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# ON SOME PROPERTIES OF STABLE BITOPOLOGICAL SPACES

I. DOCHVIRI AND T. NOIRI

Abstract. In the paper new structural properties of R. Kopperman's stable bitopological spaces in relation with QHC and locally compact spaces are obtained. It is shown that a $p$-stable $p$-Hausdorff bispace is $p$-normal. The conditions under which stability imply compactness and Lindelöfness are established.

## 1. Introduction

Since J. C. Kelly's pioneering work [8] bitopological spaces play an important role for the study of objects that lie outside of the framework of classical general topology. Like as [8], under the term a bitopological space (brief. bispace) we mean an ordered triple $(X, \tau_1, \tau_2)$, where $X$ is a set with two independent topologies $\tau_1$ and $\tau_2$. We are concerned in this paper with the idea of stable bitopological spaces and give some results. The notion of the stable bispace and some basic properties of such spaces were proposed by R. D. Kopperman [9]. Stable bispaces are structures in which any proper closed subset relative to first topology, is compact with respect to the second one (for brevity we use terminology of $(i, j)$-stable bispace, where $i, j \in \{1, 2\}, i \neq j$). Our motivation for the investigation of the stable bispaces was inspired by a small number of theoretical results in this direction.

In section 2, we obtain several properties of $(i, j)$-stable bitopological spaces and also show that every $p$-stable $p$-Hausdorff bispace is $p$-normal.

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*Key words and phrases.* Stable bispace, submaximal bispace, extremally disconnected bispace, almost Baire bispace, quasi $H$-closed bispace, multivalued map, locally compact bispace.

Using the notion of quasi-*H*-closed bispaces due to M. N. Mukherjee [11], we obtain some sufficient conditions for a $(j,i)$-stable bispace $(X,\tau_1,\tau_2)$ to be $\tau_i$-compact or $\tau_i$-Lindelöf. Moreover, for the stable bispaces we obtain A. V. Arhangelski's-like theorem. The last two theorems deal with dynamic behaviors of stable bispaces.

Throughout the paper, for a bitopological space $(X,\tau_1,\tau_2)$ we use the following notations: the interior and closure of a set $A\subset X$ with respect to the topology $\tau_i$ are denoted by $\tau_i intA$ and $\tau_i clA$, respectively, where $i\in\{1,2\}$. If $O$ is open in $\tau_i$, then we write $O\in\tau_i$, while, for the $\tau_i$-closed set $F$, we use the notation $F\in co\tau_i$ (in this case, for brevity, $O$ and $F$ are meant also as an $i$-open and an $i$-closed set, respectively).We denote by $\tau_i^A=\{A\cap U|U\in\tau_i\}$ the topology induced on the set $A$ from the $\tau_i$. Next, in several results, we apply a few important notions on bitopological structures, which are completely concerned in [5], but for classical topological ones see e.g. [6]. A set $A\subset X$ is said to be $(i,j)$-nowhere dense if $\tau_i int(\tau_j clA)=\emptyset$. If $A=\bigcup_{n=1}^{\infty}A_n$ and $A_n$ is $(i,j)$-nowhere dense for $n\in\mathbb{N}$, then $A$ is said to be a set of the $(i,j)$-first category. Otherwise, a set $A$ is said to be of the $(i,j)$-second category [5]. A bispace $(X,\tau_1,\tau_2)$ is said to be $p$-Hausdorff, if for any pair of different points $x_1$, $x_2\in X$ there exist $U\in\tau_i$ and $V\in\tau_j$, such that $x_1\in U, x_2\in V$ and $U\cap V=\emptyset$ [8]. A bispace $(X,\tau_1,\tau_2)$ is said to be $(i,j)$-regular if, for a point $x\in X$ and a set $F\in co\tau_i$ not containing $x$, there are disjoint sets $U\in\tau_i$ and $V\in\tau_j$ such that $x\in U$ and $F\subseteq V$ [5]. It can be easy to verify that a bispace $(X,\tau_1,\tau_2)$ is $(i,j)$-regular if and only if, for every point $x\in X$ and any neighborhood $U\in\tau_i$ of $x$, there exists $V\in\tau_i$ neighborhood of $x$, such that $\tau_j clV\subset U$. A bispace $(X,\tau_1,\tau_2)$ is said to be $p$-normal if for each pair of nonempty disjoint sets $F_1\in co\tau_i$ and $F_2\in co\tau_j$ there exist disjoint sets $U\in\tau_j$ and $V\in\tau_i$ such that $F_1\subseteq U$ and $F_2\subseteq V$. $(X,\tau_1,\tau_2)$ is $p$-normal if and only if for each set $F\in co\tau_i\setminus\{\emptyset\}$ and any neighborhood $U\in\tau_j$ of $F$, there exists a neighborhood $G\in\tau_j$ of $F$, such that $F\subset G\subset\tau_i clG\subset U$. A map $f:(X,\tau_1,\tau_2)\to(Y,\gamma_1,\gamma_2)$ is said to be $p$-continuous (resp. $p$-open) if both $f:(X,\tau_1)\to(Y,\gamma_1)$ and $f:(X,\tau_2)\to(Y,\gamma_2)$ are continuous (resp. open) [8].

It is well known that the peculiarity of extremal disconnectedness makes important connections between several topological constructions. Bitopologically modified spaces of this topological notion have many interesting applications (see, e.g., [10]). According to [10], a bispace $(X,\tau_1,\tau_2)$ is said to be $p$-extremally disconnected if $\tau_j clO\in\tau_i$, for any $O\in\tau_i$. It can be easy to verify that $(X,\tau_1,\tau_2)$ is $p$-extremally disconnected if and only if $\tau_i clO_1\cap\tau_j clO_2=\emptyset$ for any pair of disjoint sets $O_1\in\tau_j$ and $O_2\in\tau_i$. A bispace $(X,\tau_1,\tau_2)$ is said to be $(i,j)$-submaximal $(i,j=1,2,i\neq j)$ if every $\tau_j$-dense set is $\tau_i$-open [4].

## 2. Stable Bispaces in Statics and Dynamics

Following to the work of [9], a bispace $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-stable if each proper subset $A \in co\tau_i$ of $X$ is a $\tau_j$-compact subset in $X$. If $(X, \tau_1, \tau_2)$ is both $(i, j)$-stable and $(j, i)$-stable, then it is said to be $p$-stable, as usually is the convention for bitopological properties. Here especially notice that in the paper [7] the notion of a $p$-compact bispace was introduced. According to [7], a bispace $(X, \tau_1, \tau_2)$ is said to be $p$-compact if any cover $U$ of $X$, such that $U \subset \tau_1 \cup \tau_2$ and both $U \cap \tau_k \neq \emptyset$, $k = 1, 2$, contains a finite subcover. In the work [3] it was shown that $p$-compactness and $p$-stability are equivalent. Also, some interesting results concerning $p$-compactness are obtained in the works [17], [13] and [14].

**Theorem 2.1.** *Let a bispace $(X, \tau_1, \tau_2)$ be $p$-stable and $p$-Hausdorff, then it is $p$-normal.*

*Proof.* Consider a pair of the disjoint sets $A \in co\tau_i$, $B \in co\tau_j$ and the points $x \in A$, $y \in B$. Because $(X, \tau_1, \tau_2)$ is a $p$-Hausdorff bispace, then there exist disjoint sets $O_y(x) \in \tau_j$ and $O_x(y) \in \tau_i$ such that $x \in O_y(x)$ and $y \in O_x(y)$. Obviously, the sets $A$ and $B$ are $j$-compact and $i$-compact, respectively. If we fix a point $x \in A$ and $y$ takes various values of the points of $B$, then the collection $\{O_x(y) \in \tau_i\}_{y \in B}$ covers the set $B$. Hence it follows that there are a subset $\{y_1, y_2, \ldots, y_n\}$ of $B$ and the sets $O_x(y_k) \in \tau_i$, where $y_k \in O_x(y_k)$ and $k = \overline{1;n}$, such that $B \subset \bigcup_{k=1}^{n} O_x(y_k) \equiv O_x(B)$. Let us $O(x) \equiv \bigcap_{k=1}^{n} O_{y_k}(x) \in \tau_j$. Consider a cover $\{O(x) \in \tau_j\}_{x \in A}$ of $A$, then there exists a finite set $\{x_1, x_2, \ldots, x_p\}$ such that $A \subset \bigcup_{l=1}^{p} O(x_l) \equiv O(A) \in \tau_j$. If $O(B) \equiv \bigcap_{l=1}^{p} O_{x_l}(B)$, then $O(A) \cap O(B) = \emptyset$, i.e. the bispace $(X, \tau_1, \tau_2)$ is $p$-normal. □

A bispace $(X, \tau_1, \tau_2)$ is said to be almost $(i, j)$-Baire if any countable sequence of sets which are both $\tau_j$-open and $\tau_i$-dense, have the $\tau_i$-dense intersection [5], [1]. In [5], it is shown that $(X, \tau_1, \tau_2)$ is almost $(i, j)$-Baire if and only if for each set $A$ of the $(i, j)$-first category the set $X \setminus A$ is $\tau_i$-dense. Consequently, it may be easily to note, that if an almost $(i, j)$-Baire bispace $(X, \tau_1, \tau_2)$ is the $(j, i)$-submaximal and $(j, i)$-stable, then every set of the $(i, j)$-first category is a $\tau_i$-compact subset of $X$.

A bispace $(X, \tau_1, \tau_2)$ is said to be $(i, j) - QHC$ if any covering $V = \{O_\alpha \in \tau_i\}_{\alpha \in \Lambda}$ of $X$ contains a finite subfamily $V' = \{O_{\alpha_k} \in V\}_{k=\overline{1;n}}$ such that $X = \bigcup_{k=1}^{n} \tau_j cl O_{\alpha_k}$ [11],[15]. It is obvious that every $\tau_i$-compact bispace is an $(i, j) - QHC$ but the converse does not hold in general. Obviously, if a bispace is both $(i, j) - QHC$ and $(j, i)$-stable, then it is $\tau_i$-compact.

**Theorem 2.2.** *If $(X, \tau_1, \tau_2)$ is $(j, i)$-stable and $(Y, \gamma_1, \gamma_2)$ is an $(i, j) - QHC$ bispace, respectively, and a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is a $j$-continuous $i$-open surjection, then $(X, \tau_1, \tau_2)$ is $i$-compact.*

*Proof.* Let us consider any cover $U = \{O_\alpha \in \tau_i\}_{\alpha \in \Lambda}$ of $X$, i.e. $X = \bigcup_{\alpha \in \Lambda} O_\alpha$. Hence, $Y = f(X) = f(\bigcup_{\alpha \in \Lambda} O_\alpha) = \bigcup_{\alpha \in \Lambda} f(O_\alpha)$ and $\{f(O_\alpha)\}_{\alpha \in \Lambda}$ is a $\gamma_i$-open cover of $Y$. Since $(Y, \gamma_1, \gamma_2)$ is $(i, j) - QHC$, then we can choose a finite subfamily $f(O_{\alpha_1}), f(O_{\alpha_2}), ..., f(O_{\alpha_n})$ such that $Y = \bigcup_{k=1}^{n} \gamma_j clf(O_{\alpha_k})$. It is obvious that $X = f^{-1}(Y) = f^{-1}(\bigcup_{k=1}^{n} \gamma_j clf(O_{\alpha_k})) = \bigcup_{k=1}^{n} f^{-1}(\gamma_j clf(O_{\alpha_k}))$. Using $j$-continuity of $f$, we get $f^{-1}(\gamma_j clf(O_{\alpha_k})) \in co\tau_j$, for each $k = 1, 2, ..., n$. From $(j, i)$-stability of $(X, \tau_1, \tau_2)$ it follows that the sets $f^{-1}(\gamma_j clf(O_{\alpha_k}))$ are $\tau_i$-compact for each $k = 1, 2, ..., n$. Combine the last conclusion with the fact that: the finite union of compact sets is compact, we come to the $i$-compactness of $(X, \tau_1, \tau_2)$. □

Using the notion of an $(i, j) - QHC$ bispace we claim

**Theorem 2.3.** *Let $(X, \tau_1, \tau_2)$ be a $(j, i)$-stable bispace and suppose for any $A \in co\tau_i$, $(A, \tau_1^A, \tau_2^A)$ is an $(i, j) - QHC$ bispace. Then the space $(X, \tau_i)$ is compact.*

*Proof.* Let $O = \{O_\alpha \in \tau_i\}_{\alpha \in \Lambda}$ be any cover of $X$, then for each fixed $\alpha_0 \in \Lambda$ the set $X \setminus O_{\alpha_0} \in co\tau_i$. By the conditions of this theorem, there exists a subfamily $\{O_{\alpha_k} \in O\}_{k=\overline{1;n}}$ such that $X \setminus O_{\alpha_0} \subset \bigcup_{k=1}^{n} \tau_j clO_{\alpha_k}$. Because $(X, \tau_1, \tau_2)$ is a $(j, i)$-stable bispace, then $\bigcup_{k=1}^{n} \tau_j clO_{\alpha_k}$ is a $\tau_i$-compact subset of $X$. Observe that there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $\bigcup_{k=1}^{n} \tau_j clO_{\alpha_k} \subset \bigcup_{\alpha \in \Lambda_0} O_\alpha$. It is easy to see that a family $\{O_\alpha\}_{\alpha \in \{\alpha_0\} \cup \Lambda_0}$ represents the desirable cover of $X$, i.e. $(X, \tau_i)$ is a compact space. □

**Theorem 2.4.** *Let a $(j, i)$-stable bispace $(X, \tau_1, \tau_2)$ be $(i, j)$-regular and $i$-regular. If $A \in \tau_i \setminus \{\emptyset\}$, then the space $(A, \tau_i^A)$ is locally compact.*

*Proof.* Consider any point $a \in A$, then there are its neighborhoods $U; V \in \tau_i$, such that $\tau_i clV \subset \tau_j clU \subset A$. The $(j, i)$-stability of $(X, \tau_1, \tau_2)$ implies the $\tau_i$-compactness of the set $\tau_j clU$. Moreover, the set $\tau_i clV$ is a $\tau_i$-compact subset of the $\tau_i$-compact set $\tau_j clU$. Therefore the space $(A, \tau_i^A)$ is locally compact. □

**Theorem 2.5.** *Let a bispace $(X, \tau_1, \tau_2)$ be $(j, i)$-stable, $i$-Hausdorff and $i$-locally compact and let $(Y, \gamma_1, \gamma_2)$ be a $(j, i)$-submaximal bispace. Suppose that there exists a $j$-continuous surjective map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ such that $Y = \bigcup_{n=1}^{\infty} A_n$, where $f^{-1}(A_n) \subset \tau_i intf^{-1}(A_{n+1})$ and $A_n$ is $(i, j)$-nowhere dense in $Y$ for every $n \in \mathbb{N}$. Then the space $(X, \tau_i)$ is Lindelöf.*

*Proof.* Since $(Y, \gamma_1, \gamma_2)$ is $(j,i)$-submaximal, then the $\gamma_i$-dense sets $X \setminus A_n$ are $\gamma_j$-open, or equivalently $A_n \in co\gamma_j$. The $j$ -continuity of $f$ implies that $f^{-1}(A_n) \in co\tau_j$ for any $n \in \mathbb{N}$. Therefore, from $(j,i)$-stability of $(X, \tau_1, \tau_2)$ it follows that $f^{-1}(A_n)$ are $\tau_i$-compact for any $n \in \mathbb{N}$. Moreover, $X = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$. It is shown in [6] (p. 294-295) that a Hausdorff locally compact topological space $(X, \tau)$ is Lindelöf if and only if there exists a sequence of compact subsets $F_1, F_2, ...$ such that $F_n \subset \tau int F_{n+1}$ and $X = \bigcup_{n=1}^{\infty} F_n$. Using this fact we see that $(X, \tau_j)$ is a Lindelöf space. □

According to [6], a nonempty family $\{A_\alpha\}_{\alpha \in \Lambda}$ satisfies the finite intersection property if $A_{\alpha_1} \cap A_{\alpha_2} \cap ... \cap A_{\alpha_k} \neq \emptyset$ for each finite system $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Lambda$.

**Theorem 2.6.** *Let a $(j,i)$-stable bispace $(X, \tau_1, \tau_2)$ be a p-Hausdorff, i-bicompact and i-first countable bispace. Then it is $(i,j)$-regular, $\tau_j \subseteq \tau_i$ and $|X| \leq 2^{\aleph_0}$.*

*Proof.* In [2], A.V. Arhangel'skii proves that every first countable, bicompact (i.e. compact and Hausdorff) topological space $(X, \tau)$ satisfies the cardinal inequality $|X| \leq 2^{\aleph_0}$. Furthermore the $(j,i)$-stability and $i$-Hausdorffness of $(X, \tau_1, \tau_2)$ yield that $\tau_j \subseteq \tau_i$.

Now we must show that a bispace $(X, \tau_1, \tau_2)$ represents an $(i,j)$-regular space. By first countability of $(X, \tau_i)$, we choose a countable $T_x = \{O_n(x)\}_{n \in \mathbb{N}} \subseteq \tau_i$ such that for each $n \in \mathbb{N}, x \in O_n(x)$ and whenever $x \in O(x) \in \tau_i$ then some $O_n(x) \subseteq O(x)$. Now consider the sets $\widetilde{O}_n(x) = \bigcap_{k=1}^{n} O_k(x)$, where $n \in \mathbb{N}$ and note that a family $T'_x = \{\widetilde{O}_n(x)\}_{n \in \mathbb{N}}$ represents local base for a point $x \in X$. Moreover, it is obvious that $\widetilde{O}_1(x) \supset \widetilde{O}_2(x) \supset \ldots$ . Therefore we can begin reasoning by taking as a local base the family $T'_x$ for a point $x \in X$.

Suppose that the bispace $(X, \tau_1, \tau_2)$ is not $(i,j)$-regular. Then for each $\widetilde{O}_n(x) \in T'_x$ and $\widetilde{O}_{n+k}(x) \in T'_x$, where $k \in \mathbb{N}$, we have $\tau_j cl \widetilde{O}_{n+k}(x) \not\subset \widetilde{O}_n(x)$, i.e. $\tau_j cl \widetilde{O}_{n+k}(x) \cap (X \setminus \widetilde{O}_n(x)) \neq \emptyset$. Consider the sets $\Phi_k = (\tau_j cl \widetilde{O}_{n+k}(x)) \setminus \widetilde{O}_n(x)$ and we observe that $(j,i)$-stability and $i$-Hausdorff of $(X, \tau_1, \tau_2)$ implies $\Phi_k \in co\tau_i$ for each $k \in \mathbb{N}$. It is obvious that $\{\Phi_k\}_{k \in \mathbb{N}}$ represents a decreasing sequence of sets $\Phi_k$ and hence $\{\Phi_k\}_{k \in \mathbb{N}}$ is a family with finite intersection property. The compactness of $(X, \tau_i)$ implies that $\bigcap_{k=1}^{\infty} \Phi_k \neq \emptyset$. Therefore, there exists $y \in \bigcap_{k=1}^{\infty} \Phi_k$ such that $y \neq x$. Since $\Phi_k \subset \tau_j cl \widetilde{O}_{n+k}(x)$, then we have $y \in \bigcap_{k=1}^{\infty} \tau_j cl \widetilde{O}_k(x)$. Since $(X, \tau_1, \tau_2)$ is $p$-Hausdorff, there exist a $\tau_j$-open neighborhood $O(y)$ of $y$ and $\widetilde{O}_m(x) \in T'_x$ such that $O(y) \cap \widetilde{O}_m(x) = \emptyset$. Therefore, $y \notin \tau_j cl \widetilde{O}_m(x)$ and $y \notin \bigcap_{k=1}^{\infty} \tau_j cl \widetilde{O}_k(x)$. This contradiction ends the proof of the theorem. □

The following result is shown in [11, Theorem 2.3 ]: $(X, \tau_1, \tau_2)$ is $(i,j)-QHC$ if and only if any family $\{U_\alpha \in \tau_i\}_{\alpha\in\Lambda}$ of $\tau_i$-open sets with the finite intersection property satisfies the condition $\bigcap_{\alpha\in\Lambda} \tau_j clU_\alpha \neq \emptyset$.

Next we say that $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is $(i,j) - \Delta$ continuous if the map $f : (X, \tau_i) \to (Y, \gamma_j)$ is continuous. Moreover, we say that $(X, \tau)$ is a stable space if every proper closed subset is compact.

Using the constructions given by A. Taimanov [18], we claim

**Theorem 2.7.** *Let a bispace $(X, \tau_1, \tau_2)$ be $(i,j)$-submaximal and $(Y, \gamma_1, \gamma_2)$ be a p-extremally disconnected, $(i,j)-QHC$, j-stable, j-Urysohn bispace. Suppose that for a $\tau_1$-dense and $\tau_2$-dense set A a map*

$$f : (A, \tau_1^A, \tau_2^A) \to (Y, \gamma_1, \gamma_2)$$

*is i-open and $(i,j) - \Delta$ continuous. Then there exists an $(i,j) - \Delta$ continuous extension*

$$\varphi : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$$

*of the map f if and only if for any pair of disjoint sets $F_1, F_2 \in co\gamma_j$ implies $\tau_i clf^{-1}(F_1) \cap \tau_i clf^{-1}(F_2) = \emptyset$.*

*Proof.* Necessity-Let a map $\varphi : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is an extension of $f$. If $P_k \in co\gamma_j$ for $k = 1, 2$ and $P_1 \cap P_2 = \emptyset$, then by $(i,j) - \Delta$ continuity of $\varphi$ it follows that $\varphi^{-1}(P_k) = \tau_i cl\varphi^{-1}(P_k)$, where $k = 1, 2$ and $\varphi^{-1}(P_1) \cap \varphi^{-1}(P_2) = \emptyset$. It is obvious that $\tau_i clf^{-1}(P_1) \cap \tau_i clf^{-1}(P_2) = \varphi^{-1}(P_1) \cap \varphi^{-1}(P_2) = \emptyset$.

Sufficiency-Let us consider a family $B(x) = \{\gamma_j clf(A \cap U) | x \in U \in \tau_i\}$ for each $x \in X$. The $p$-extremal disconnectedness of $(Y, \gamma_1, \gamma_2)$ and $i$-openess of the map $f$ imply that $\gamma_j clf(A \cap U) \in \gamma_i$. First we shall show that the family $B(x)$ has the finite intersection property. Indeed, it is obvious that $x \in \bigcap_{k=1}^{l} U_k \in \tau_i, A \cap U_k \neq \emptyset$ and $A \cap (\bigcap_{k=1}^{l} U_k) \neq \emptyset$, where $x \in U_k \in \tau_i$ for any $k = \overline{1;l}$. Note that $\emptyset \neq \gamma_j clf(A \cap \bigcap_{k=1}^{l} U_k) \subset \bigcap_{k=1}^{l} \gamma_j clf(A \cap U_k)$, i.e. $B(x)$ is a family of $\gamma_i$-open sets with the finite intersection property. Since $(Y, \gamma_1, \gamma_2)$ is an $(i,j) - QHC$ bispace, then the set $\varphi(x) = \cap B(x) \neq \emptyset$ for every $x \in X$ by [11, Theorem 2.3]. We shall show that $|\varphi(x)| = 1$. For this purpose we suppose $y_1, y_2 \in \varphi(x)$ and $y_1 \neq y_2$. Since the space $(Y, \gamma_j)$ is Urysohn, then there are the neighborhoods $V_1 \in \gamma_j$ of $y_1$ and $V_2 \in \gamma_j$ of $y_2$, such that $\gamma_j clV_1 \cap \gamma_j clV_2 = \emptyset$. By the conditions of our Theorem it follows that $\tau_i clf^{-1}(V_1) \cap \tau_i clf^{-1}(V_2) \subset \tau_i clf^{-1}(\gamma_j clV_1) \cap \tau_i clf^{-1}(\gamma_j clV_2) = \emptyset$. Therefore, $X = O_1 \cup O_2$, where $O_1 = X \setminus \tau_i clf^{-1}(V_1)$ and $O_2 = X \setminus \tau_i clf^{-1}(V_2)$. It is obvious $x \in O_1$ or $x \in O_2$. Hence, if we put $x \in O_1$ (in the other case the reasonings are analogous), then the following implication holds: $V_1 \cap f(A \cap O_1) \subset V_1 \cap f(O_1) = V_1 \cap f(X \setminus \tau_i clf^{-1}(V_1)) \subset V_1 \cap (Y \setminus V_1) = \emptyset$.

Since $V_1 \in \gamma_j$, then $V_1 \cap \gamma_j clf(A \setminus \tau_i clf^{-1}(V_1)) = \emptyset$. Therefore $y_1 \notin \gamma_j clf(A \setminus \tau_i clf^{-1}(V_1)) = \gamma_j clf(A \cap O_1) \in B(x)$. This contradiction gives the equality $|\varphi(x)| = 1$.

For every $x \in X$ the equation $\varphi(x) = \cap\{\gamma_j clf(A \cap U) | x \in U \in \tau_i\}$ defines an extension of the map $f$. We want to show the equation $\varphi|_A = f$. Suppose $\varphi(x) \neq f(x)$ for some $x \in A$, then there are neighborhoods $G_1 \in \gamma_j$ of $\varphi(x)$ and $G_2 \in \gamma_j$ of $f(x)$, such that $\gamma_j clG_1 \cap \gamma_j clG_2 = \emptyset$. By $(i,j) - \Delta$ continuity of the map $f$ there is $W \in \tau_i^A$ such that $x \in W$ and $f(W) \subseteq G_2$. Then $W = U^* \cap A$ for some $U^* \in \tau_i$, so $x \in U^*$ and $f(A \cap U^*) \subset G_2$. Obviously, $\varphi(x) \in \gamma_j clf(A \cap U^*) \subseteq \gamma_j clG_2$, but this contradicts $\gamma_j clG_1 \cap \gamma_j clG_2 = \emptyset$. Therefore we see that $\varphi|_A = f$.

Now we must show that the map $\varphi$ is $(i,j) - \Delta$ continuous. Let $x \in X$ and $V \in \gamma_j$ be a neighborhood of $\varphi(x)$. Then $j$-stability of $(Y, \gamma_1, \gamma_2)$ implies $\gamma_j$-compactness of $Y \setminus V$. Since $\varphi(x) = \cap\{\gamma_j clf(A \cap U) | x \in U \in \tau_i\} \subset V$, then $Y \setminus V \subset \cup\{Y \setminus \gamma_j clf(A \cap U) | x \in U \in \tau_i\}$. Therefore there are the neighborhoods $U_m \in \tau_i$ of $x$, $m = \overline{1;n}$ such that $\bigcap_{m=1}^n \gamma_j clf(A \cap U_m) \subset V$. It is obvious $x \in \bigcap_{m=1}^n U_m = U \in \tau_i$ and $\gamma_j clf(A \cap U) \subset V$. Moreover, $\varphi(\xi) = \cap\{\gamma_j clf(A \cap G) | \xi \in G \in \tau_i\} \subset \gamma_j clf(A \cap U) \subset V$ for every $\xi \in U$, i.e. $\varphi(U) \subset V$. □

Let $F : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be a multi-valued map, then we denote by $F' : (Y, \gamma_1, \gamma_2) \to (X, \tau_1, \tau_2)$ a multi-valued map defined as $F'(y) = \{x \in X | y \in F(x)\}$ for each $y \in Y$. A multi-valued map $F : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be: (a) $i - X$ compact if the subset $F'(y) \subset X$ is $i$-compact for any point $y \in Y$ [12]; (b) $(i,j) - \Delta$ closed if $F : (X, \tau_i) \to (Y, \gamma_j)$ is closed; (c) $i$-closed (resp. $i$-open) if $F : (X, \tau_i) \to (Y, \gamma_i)$ is a closed (resp. open) mapping.

In [12] it was established that a multi-valued map $F : (X, \tau) \to (Y, \gamma)$ is closed (or equivalently $F'$ is continuous) if and only if $F(\tau clA) \subseteq \gamma clF(A)$ for any $A \subseteq X$.

Recall that a bispace $(X, \tau_1, \tau_2)$ is said to be $(i,j)$-locally compact if every point $x \in X$ admits an $i$-open neighborhood $U$ such that $\tau_j clU$ is $\tau_j$-compact [16]. It is obvious that every $j$-compact bispace is $(i,j)$-locally compact.

**Theorem 2.8.** *Let $(X, \tau_1, \tau_2)$ be $(i,j)$-locally compact and $(Y, \gamma_1, \gamma_2)$ be an $(i,j)$-stable bispace. Moreover, let a multi-valued map $F : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be $i - X$ compact, $i$-open, $(j,i) - \Delta$-closed and $j$-closed. Then $(Y, \gamma_1, \gamma_2)$ is an $(i,j)$-locally compact bispace.*

*Proof.* Let us consider any point $y_0 \in Y$. Then $F'(y_0)$ is an $i$-compact subset of an $(i,j)$-locally compact bispace $(X, \tau_1, \tau_2)$. Note that, for each $\xi \in F'(y_0)$ there exists a neighborhood $U(\xi) \in \tau_i$ of $\xi$ such that $\tau_j clU(\xi)$ is a $\tau_j$-compact subset of $X$. It is obvious that $\{U(\xi)\}_{\xi \in F'(y_0)}$ represents a $\tau_i$-open cover of $F'(y_0)$. Because $F'(y_0)$ is a $\tau_i$-compact subset of $X$, then the cover $\{U(\xi)\}_{\xi \in F'(y_0)}$ contains a finite subfamily $\{U(\xi_k)\}_{k=\overline{1;n}}$ such that $F'(y_0) \subset \bigcup_{k=1}^{n} U(\xi_k)$ and $\tau_j clU(\xi_k)$ is a $\tau_j$-compact subset of $X$ for any $k = \overline{1;n}$. Set $O(F'(y_0)) \equiv \bigcup_{k=1}^{n} U(\xi_k) \in \tau_i$, then the set $\tau_j clO(F'(y_0)) = \bigcup_{k=1}^{n} \tau_j clU(\xi_k)$ represents a $\tau_j$-compact subset of $X$. From $(j,i) - \Delta$ closedness of $F$ and $(i,j)$-stability of $(Y, \gamma_1, \gamma_2)$ it follows the $\gamma_j$-compactness of the set $K \equiv F(\tau_j clO(F'(y_0))) \subset Y$. Since $F$ is an $i$-open map, then $y_0 \in O(y_0) \equiv FO(F'(y_0)) \in \gamma_i$. The $j$-closedness of $F$ implies that $\gamma_j clO(y_0) \subseteq F(\tau_j clO(F'(y_0))) = K$. Because $K \subseteq Y$ is $\gamma_j$-compact and $K \supset A \equiv \gamma_j clO(y_0) \in co\gamma_j$, then $A$ represents $\gamma_j$-compact neighborhood of $y_0$, i.e. $(Y, \gamma_1, \gamma_2)$ is $(i,j)$-locally compact. □

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## References


[1] C. Alegre, J. Ferrer, and V. Gregori, *On pairwise Baire bitopological spaces*, Publ. Math. Debrecen **55** (1999), 3–15.

[2] A. V. Arhangel'skii, *On cardinality of bicompacts with first axiom of countability*, Soviet. Math. Dokl. **187** (1969), 967–970.

[3] I. E. Cooke and I. L. Reilly, *On bitopological compactness*, J. London Math. Soc. **9** (1975), 518–522.

[4] I. Dochviri, *On submaximality of bitopological spaces*, Kochi J. Math. **5** (2010), 121–128.

[5] B. P. Dvalishvili, *Bitopological spaces: theory, relations with generalized algebraic structures and applications.* [Math. Studies 199]. North-Holland, Elsevier, 2005.

[6] R. Engelking, *General topology.* (Russian Edit.) Moscow, Publ. House Mir, 1986.

[7] P. Fletcher, H. B. Hoyle and C. W. Patty, *The comparision of topologies*, Duke Math. J. **36** (1969), 325–331.

[8] J. C. Kelly, *Bitopological spaces*, Proc. London Math Soc. **9(13)** (1963), 71–89.

[9] R. D. Kopperman, *Assymetry and duality in topology*, Topol. Appl. **66** (1995), 1–39.

[10] S. Lal, *Pairwise concepts in bitopological spaces*, J. Austral. Math. Soc., Ser. A **26** (1978), 241–250.

[11] M. N. Mukherjee, *On pairwise almost compactness and pairwise $H$-closedness in bitopological spaces*, Ann. Soc. Sci. Bruxelles **96(11)** (1982), 98–106.

[12] V. I. Ponomarev, *Properties of topological spaces preserved under continuous multivalued mappings*, Russ. Mat. Sb. **51(93)** (1960), 515–536.

[13] I. L. Reilly, *Bitopological local compactness*, Indag. Math. **34** (1972), 407–411.

[14] ———, *Zero dimensional bitopological spaces*, Indag. Math. **35** (1973), 127–131.

[15] K. Sen and M. N. Mukherjee, *Some characterization of bitoplogical QHC-spaces*, Indian J. Pure Appl. Math. **24(6)** (1993), 389–399.

[16] R. Stoltenberg, *On quasi-metric spaces*, Duke Math. J. **36** (1969), 65–71.

[17] J. Swart, *Total disconnectedness in bitopological spaces and product bitopological spaces*, Indag. Math. **33** (1971), 135–145.

[18] A. D. Taimanov, *On extension of continuous maps of topological spaces*, Russ. Math. Sb. **31(73)** (1952), 65–71.

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