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AN SSGP TOPOLOGY FOR $\mathbb{Z}^\omega$

FRANKLIN R. GOULD

Respectfully dedicated to William Wistar Comfort,
mathematician and educator,
on the occasion of his 80th birthday.

Abstract. A topological group $G = (G, \mathcal{T})$ has the *small subgroup generating property* (briefly, *SSGP property*, or is an *SSGP group*) if, for each neighborhood U of 1_G, there is a family $\mathcal{H} = \{H_i : i \in I\} \subseteq \mathcal{P}(U)$ of subgroups of G such that $\langle \bigcup_{i \in I} H_i \rangle$ is dense in G. It is shown by explicit construction that there exist group topologies with this property for the group $\mathbb{Z}^\omega$.

1. Introduction

This paper resolves an issue which I had been unable to settle in my doctoral dissertation [2] written at Wesleyan University under the guidance of W. W. Comfort. That dissertation gives the following definition.

Definition 1.1. A topological group $G = (G, \mathcal{T})$ has the *small subgroup generating property* (briefly: has the *SSGP property*, or is an *SSGP group*) if for each neighborhood U of 1_G there is a family $\mathcal{H} = \{H_i : i \in I\} \subseteq \mathcal{P}(U)$ of subgroups of G such that $\langle \bigcup_{i \in I} H_i \rangle$ is dense in G.

My study of SSGP groups was motivated by the easily demonstrated fact that every SSGP group is a minimally almost periodic group (as defined in [4]). In a forthcoming joint paper, some of the themes developed in [2] are elaborated on and pursued more deeply.

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2. Main Results

We construct a Hausdorff SSGP topology for the product group $\mathbb{Z}^\omega$ by defining an appropriate 2-parameter set of basic neighborhoods of 0, $U_n(\varepsilon)$. This will be facilitated by a sequence of subgroups and quotient groups. Let $p_1, p_2, p_3, \ldots$ be the primes $2, 3, 5, \ldots$, respectively. We define $H_1 := 2\mathbb{Z}^\omega$ and $H_{n+1} := p_{n+1}H_n$ where, by nH, we mean $\{x \in H : x = ny$ for some $y \in H\}$. For each of these subgroups, we let $G_n := \mathbb{Z}^\omega/H_n$ be the corresponding quotient group. This leads to the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{n+1} & \xrightarrow{\rho_{n+1}} & \mathbb{Z}^\omega & \xrightarrow{\pi_{n+1}} & G_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_{n+1}} & & \| & & \downarrow{\scriptstyle \psi_{n+1}} & & \\ 0 & \longrightarrow & H_n & \xrightarrow{\rho_n} & \mathbb{Z}^\omega & \xrightarrow{\pi_n} & G_n & \longrightarrow & 0 \end{array}$$

Note that ϕ_{n+1} is an injection and ψ_{n+1} is a surjection. We fix ε and we define $U_n(\varepsilon) = \pi_n^{-1}[V_n(\varepsilon)]$ where $V_n(\varepsilon) \subseteq G_n$ will be defined inductively, starting with $V_1(\varepsilon) \subseteq G_1$. Since $G_1 = \mathbb{Z}^\omega/(2\mathbb{Z}^\omega)$ is a vector space over $\mathbb{Z}_2$, we can select a basis of c-many vectors, which we reorganize into c sets of ω basis vectors, $\{u_{i,k}^{(1)} : 1 \leq i < c; 1 \leq k < \omega\}$. Then every $x \in G_1$ can be expressed uniquely as a linear combination of a finite number of these basis vectors. Now we assign a norm $N^{(1)}(x)$ to each $x \in G_1$ by first assigning a value $N_i(x)$ to the set of components $\{x_{i,k} : 1 \leq i < c; 1 \leq k < \omega\}$ for each fixed i as follows:

(1) We choose a bijection $\eta : \mathbb{N} \to D$ where D is the set of rational numbers between 0 and 1.
(2) With each i and k, we associate the function $\varphi_{i,k} : I \to \mathbb{Z}_2$ from the unit interval to $\mathbb{Z}_2$ where $\varphi_{i,k}(r) = 0$ for $0 \leq r < \eta(k)$ and $\varphi_{i,k}(r) = x_{i,k}$ for $\eta(k) \leq r \leq 1$.
(3) We let $f_i = \sum_{k=1}^{M_i} \varphi_{i,k}$ where M_i is the maximum value of k for which $x_{i,k}$ is non-zero and the functions are summed pointwise.
(4) We define $N_i(x)$ to be the Lebesgue measure of the support of f_i.

One easily checks that (1), (2), and (3) together establish an isomorphism between G_1 and step functions on the disjoint union of c-many unit intervals where the steps are between rational points within an interval, where each function has a finite number of steps in an interval, and where the total support of each function has finite Lebesgue measure, which we label $N^{(1)}(x)$, the sum of the finite number of $N_i(x)$ which are non-zero. Finally, $V_1(\varepsilon) := \{x \in G_1 : N^{(1)}(x) < \varepsilon\}$.

For the induction step, suppose that we have already defined $N^{(n)}(x)$ for each $x \in G_n$ giving us $V_n(\varepsilon) := \{x \in G_n : N^{(n)}(x) < \varepsilon\}$. Since a torsion group is the direct sum of its p-groups with the summands uniquely determined, G_{n+1} can be expressed uniquely as $G_{n+1} = A \oplus B$ where A is a group all of whose non-zero elements have order p_{n+1} and where B is isomorphic to G_n. Again, the group A is a vector space over $\mathbb{Z}_{p_{n+1}}$ and we can choose a basis $\{u_{i,k}^{(n+1)} : 1 \le i < c; 1 \le k < \omega\}$, as before. Now we repeat steps (1) through (4), above, for the group A in place of G_1, obtaining a measure $N_A^{(n+1)}(x_A)$ for $x_A \in A$. For the measure $N^{(n+1)}(x)$ of an element $x \in G_{n+1}$, we define $N^{(n+1)}(x) = N_A^{(n+1)}(x_A) + N^{(n)}(\psi_{n+1}(x))$ where $x = x_A + x_B$ with $x_A \in A$ and $x_B \in B$. This gives us $V_{n+1}(\varepsilon) := \{x \in G_{n+1} : N^{(n+1)}(x) < \varepsilon\}$.

It follows that the $U_n(\varepsilon)$'s generate a Hausdorff group topology on $\mathbb{Z}^\omega$ once we establish the following facts.

(1) $U_n(\varepsilon_1) \subseteq U_n(\varepsilon_2)$ for $\varepsilon_1 < \varepsilon_2$.
(2) $U_{n+1}(\varepsilon) \subseteq U_n(\varepsilon)$.
(3) $U_n(\varepsilon/2) + U_n(\varepsilon/2) \subseteq U_n(\varepsilon)$.
(4) If $x \in U_n(\varepsilon)$, then $nx \in U_n(\varepsilon)$ for $n \in \mathbb{Z}$.
(5) $\bigcap_n \bigcap_{\varepsilon>0} U_n(\varepsilon) = \{0\}$.

The first fact is an immediate consequence of the definition of $V_n(\varepsilon)$ and the preservation of subset containment under an inverse map. For the second fact, suppose $x \in U_{n+1}(\varepsilon)$. By definition, we have $N^{(n+1)}(\pi_{n+1}(x)) < \varepsilon$. Then clearly, $N^{(n)}(\psi_{n+1} \circ \pi_{n+1}(x)) < \varepsilon$, as well, because the p_{n+1} component of $\pi_{n+1}(x)$ can only add to the measure. But $\psi_{n+1} \circ \pi_{n+1} = \pi_n$, so $x \in U_n(\varepsilon)$. Fact (4) follows from the obvious fact that $N^{(n)}(mx) \le N^{(n)}(x)$ for $x \in G_n$ and $m \in \mathbb{Z}$. This guarantees that the neighborhoods $U_n(\varepsilon)$ are symmetric about 0.

For (3), suppose that $x, y \in G_n$. The measures $N^{(n)}(x)$ and $N^{(n)}(y)$ are each given by the sum of the Lebesgue measures of the support of functions on a finite number of unit intervals. $N^{(n)}(x + y)$ cannot exceed the sum of the two separate measures because anywhere that a function representing x overlaps with a function representing y, the sum of the two functions cannot have any greater support than the union of the support of the two functions separately. It follows that if $N^{(n)}(x)$ and $N^{(n)}(y)$ are each less than $\varepsilon/2$, then $N^{(n)}(x + y) < \varepsilon$.

To demonstrate (5) and the Hausdorff property, let $x \in \mathbb{Z}^\omega$ and let x_i represent the i^{th} coordinate of x in the canonical representation. Suppose that x_m is the smallest non-zero coordinate and that p_{n_0} is the smallest prime which does not divide x_m. Then $N^{(n_0)}(\pi_{n_0}(x))$ will have a non-zero contribution, ε_0, from the p_{n_0}-component of $\pi_{n_0}(x)$. We can conclude that $x \notin U_n(\varepsilon)$ for $n \geq n_0$ and $\varepsilon < \varepsilon_0$.

Finally, we need to show that the topology defined by the neighborhoods $U_n(\varepsilon)$ is an SSGP topology. We already know from (4) that every element in $U_n(\varepsilon)$ is a member of an entire subgroup contained in $U_n(\varepsilon)$. It remains only to show that the subgroup generated by $U_n(\varepsilon)$ is dense in $\mathbb{Z}^\omega$. In fact, we will show than any $x \in \mathbb{Z}^\omega$ is a finite combination of elements from $U_n(\varepsilon)$. We know that $\pi_n(x)$ is a linear combination of a finite number of basis elements for G_n from a list of length $n \times c \times \omega$. This corresponds to a function which has support on a finite number of unit intervals. The range within each interval is one of the groups $\mathbb{Z}_{p_m}$ with $m \leq n$. Let $f_{m,i} : I \to \mathbb{Z}_{p_m}$ be a component of the function on one such interval. It should be clear that $f_{m,i}$ can be decomposed into a finite sum of step functions, each of which has support only on one small interval, of measure less than ε, where $f_{m,i}$ has a constant value. Let $k, k' < \omega$ be such that $0 < \eta(k') - \eta(k) < \varepsilon$ and such that $f_{m,i}(r) = z \in \mathbb{Z}_{p_m}$ for $\eta(k) \leq r < \eta(k')$. Then the function on I which agrees with $f_{m,i}$ on the interval $[\eta(k),\ \eta(k')\ )$ and is zero elsewhere in I corresponds to an element $g \in V_n(\varepsilon) \subseteq G_n$ whose only non-zero components are given by $g_{m,i,k} = z$ and $g_{m,i,k'} = -z$. Since $\pi_n(x)$ is a finite sum of such $g \in V_n(\varepsilon)$ and π_n is a quotient map, it follows that for each such g there is a member of the coset $\pi_n^{-1}(g) \subseteq U_n(\varepsilon)$ such that their sum is x, and we are done.

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