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TOPOLOGY PROCEEDINGS

Volume 45, 2015

Pages 5–8

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Electronically published on January 31, 2014

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 45 (2015)
Pages 5-8

E-Published on January 31, 2014

ON THE BOOLEAN PRIME IDEAL THEOREM AND CERTAIN COVERINGS OF CANTOR CUBES

KYRIAKOS KEREMEDIS

ABSTRACT. We show that the Boolean prime ideal theorem **BPI** is equivalent to the assertion "for every set X, the Tychonoff product $\mathbf{2}^X$ is compact-3" (every open cover $\mathcal{U}$ of $\mathbf{2}^X$ consisting in basic open sets of length 3 has a finite subcover).

1. NOTATION AND TERMINOLOGY AND SOME KNOWN RESULTS

(1) Let $\mathbf{X} = (X, T)$ be a topological space. $\mathbf{X}$ is *compact* iff every open cover of X has a finite subcover. Equivalently, $\mathbf{X}$ is compact iff for every family $\mathcal{G}$ of closed subsets of $\mathbf{X}$ having the finite intersection property (fip for abbreviation), $\bigcap \mathcal{G} \neq \varnothing$.

(2) Let X be a non empty set.

(a) $\mathbf{2}^X$ denotes the Tychonoff product of the discrete space $\mathbf{2}(2 = \{0, 1\})$ and, $\mathcal{B}_X = \{[p] : p \in \mathrm{Fn}(X, 2)\}$, where $\mathrm{Fn}(X, 2)$ is the set of all finite partial functions from X into 2 and $[p] = \{f \in 2^X : p \subset f\}$, will denote the standard clopen (= simultaneously closed and open) base for the topology on $\mathbf{2}^X$. For every $n \in \mathbb{N}$, let $\mathcal{B}_X^n = \{[p] \in \mathcal{B}_X : |p| = n\}$. We call the elements of $\mathcal{B}_X^n$, $n \in \mathbb{N}$, *n-basic open sets of* 2^X. The complement $[p]^c = \cup\{[\{(i, 1 - p(i))\}] : i \in Dom(p)\}$ of the n-basic open set $[p]$ of $\mathbf{2}^X$ is called *n-basic closed set of* $\mathbf{2}^X$.

(b) Let G be a subset of $\mathbf{2}^X$. For $n \in \mathbb{N}$, G is *closed-n* iff for every $h \in G^c$ there exists $[p] \in \mathcal{B}_X^n$ with $h \in [p]$ and $[p] \cap G = \varnothing$.

2010 *Mathematics Subject Classification.* Primary 03E25; Secondary 03E35, 54B10, 54D30.

Key words and phrases. Boolean prime ideal theorem, compactness, Tychonoff products.

(c) For $n \in \mathbb{N}$, $\mathbf{2}^X$ is *compact-n* iff every cover $\mathcal{U} \subset \mathcal{B}_X^n$ of $\mathbf{2}^X$ has a finite subcover. Equivalently, every $\mathcal{G} \subset \mathcal{C}_X^n = \{[p]^c : |p| = n\}$ with the fip has a non empty intersection.

(d) Let $\mathbf{Y}$ be a subspace of $\mathbf{2}^X$. For $n \in \mathbb{N}$, $\mathbf{Y}$ is *compact-n* iff every cover $\mathcal{U} \subset \mathcal{B}_X^n$ of $\mathbf{Y}$ has a finite subcover. Equivalently, for every $\mathcal{G} \subset \mathcal{C}_X^n$ if $\{G \cap Y : G \in \mathcal{G}\}$ has the fip then $\cap\{G \cap Y : G \in \mathcal{G}\} \neq \varnothing$.

(3) **BPI** : Every Boolean algebra has a prime ideal.

We list the next theorem for future reference in the main result section.

Theorem 1.1. *(i) ([3]) In **ZF**, **BPI** iff "for every X, the product $\mathbf{2}^X$ is compact".*
(ii) ([1]) For all $m, n \in \mathbb{N}, m \leq n, (\mathbf{2}^X$ is compact-$n \rightarrow \mathbf{2}^X$ is compact-$m)$.

Proposition 1.2. *(i) For every set X, if $K \subseteq \mathbf{2}^X$ is a closed-n set and $\mathbf{2}^X$ is compact-n then K is compact-n.*
(ii) For every set X, for all $K \subseteq \mathbf{2}^X$, for all $m, n \in N, m < n$, if K is closed-m then K is closed-n.

Proof. (i) To see that K is compact-n, fix $\mathcal{W} \subset \mathcal{B}_X^n$ a cover of K and let $\mathcal{U} = \mathcal{W} \cup \{V \in \mathcal{B}_X^n : V \subseteq 2^{\mathcal{P}(X)} \backslash K\}$. By our hypothesis, it follows that $\mathcal{U} \subset \mathcal{B}_X^n$ is a cover of $\mathbf{2}^{\mathcal{P}(X)}$. Since $\mathbf{2}^X$ is compact-n it follows that $\mathcal{U}$ has a finite subcover $\mathcal{V}$. Clearly, $\mathcal{V} \cap \mathcal{W}$ is a finite subcover of $\mathcal{W}$ and K is compact-n as required.

(ii) This is straightforward. □

2. Main result

In [2] it has been established that:

BPI iff *for every X, for every $n \in N$, $\mathbf{2}^X$ is countably compact and compact-n.*

Subsequently, in [4] it was shown that

BPI iff *for all $n \geq 6$, $\mathbf{Q}(n)$ = "for every X, 2^X is compact-n"*

and it has been asked the status of the statements $\mathbf{Q}(2)$,$\mathbf{Q}(3)$,$\mathbf{Q}(4)$ and $\mathbf{Q}(5)$. The aim of this short note is to show that the statement $\mathbf{Q}(3)$ hence, $\mathbf{Q}(4)$ and $\mathbf{Q}(5)$ also, are equivalent to **BPI**.

Theorem 2.1. ***BPI** iff "for every set X, the product $\mathbf{2}^X$ is compact-3".*

Proof. ($\rightarrow$) This follows from Theorem 1.1.

($\leftarrow$) It suffices to show that for every set X and every filter $\mathcal{H}$ of X there exists a maximal filter $\mathcal{F}$ of X extending $\mathcal{H}$. Fix X and let $\mathcal{H}$ be a filter of X. Put $K = \{f \in \mathbf{2}^{\mathcal{P}(X)} : (f(\varnothing) = 0) \wedge (\mathcal{H} \subset f^{-1}(1)) \wedge (\forall A, B \in f^{-1}(1), f(A \cap B) = 1)\ \}$.

Claim. K is closed-3 and compact-3.

Proof of the claim. Fix $h \in \mathbf{2}^{\mathcal{P}(X)} \backslash K$. It suffices, in view of in view of Proposition 1.2 (ii), to show that h has a neighborhood $V_h \in \mathcal{B}_X^j, j \leq 3$ avoiding K. We consider the following cases:

(i) $h(\varnothing) = 1$. Clearly, $V_h = [\{(\varnothing, 1)\}]$ is a neighborhood of h missing K.

(ii) $h(H) = 0$ for some $H \in \mathcal{H}$. Clearly, $V_h = [\{(H, 0)\}]$ is a neighborhood of h satisfying $V_h \cap K = \varnothing$.

(iii) $(h(\varnothing) = 0) \wedge (\exists A, B \in h^{-1}(1), h(A \cap B) = 0)$. In this case $V_h = [\{(A, 1), (B, 1), ((A \cap B), 0)\}]$ is a neighborhood of h avoiding K.

The second assertion of the claim follows at once from Proposition 1.2 (i).

Let $\mathcal{G} = \{G_A = [\{(A, 1)\}] \cup [\{(A^c, 1)\}] \cup [\{(\varnothing, 1)\}] : A \in \mathcal{P}(X))\} \subset \mathcal{C}_X^3$. We show that $\{G \cap K : G \in \mathcal{G}\}$ has the fip. Fix $\{A_i : i \leq n\}$ a finite family of subsets of X and pick, via a straightforward induction, for every $i \leq n$, $C_i \in \{A_i, A_i^c\}$ such that $\mathcal{Q} = \mathcal{H} \cup \{C_i : i \leq n\}$ has the fip. It can be readily verified that the characteristic function $\mathcal{X}_F$ of the set $F = \{\cap Q : Q$ is a finite non empty subset of $\mathcal{Q}\}$ belongs to $\cap\{G_{A_i} \cap K : i \leq n\}$. Thus, $\{G_A \cap K : A \in \mathcal{P}(X)\}$ has the fip as required.

Since K compact-3, it follows that $H = \cap\{G_A : A \in \mathbf{2}^{\mathcal{P}(X)}\} \neq \varnothing$. Fix $h \in H$ and let $\mathcal{F} = h^{-1}(1)$. Clearly, $\mathcal{H} \subset \mathcal{F}$ and $\mathcal{F}$ is a maximal family of subsets of X with respect to $\subseteq$ having the fip. Hence, $\mathcal{F}$ is the required ultrafilter finishing the proof of the theorem. □

Corollary 2.2. *The following are equivalent: (i)* ***BPI****.*
(ii) $\boldsymbol{Q}(3)$.
(iii) $\boldsymbol{Q}(4)$.
(iv) $\boldsymbol{Q}(5)$.
*(v) "For every set X, every closed-*3 *subspace of the product* $\mathbf{2}^X$ *is compact-*2*".*

Proof. (i)↔(ii)↔(iii)↔(iv). These follow at once from Theorems 1.1 (ii) and 2.1.

(i)↔(v). Let K be as in the proof of Theorem 2.1. Clearly, $\{G \cap K : G \in \mathcal{G} = \{G_A = [\{(A, 1)\}] \cup [\{(A^c, 1)\}]\}] : A \in \mathcal{P}(X))\}\}$ has the fip. For the rest of the proof mimic the proof of Theorem 2.1. □

References

[1] K. Keremedis, *Tychonoff products of compact spaces in* **ZF** *and closed ultrafilters*, Math. Logic Quart. **56 (5)** (2010), 475–487.

[2] K. Keremedis and E. Tachtsis, *Extensions of compactness of Tychonoff powers of* **2** *in* **ZF**, Topology Proc. **37** (2011), 15–31.

[3] J. Mycielski, *Two remarks on Tychonoff's Product Theorem*, Bull. Acad. Polon. Sci., **Vol. XII** No **8** (1964), 439–441.

[4] E. Tachtsis, **BPI** *is equivalent to compactness-*$n, n \geq 6$*, of Tychonoff powers of* 2, Topology Proc. **40** (2012), 21–28.

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