contained in $M_n - M_{n-1}$. Hence, we may assume that each term of the sequence $D_2, D_3, D_4, \ldots$ is connected.

Note that, for each n, $M - M_n$ contains cofinitely many terms of the sequence $D_1, D_2, D_3, \ldots$. Thus, there is a subsequence $D_1^1, D_2^1, D_3^1, \ldots$ of $D_2, D_3, D_4, \ldots$, each term of which is contained within a single one of the finitely many components of $M - M_1$. Denote this component by C_1. Similarly, there is a subsequence $D_1^2, D_2^2, D_3^2, \ldots$ of $D_1^1, D_2^1, D_3^1, \ldots$, each term of which is contained within a single component C_2 of $M - M_2$.

Proceeding inductively, we may define a sequence $C_1, C_2, C_3, \ldots$ and a sequence of sequences $\{D_1^n, D_2^n, D_3^n, \ldots\}_{n \in \mathbb{N}}$ with the following properties.

- $D_1^{n+1}, D_2^{n+1}, D_3^{n+1}, \ldots$ is a subsequence of $D_1^n, D_2^n, D_3^n, \ldots$ for each n, and $D_1^1, D_2^1, D_3^1, \ldots$ is a subsequence of $D_1, D_2, D_3, \ldots$.
- C_n is a component of $M - M_n$ for each n.
- C_n contains each term of $D_1^n, D_2^n, D_3^n, \ldots$ for each n.

Since C_{n+1} is a connected subset of $M - M_n$ that contains a point of the component C_n of $M - M_n$ (any point of D_1^{n+1}, for example), it follows that $C_{n+1} \subset C_n$ for each n.

Note that, for each n, C_n is a connected, open, and, by Lemma 2.1(1), non-separating subset of M. Define X_n to be $M - C_n$ for each n. Then $X_1, X_2, X_3, \ldots$ is a nondecreasing sequence of non-separating subcontinua of M. Furthermore, $D_n \subset M_n \subset X_n$ for each n, and infinitely many terms of $D_2, D_3, D_4, \ldots$ lie in the complement of X_n for each n. It follows that some subsequence of $X_1, X_2, X_3, \ldots$ satisfies (7).

Proof of (7) $\Rightarrow$ (8). If M is an ∞-od, then (8) follows. Suppose there is an increasing sequence $M_1, M_2, M_3, \ldots$ of non-separating subcontinua of M such that $M_{n+1} - M_n$ contains a non-separating open subset of M for each $n \in \mathbb{N}$. If $M_1 \cup M_2 \cup \ldots = M$, then (8) follows. Suppose that $M_1 \cup M_2 \cup \ldots \neq M$. Denote $(\overline{M - M_1}) \cap (\overline{M - M_2}) \cap (\overline{M - M_3}) \cap \ldots$ by C. Then, as the common part of a monotonic collection of continua, C is a continuum. Every point of M that does not belong to some term of $M_1, M_2, M_3, \ldots$ belongs to C. Denote by I a subcontinuum of M that is irreducible about $M_1 \cup C$. Then $I \cup M_n$ is a continuum for each n and $M = (I \cup M_1) \cup (I \cup M_2) \cup \ldots$.

If $I \cup M_n$ is a proper subcontinuum of M for each positive integer n, then (8) holds. Suppose to the contrary that for some positive integer N, $I \cup M_N = M$. Denote by D_{N+1} the hypothesized non-separating open subset of M contained by $M_{N+1} - M_N$. Notice that D_{N+1} fails to intersect both C and M_N. Then $M - D_{N+1}$ is a subcontinuum of M that contains both C and the boundary of M_N. Hence, it has a subcontinuum

J_1 that is irreducible about $C \cup \mathrm{bd}(M_N)$. Since C fails to intersect the interior of M_N, so does J_1. Hence, J_1 is a subcontinuum of I that does not contain D_{N+1}.

Since I contains both M_1 and the boundary of M_N, it has a subcontinuum J_2 that is irreducible about $M_1 \cup \mathrm{bd}(M_N)$. Since M_1 fails to intersect the complement of M_N, so does J_2. Hence, J_2 also fails to contain D_{N+1}. Then $J_1 \cup J_2$ is a proper subcontinuum of I that contains $M_1 \cup C$, contradicting the irreducibility of I. Thus, $I \cup M_n$ is a proper subcontinuum of M for each n, and the result follows. □

2.3. **Finite irreducibility**.

Reformulating (1), (4), (5), (6), and (7) of Theorem 1.1 yields the following new characterizations of continua irreducible about finitely many points.

Theorem 2.2. *Suppose M is a continuum. The following are equivalent.*

1. *M is irreducible about a finite set.*
2. *Every collection of subcontinua of M each of which contains a non-separating open subset of M that fails to intersect any of the other continua is finite.*
3. *Every monotonic collection of subcontinua of M such that the difference between any two members contains a non-separating open subset of M is finite.*
4. *Every increasing sequence $M_1, M_2, M_3, \ldots$ of subcontinua of M such that $M_{n+1} - M_n$ contains a non-separating open subset of M for each n is finite.*
5. *M is not an ∞-od, and every increasing sequence $M_1, M_2, M_3, \ldots$ of non-separating subcontinua of M such that $M_{n+1} - M_n$ contains a non-separating open subset of M for each n is finite.*

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