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A NOTE ON AN UNUSUAL CHARACTERIZATION OF THE PSEUDO-ARC

JAN VAN MILL

Abstract. Lewis showed that the pseudo-arc is the unique non-degenerate continuum having the property that any two copies of it that are setwise near each other in terms of the Hausdorff distance are homeomorphically near each other. We present a new proof of this fact based on a well-known result of Bing, standard facts from infinite-dimensional topology and the Effros Theorem.

1. Introduction

All spaces under discussion are separable metric. For all undefined notions, see Nadler [7] and van Mill [5].

A compactum X is said to have *property* HN (for 'homeomorphically near'), Lewis [4], if for any copy X_0 of X in the Hilbert cube Q and any $\varepsilon > 0$ there exists $\delta > 0$ such that for any copy X_1 of X in Q such that $d_H(X_0, X_1) < \delta$ there exists a homeomorphism $h\colon X_0 \to X_1$ such that $d\big(x, h(x)\big) < \varepsilon$ for each $x \in X_0$. In [4], Lewis proved the following:

Theorem 1.1. *The pseudo-arc is the only non-degenerate continuum with property* HN.

The aim of this note is to present a new proof of this fact, based on Bing's Theorem from [1] that the space of pseudo-arcs is a dense G_δ-subset of $C(Q)$, standard facts from infinite-dimensional topology and the Effros Theorem from [2] (see also [6]). In fact, besides Bing's result, we need no specifics in our proof about the pseudo-arc. This is rather curious and it may make our method applicable in different situations.

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2. Proofs

For a finite collection $\mathscr{U}$ of open subsets of Q we put

$$N(\mathscr{U}) = \{B \in C(Q) : (B \subseteq \bigcup \mathscr{U}) \,\&\, (\forall U \in \mathscr{U})(B \cap U \neq \emptyset)\}.$$

Let $\mathscr{Z}(Q)$ denote the collection of all Z-sets in Q. For a non-degenerate continuum X, put

$$C(Q, X) = \{A \in C(Q) : A \text{ is homeomorphic to } X\},$$

and

$$C_{\mathscr{Z}}(Q, X) = C(Q, X) \cap \mathscr{Z}(Q),$$

respectively.

Proposition 2.1. *If X is a non-degenerate continuum with property* HN*, then $C_{\mathscr{Z}}(Q, X)$ is a dense G_δ-subset of $C(Q)$.*

Proof. We will first show that $C_{\mathscr{Z}}(Q, X)$ is dense. To this end, take arbitrary $B \in C(Q)$ and $\varepsilon > 0$. There is a finite collection of open subsets $\mathscr{U}$ of Q such that $B \in N(\mathscr{U})$ while $d_H(B, C) < \varepsilon$ for each $C \in N(\mathscr{U})$. By [7, Theorem 19.2] we may pick $I \in N(\mathscr{U})$ such that $I \approx [0, 1]$. Let $r\colon Q \to I$ be a retraction. By the Mapping Replacement Theorem [5, Theorem 6.4.8], r can be approximated arbitrarily closely by a Z-imbedding. Hence we may assume that there is an element $Y \in \mathscr{Z}(Q) \cap N(\mathscr{U})$ such that $Y \approx Q$. There is a topological copy Z of X which is contained in Y. By the Homeomorphism Extension Theorem [5, Theorem 6.4.6], homeomorphisms between finite subsets of Q can be extended to homeomorphisms of Q. Hence we may assume that $Z \cap U \neq \emptyset$ for every $U \in \mathscr{U}$, i.e., $Z \in N(\mathscr{U})$.

We will next show that $C_{\mathscr{Z}}(Q, X)$ is a second category subset of $C(Q)$. Indeed, assume that for every i, $\mathscr{N}_i$ is a closed and nowhere dense subset of $C(Q)$. Pick any element $Z \in C_{\mathscr{Z}}(Q, X)$.

Claim 1. Fix $i \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists a homeomorphism $f\colon Q \to Q$ such that $d(f, 1_Q) < \varepsilon$ and $f(Z) \notin \mathscr{N}_i$.

Using our assumptions, pick $\delta > 0$ such that for any copy X_0 of X in Q such that $d_H(X_0, Z) < \delta$ there exists a homeomorphism $h\colon Z \to X_0$ such that $d(h, 1_Z) < \varepsilon$. Since $C_{\mathscr{Z}}(Q, X)$ is dense in $C(Q)$ we may pick $X_0 \in C_{\mathscr{Z}}(Q, X) \setminus \mathscr{N}_i$ such that $d_H(Z, X_0) < \delta$. Hence there exists a homeomorphism $h\colon Z \to X_0$ such that $d(h, 1_Z) < \varepsilon$. By the Homeomorphism Extension Theorem [5, Theorem 6.4.6], we may extend this homeomorphism to a homeomorphism $f\colon Q \to Q$ such that $d(f, 1_Q) < \varepsilon$.

Hence we can 'free' Z from $\mathscr{N}_i$ by an arbitrarily small move. This means that in an inductive process we can free Z from all the $\mathscr{N}_i$. This has to be done with a little care so that once Z is free from some $\mathscr{N}_i$, the limit homeomorphism does not carry it back to that $\mathscr{N}_i$. But this can easily be achieved by the Claim and a standard application of the Inductive Convergence Criterion [5, Theorem 6.1.2] (cf., [5, the proof of Theorem 6.4.5]).

Let $\mathscr{H}(Q)$ denote the group of homeomorphisms of Q endowed with the standard compact-open topology. Then $\mathscr{H}(Q)$ is Polish, and the Homeomorphism Extension Theorem [5, Theorem 6.4.6], shows that it acts transitively on the second category space $C_{\mathscr{Z}}(Q,X)$. By the Effros Theorem from [2] (see also [6]), it follows that $C_{\mathscr{Z}}(Q,X)$ is Polish and hence a G_δ-subset of $C(Q)$. □

This leads us to a proof of Lewis' result from [4].

Theorem 2.2. *Let X be a non-degenerate continuum. Then the following statements are equivalent:*

(1) *X has property* HN*,*
(2) *$C_{\mathscr{Z}}(Q,X)$ is a dense G_δ-subset of $C(Q)$,*
(3) *$C(Q,X)$ is a dense G_δ-subset of $C(Q)$,*
(4) *$C(Q,X)$ contains a dense G_δ-subset of $C(Q)$,*
(5) *X is homeomorphic to the pseudo-arc.*

Proof. For (1) ⇒ (2), we simply apply Proposition 2.1. For (2) ⇒ (5), recall Bing's Theorem [1] quoted above that the collection of pseudo-arcs is a dense G_δ-subset of $C(Q)$. Since by the Baire Category Theorem any two dense G_δ-subsets of $C(Q)$ intersect, we conclude that X is homeomorphic to the pseudo-arc. We achieve (5) ⇒ (1) by another application of the Effros Theorem. Indeed, first note that the connected Z-sets form a dense G_δ-subset of $C(Q)$ (Kroonenberg [3, Lemma 2.1(b)]). Hence if P denotes the pseudo-arc, then by Bing's Theorem just quoted and the Baire Category Theorem we obtain that $C_{\mathscr{Z}}(Q,P)$ is a dense G_δ in $C(Q)$. Now observe that $\mathscr{H}(Q)$ acts transitively on $C_{\mathscr{Z}}(Q,P)$. By the Effros Theorem from [2] (see also [6]), $\mathscr{H}(Q)$ acts micro-transitively on $C_{\mathscr{Z}}(Q,P)$. Pick an arbitrary element $S\in C_{\mathscr{Z}}(Q,P)$, and let $\varepsilon>0$. The evaluation function $\gamma_S\colon \mathscr{H}(Q)\to C_{\mathscr{Z}}(Q,P)$ defined by $\gamma_S(h)=h(S)$ is a continuous, open surjection. By continuity of γ_S there exists $\theta>0$ such that

$$\gamma_S(\{g\in\mathscr{H}(Q): d(g,1_Q)<\theta\})\subseteq\{A\in C_{\mathscr{Z}}(Q,P): d_H(S,A)<\varepsilon\}.$$

Since γ_S is open, there exists $\delta>0$ such that

$$\{A\in C_{\mathscr{Z}}(Q,P): d_H(A,S)<\delta\}\subseteq\gamma_S(\{g\in\mathscr{H}(Q): d(g,1_Q)<\theta\}).$$

Hence this δ has the following property: if $T\in C_{\mathscr{Z}}(Q,P)$ and $d_H(S,T)<\delta$, then there is a homeomorphism $f\colon Q\to Q$ such that $f(S)=T$ and $d(f,1_Q)<\varepsilon$.

To prove that P has property HN, take arbitrary $P_0 \in C(Q,P)$ and $\varepsilon > 0$. We assume without loss of generality that $\varepsilon < 1$. Define $f\colon Q \to Q$ by $f(x) = (1 - {}^1\!/_3\varepsilon)x$. Then f is a Z-imbedding and $d_H(f(A), f(B)) \le d_H(A,B)$ for all $A, B \in C(Q)$. Put $S = f(P_0)$ and let $\delta > 0$ be as above for S and ${}^1\!/_3\varepsilon$. Now take an arbitrary $P_0 \in C(Q,P)$ such that $d_H(P_0,P_1) < \delta$. Then $d_H(S, f(P_1)) < \delta$. Hence there is a homeomorphism $\alpha\colon Q \to Q$ such that $d(\alpha, 1_Q) < {}^1\!/_3\varepsilon$ and $\alpha(S) = f(P_1)$. Hence the function $\beta\colon P_0 \to P_1$ defined by $\beta(x) = f^{-1}(\alpha(f(x)))$ is a homeomorphism such that for every $x \in P_0$, $d(x, \beta(x)) < \varepsilon$.

The statements (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are a direct consequence of Bing's Theorem. □

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