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by

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# PARTIALLY PERIPHERAL HYPERBOLIC LINKS AND SPATIAL GRAPHS

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Abstract. The partially peripheral property of links and spatial graphs in 3-manifolds indicates a certain level of complexity which shows the similarity with links and spatial graphs in $S^3$ in some peripheral property of the exteriors. The aim of this paper is to prove that every closed connected orientable 3-manifold contains infinitely many partially peripheral hyperbolic links and spatial graphs.

## 1. Introduction

The knot theory has provided various tools and ideas mainly from the study of knots in $S^3$. However, in many cases it is a difficult problem to find a generalizing method which makes them applicable to sufficiently many 3-manifolds not contained in $S^3$. The author defined in [3] a new class of 3-manifolds, called partially peripheral 3-manifolds, with a certain peripheral property similar to the exteriors of knots in $S^3$, and studied the case of graph manifolds. Moreover, the partially peripheral property plays an important role in the study of symmetries of 3-manifolds (see [4]), which is applied to the study of symmetries of spatial graphs in 3-manifolds (see [5]). In this paper, we show that any closed orientable 3-manifold contains infinitely many links and spatial graphs whose exteriors are partially peripheral hyperbolic 3-manifolds.

We grade 3-manifolds on their peripheral properties as follows. We say that a 3-manifold $M$ is *peripheral* if the boundary contains a compact

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connected surface $F$ with a surjective inclusion-induced homomorphism $\pi_1 F \to \pi_1 M$. We say that $M$ is *totally peripheral* if every loop in $M$ is freely homotopic into $\partial M$. We say that $M$ is *partially peripheral* if every loop in $M$ is a band sum of a finite number of loops freely homotopic into the boundary. Brin, Johannson and Scott [1] proved that compact orientable totally peripheral 3-manifolds are peripheral, and therefore they are compression bodies. However, partially peripheral 3-manifolds form a sufficiently large class of 3-manifolds containing the exteriors of links in $S^3$. This is verified by considering the Wirtinger presentations of the link groups.

Several properties of partially peripheral 3-manifolds $M$ are easily verified. Suppose that a loop $l$ in $M$ is the band sum of loops $l_1, \dots, l_n$ freely homotopic into $\partial M$ along bands $\beta_1, \dots, \beta_{n-1}$. For each $l_i$ there exists a continuous map $f_i$ of the annulus $S^1 \times I$ into $M$ which takes $S^1 \times \{0\}$ onto $l_i$ homeomorphically and $S^1 \times \{1\}$ into $\partial M$. Then $f_1, \dots, f_n$ extend by using $\beta_1, \dots, \beta_{n-1}$ to a continuous map $f$ of a compact planar surface with boundary components $b_0, \dots, b_n$ into $M$ which takes $b_0$ onto $l$ homeomorphically and $b_1 \cup \dots \cup b_n$ into $\partial M$. We call $f$ a *$P$-map* for $l$. Since the inclusion induced homomorphism $H_1(M) \to H_1(M, \partial M)$ is the zero map, the duality $H_1(M, \partial M) \cong H^2(M)$ implies that every closed surface in $M$ separates $M$ into two pieces. Every compact submanifold $N$ of $M$ is partially peripheral, since the $P$-map for a loop $L \subset N$ has a restriction in $N$ connecting $L$ and $\partial N$. Thus the partially peripheral property of a compact 3-manifold survives the boundary compression, the prime factorization and the JSJ decomposition [6, 7]. In addition, partially peripheral graph manifolds are characterized in [3].

Let $M$ be a compact connected 3-manifold with several boundary components, and $F$ a compact connected surface of Euler characteristic $\chi$ properly embedded in $M$ which meets $2 - \chi$ components of $\partial M$. Suppose either $F$ is separating or every component of $\partial F$ but one is a separating loop on the corresponding component of $\partial M$. Suppose that splitting $M$ along $F$ yields $M_1$ and $M_2$ (possibly $M_1 = M_2$). Then $M$ is called the *$P_1$-sum* of $M_1$ and $M_2$. We first prove the following theorem, which provides a method for constructing partially peripheral 3-manifolds.

**Theorem 1.1.** *The $P_1$-sum of one or two partially peripheral 3-manifolds is partially peripheral.*

Let $F$ be a compact connected surface properly embedded in $M$. Suppose either $F$ is separating or there is an oriented link on $\partial M$ intersecting $F$ algebraically once. Suppose that splitting $M$ along $F$ yields $M_1$ and $M_2$ (possibly $M_1 = M_2$). Then $M$ is called the *$P_2$-sum* of $M_1$ and $M_2$. We prove the following theorem.

**Theorem 1.2.** *The $P_2$-sum $M$ of one or two partially peripheral 3-manifolds along a surface $F$ is partially peripheral, if every loop in $M - F$ admits a $P$-map whose image avoids $F$.*

We say that a spatial graph $G$ in a closed 3-manifold $M$ is *partially peripheral* if the exterior $E(G) = M - \mathrm{int}N(G)$ is partially peripheral. Theorems 1.1 and 1.2 are useful for constructing partially peripheral hyperbolic links and spatial graphs in closed connected orientable 3-manifolds, if the gluing surface has negative Euler characteristic. By starting with the exterior of hyperbolic spatial graph in $S^3$ and applying the $P_1$- and $P_2$-sum operations, we prove the following theorem.

**Theorem 1.3.** *For any integer $\chi \leq 0$, every closed connected orientable 3-manifold contains infinitely many partially peripheral hyperbolic spatial graphs, each of whose components has Euler characteristic $\chi$.*

Note that the case $\chi = 0$ implies the existence of infinitely many partially peripheral hyperbolic links. The possibly not partially peripheral version of Theorem 1.3 is known as a result of Myers [11]. But the proof of Theorem 1.3 takes a different approach from Myers' argument.

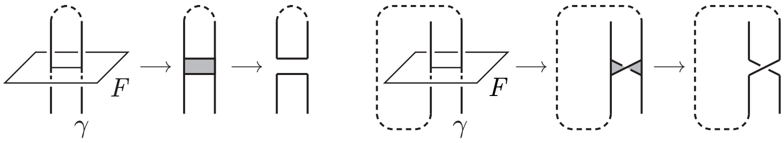
This paper is arranged as follows. In Section 2, we study the $P_1$- and $P_2$-sum of partially peripheral 3-manifolds to prove Theorems 1.1 and 1.2. In Section 3, we apply these theorems to some operations on spatial graphs like vertex connected sum, and prove Theorem 1.3.

## 2. Construction of partially peripheral 3-manifolds

In this section, we prove Theorems 1.1 and 1.2 to show that the $P_1$- and $P_2$-sum operations are useful for constructing partially peripheral hyperbolic 3-manifolds. We start with the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Let $M$ be a compact connected 3-manifold which is the $P_1$-sum of one or two partially peripheral 3-manifolds along a compact connected surface $F$ of Euler characteristic $\chi$. Then $F$ is a genus zero surface with $2 - \chi$ boundary components.

We first consider the case where $F$ splits $M$ into two partially peripheral 3-manifolds $M_1$ and $M_2$. For any loop $\gamma$ in $M$ which meets $F$ in $2n$ points $x_1, \ldots, x_{2n}$ in order, a system of $n$ disjoint arcs connecting $x_{2i-1}$ and $x_{2i}$ for $1 \leq i \leq n$ is found on $F$, and induces a band surgery of $\gamma$ into $n$ loops disjoint from $F$, as illustrated in Figure 1 (1). Since $\gamma$ is the band sum of these loops, it is enough without loss of generality to consider the case $\gamma \subset \mathrm{int}M_1$.




(1) A band surgery which produces a loop disjoint from $F$

(2) A band surgery which produces a loop intersecting $F$ in a point

FIGURE 1. Band surgeries of $\gamma$ along arcs on $F$.

By the partially peripheral property of $M_1$, there is a $P$-map $f$ for $\gamma$ into $M_1$. Suppose that $f$ is a continuous map of a compact planar surface bounded by loops $b_0, \dots, b_m$ into $M_1$ which takes $b_0$ onto $\gamma$ homeomorphically and $b_1 \cup \cdots \cup b_m$ into $\partial M_1$. Since $\pi_1 F$ is generated by loops homotopic into $\partial F$, for each $f(b_i)$ lying on $F$ there is a $P$-map whose image lies on $F$. By extending $f$ by using such $P$-maps, $f$ can be modified so that each $f(b_i)$ meets $F$ in a system of arcs $a_1, \dots, a_k$, where $f^{-1}(a_1), \dots, f^{-1}(a_k)$ appear in order as we go along $b_i$. Since $F$ intersects distinct $2 - \chi$ components of $\partial M$, the end point of $a_j$ and the start point of $a_{j+1}$, where $1 \leq j < k$, lie on the same component of $\partial F$. Therefore, $a_j$ and $a_{j+1}$ are connected by an arc $\tau_j$ on $\partial F$. Similarly, an arc $\tau_k$ connecting $a_k$ and $a_1$ is found on $\partial F$. Let $L_i$ be the closed path on $F$ which traverses $a_1, \tau_1, a_2, \tau_2, \dots, a_{k-1}, \tau_{k-1}, a_k, \tau_k$ in order. Then $L_i$ admits a $P$-map $f_i$ whose image lies on $F$. Therefore, $f$ extends by using $f_1, \dots, f_m$ to a $P$-map for $\gamma$ into $M$.

Next, we consider the case where $M$ is the $P_1$-sum of a single partially peripheral 3-manifold $M'$. Let $F_1$ and $F_2$ be two disjoint copies of $F$ lying on $\partial M'$, where the $P_1$-sum is performed. Then each $F_i$ connects distinct $2 - \chi$ components of $\partial M' - \operatorname{int}(F_1 \cup F_2)$, where exactly one of them connects $F_1$ and $F_2$. The argument presented for the previous case implies that any loop in $M'$ admits a $P$-map whose image avoids $F_1 \cup F_2$. Therefore, it is enough to show that any loop $\gamma$ in $M$ admits a $P$-map if it meets $F$ transversally in $n \geq 1$ points. By considering band surgeries along arcs on $F$ illustrated in Figure 1, we may assume $n = 1$. Take a loop $L$ on $\partial M$ intersecting $F$ transversally in a point, and an arc $\alpha$ on $F$ connecting $\gamma$ and $L$. Then a band $\beta$ along $\alpha$ induces a band surgery of $\gamma \cup L$ into a loop $L'$ in $M'$. The partially peripheral property of $M'$ implies that $L'$ admits a $P$-map into $M'$, which can be modified so that the image avoids $F_1 \cup F_2$ as before. Then we obtain a $P$-map for $\gamma$ induced from the $P$-map for $L'$ and the band $\beta$. $\square$

**Corollary 2.1.** *A compact connected 3-manifold $M$ is partially peripheral if $M$ is a connected sum or a boundary connected sum of two partially peripheral 3-manifolds.*

*Proof.* We can see $M$ as a $P_1$-sum of the two partially peripheral 3-manifolds along a sphere or a disk. Hence, the conclusion is immediate from Theorem 1.1. □

*Proof of Theorem 1.2.* The case where $F$ is separating follows from the argument presented in the proof of Theorem 1.1, since every loop $\gamma$ in $M - F$ admits a $P$-map whose image avoids $F$. We therefore assume that $M$ is the $P_2$-sum of a single partially peripheral 3-manifold $M'$ along $F$. Let $F^-$ and $F^+$ be two disjoint copies of $F$ lying on $\partial M'$, where the $P_2$-sum is performed.

Let $\gamma$ be a loop in $M$. We are going to construct a $P$-map for $\gamma$. By the assumption, every loop in $M'$ admits a $P$-map whose image avoids $F^- \cup F^+$. Therefore, by considering band surgeries of $\gamma$ along arcs on $F$ illustrated in Figure 1, it is enough to consider the case where $\gamma$ intersects $F$ transversally in a point. There is an $n$-component oriented link $l_1 \cup \cdots \cup l_n$ on $\partial M$ intersecting $F$ algebraically once. Denote by $\nu_i$ the algebraic intersection number of $l_i$ and $F$. Then $\nu_1 + \cdots + \nu_n = 1$. By the argument similar to the above, we can take the link so that each $l_i$ intersects $F$ in exactly $|\nu_i| \neq 0$ points. Without loss of generality, $\nu_i > 0$ for $1 \leq i \leq k$ and $\nu_i < 0$ for $k < i \leq n$.

Let $v_i \in l_i \cap F$ for $1 \leq i \leq n$. Connect $v_i$ and $v_{i+1}$ by a path $\alpha_i$ on $F$ for $1 \leq i < n$. Denote by $\alpha_0$ the path which traverses $\alpha_1, \cdots, \alpha_{n-1}$ in order. Let $\widetilde{M}$ be the infinite cyclic covering of $M$ obtained from infinite copies $(M'_i, F_i^+, F_i^-)$ of $(M', F^+, F^-)$ for $i \in \mathbb{Z}$ by gluing each $F_i^+$ to $F_{i+1}^-$. Denote by $p\colon \widetilde{M} \to M$ the projection map and by $\tau\colon \widetilde{M} \to \widetilde{M}$ the covering transformation which takes $M'_i$ to $M'_{i+1}$. Let $s_0 = 0$ and $s_i = \sum_{j=1}^{i} \nu_j$ for $0 \leq i \leq n$. Then $s_n = 1$. Take the lifts $\tilde{l}_i$ of $l_i$ connecting $F_{s_{i-1}}^-$ and $F_{s_i}^-$ for $1 \leq i \leq n$, the lift $\tilde{\alpha}_0$ of $\alpha_0$ on $F_0^-$, and the lifts $\tilde{\alpha}_i$ of $\alpha_i$ on $F_{s_i}^-$ for $1 \leq i < n$. Let $L$ be the path in $\widetilde{M}$ which traverses $\tilde{\alpha}_0, \tilde{l}_1, \tilde{\alpha}_1, \tilde{l}_2, \ldots, \tilde{\alpha}_{n-1}, \tilde{l}_n$ in order. Then $L$ induces a closed path in $M$ which is homotopic to a band sum $\lambda$ of $l_1, \ldots, l_n$ along arcs $\alpha_1, \ldots, \alpha_{n-1}$ on $F$. Note that $\lambda$ admits a $P$-map, since $l_1, \ldots, l_n$ lie on $\partial M$.

Let $\tilde{\gamma}$ be the lift of $\gamma$ connecting $F_0^-$ and $F_1^-$ through $M'_0$. Take a path $\beta$ on $F_1^-$ connecting $L$ and $\tilde{\gamma}$. The path $L'$ which traverses $L, \beta, \tilde{\gamma}, \tau^{-1}(\beta)$ in order induces a closed path in $M$ which is homotopic to a band sum $\lambda'$ of $\lambda$ and $\gamma$ along $p(\beta)$ (see Figure 2). We can see $L'$ as a band sum of loops in $M'_0, \ldots, M'_{s_k-1}$ along bands in $N(\bigcup_{i=1}^{s_k-1} F_i^-)$. Since each summand loop in $M'_i$ admits a $P$-map whose image avoids $F_i^{\pm}$, these $P$-maps and the

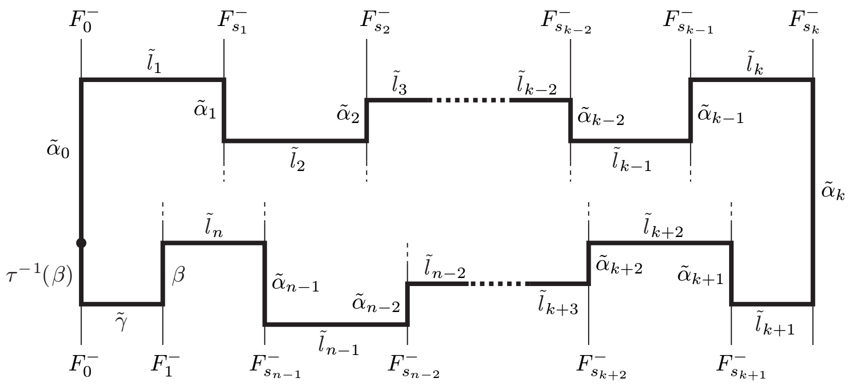


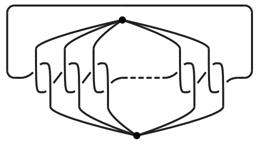

FIGURE 2. The closed path $L'$ in $\widetilde{M}$.

bands induce a $P$-map for $\lambda'$. Then the $P$-map for $\lambda'$ extends to a $P$-map for $\gamma$ by using $p(\beta)$ and the $P$-map for $\lambda$, as required. □

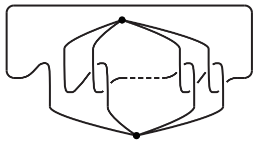
**Proposition 2.2** (Myers [11]). *Let $M$ be a compact connected 3-manifold with non-empty boundary, and $F$ a compact proper surface of negative Euler characteristic in $M$ bounded by essential loops on $\partial M$. Then $M$ is an irreducible atoroidal anannular 3-manifold with incompressible boundary if so is each piece obtained by splitting $M$ along $F$.*

*Proof.* Assume that $M$ is not an irreducible atoroidal anannular 3-manifold with incompressible boundary. There exists an essential surface $S$ in $M$ which is either a sphere, a disk, an annulus, or a torus. Then $S \cap F$ is not empty. Since the exterior $E(F)$ has incompressible boundary, $F$ is incompressible and boundary-incompressible. Therefore, the irreducibility of $E(F)$ implies that $S$ can be isotoped so as to intersect $F$ in essential loops and essential arcs on $S$. Then $S$ is a torus or an annulus which meets $E(F)$ in a system of disks or annuli. Since $S$ is essential and $F$ is neither a disk nor an annulus, some component of $S \cap E(F)$ is not parallel into $\partial E(F)$. However, this contradicts the assumption that $E(F)$ is an anannular 3-manifold with incompressible boundary. □

**Remark 2.3.** Let $M$ be a compact connected irreducible atoroidal anannular 3-manifold with incompressible boundary. Then Thurston's hyperbolization theorem [10] implies that $M$ admits a complete hyperbolic structure of finite volume so that the toral components of $\partial M$ correspond to the cusps and the others are totally geodesic surfaces.



(1) $H_n$



(2) a planar subgraph

FIGURE 3. The spatial graphs $H_n$ and a planar subgraph.

## 3. Hyperbolic links and spatial graphs

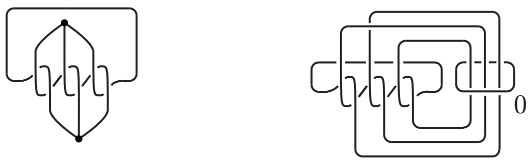
Let $G$ be a spatial graph in a closed connected 3-manifold $M$. Suppose that a sphere $S$ in $M$ intersects $G$ in $n$ points and splits $M$ into $M_1$ and $M_2$ (possibly $M_1 = M_2$). By collapsing each sphere on $\partial M_i$ to a point, $G \cap M_i$ is deformed to a spatial graph $G_i$ in a closed 3-manifold $\overline{M}_i$, where the collapsed points $v_1$ and $v_2$ are $n$-valent vertices of $G_1$ and $G_2$ respectively. We say that $G$ is a *vertex connected sum* of $G_1$ and $G_2$ at $v_1$ and $v_2$. In particular, $M = \overline{M}_1 \# \overline{M}_2$ if $\overline{M}_1 \neq \overline{M}_2$, and $M = \overline{M}_1 \# S^2 \times S^1$ otherwise. The vertex connected sum operation glues $E(G_1)$ and $E(G_2)$ along compact planar surfaces on the boundaries, where the gluing surface $F$ in $E(G)$ lies on $S$.

Assume that $G_1$ and $G_2$ are hyperbolic spatial graphs, and that $F$ has negative Euler characteristic. Then Proposition 2.2 implies that $E(G)$ is an irreducible atoroidal anannular 3-manifold. Therefore, Thurston's hyperbolization theorem [10] implies that $G$ is hyperbolic.

Paoluzzi and Zimmermann [12] proved that the spatial $\theta_n$-curve $H_n$ in $S^3$ illustrated in Figure 3 (1) is hyperbolic for $n \geq 3$. Ushijima [14, 2] showed that $H_n$ is an example of hyperbolic spatial graphs which are $n/2$-fold cyclic branched coverings of strongly invertible knots. Suzuki [13] showed that any proper subgraph of $H_n$ is planar. An example of a planar subgraph of $H_n$ is illustrated in Figure 3 (2). Moreover, $H_n$ is partially peripheral, since $\pi_1(E(H_n))$ admits the Wirtinger presentation.

Figure 4 illustrates an example of a vertex connected sum $L$ of $H_3$ at the two vertices. A disk $D$ bounded by the 0-framed trivial knot corresponds to a sphere in $S^2 \times S^1$ on which the gluing surface $F$ lies. Note that $\partial F$ consists of meridians of $L$. Since $L$ has a component which meets $D$ once, Theorem 1.2 implies that $L$ is a partially peripheral hyperbolic link in $S^2 \times S^1$.

**Proposition 3.1.** *Every closed connected orientable 3-manifold contains infinitely many partially peripheral hyperbolic links.*




(1) $H_3$ (2) $L$

FIGURE 4. A vertex connected sum $L$ of $H_3$ realizing a $P_2$-sum of $E(H_3)$.

*Proof.* Let $M$ be a closed connected orientable 3-manifold. Since $S^3$ contains infinitely many partially peripheral hyperbolic links such as non-torus 2-bridge links (see [9]), it is enough to consider the case $M \neq S^3$. The Lickorish-Wallace Theorem [8, 15] states that $M$ is obtained by performing $\pm 1$-Dehn surgery on a link $L = L_1 \cup \cdots \cup L_\mu$ in $S^3$. Take an unknotting tunnel system $\mathcal{T}$ for $L$ so that $\Gamma = L \cup \mathcal{T}$ is a connected planar trivalent graph in $S^3$. Suppose that $\Gamma$ has Euler characteristic $2-n$ where $n \geq \mu + 1$, i.e. $\Gamma$ has $n-2$ edges not lying on $L$. Then a planar $\theta_n$-curve $T_n$ is obtained from $\Gamma$ by sliding some edges. For any integer $m > 2n-2$, we consider $T_n$ to be a subgraph of $H_m$.

We first deform $H_m$ in $S^3$ to a 4-valent spatial graph containing $\Gamma$. Slide some edges of $H_m$ so as to realize the converse deformation of $T_n$ to $\Gamma$. Then we obtain $\Gamma$ with $m-n$ additional edges attached. Slide the $m-n$ attached edges to obtain a 4-valent spatial graph $H'_m$ so that $n-2$ of them and the unknotting tunnels in $\Gamma$ form $n-2$ cycle graphs of length two, and that the other $m-2n+2$ are deformed to loop edges with new vertices on $L$. Note that every vertex of $H'_m$ lies on $L$, and that each component $L_i$ of $L$ has at least one vertex which is incident to either a loop edge or an edge connected to another component. For example, Figure 5 illustrates the case where $n=2$, $m=6$, and $\Gamma$, which is highlighted with thick lines, consists of a trefoil knot and an unknotting tunnel.

Now we perform the Dehn surgery on $L$. Then $E(H'_m)$ can be regarded as the exterior of a spatial graph $H^M_m$ in $M$ isomorphic to $H'_m$. Let $L^M = L^M_1 \cup \cdots \cup L^M_\mu$ be the link on $H^M_m$ where each $L^M_i$ corresponds to $L_i$. Then $N(H^M_m)$ is obtained from $N(L^M)$ by adding $m-2$ 1-handles corresponding to the edges not lying on $L^M$. Let $D_1, \ldots, D_{2t}$ be the disks on $\partial N(L^M_i)$ along which 1-handles are attached, where $D_{2k-1}$ and $D_{2k}$ correspond to the same vertex on $L^M_i$ for $1 \leq k \leq t$.

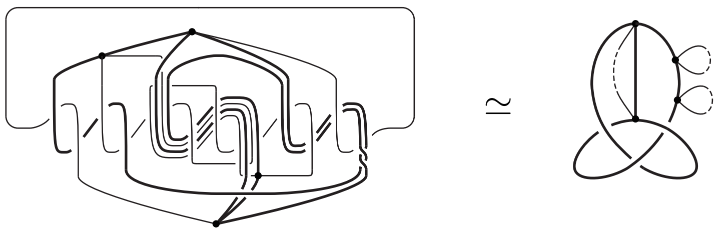


FIGURE 5. A spatial graph $H_6'$ including a trefoil knot $L$ and an unknotting tunnel $\mathcal{T}$.

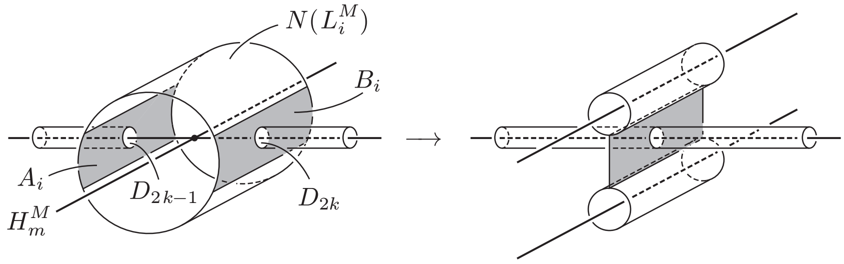
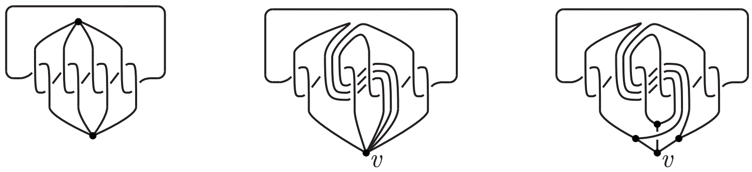



FIGURE 6. Modification in $N(L_i^M)$.

Next we perform a $P_2$-sum of $E(H_m^M)$ to obtain a hyperbolic link in $M$. Take a pair of disjoint annuli $A_i$ and $B_i$ on $\partial N(L_i^M)$ bounded by the loops originated from meridians of $L_i$ so that $D_{2k-1} \subset \text{int}A_i$ and $D_{2k} \subset \text{int}B_i$ for $1 \leq k \leq t$. Isotope $E(L_i^M)$ so as to move $A_i$ toward $B_i$ through $N(L_i^M)$ and glue them by a homeomorphism which takes $D_{2k-1}$ onto $D_{2k}$ for $1 \leq k \leq t$. The torus $\partial N(L_i^M)$ is deformed to two tori connected by the glued annulus $A_i = B_i$ (see Figure 6). Note that there is a cycle subgraph of $H_m^M$ of length one or two which meets the glued surface once. Then a $P_2$-sum of $E(H_m^M)$ gluing $A_i \cap \partial N(H_m^M)$ and $B_i \cap \partial N(H_m^M)$ is performed. Applying the similar argument to all $L_i^M$, we obtain a 3-manifold $E(L'_{\mu+m-n}) \subset M$ which is the exterior of a $(\mu+m-n)$-component link $L'_{\mu+m-n}$ in $M$. Note that $L'_{\mu+m-n}$ consists of $\mu$ components originated from $L$, $n-2$ components corresponding to the edges in $\mathcal{T}$, and $m-2n+2$ components originated from the loop edges of $H_m^M$. Theorem 1.2 implies that $E(L'_{\mu+m-n})$ is partially peripheral. Since the Euler characteristic of $A_i \cap \partial N(H_m^M)$ is negative, Proposition 2.2 implies that $E(L'_{\mu+m-n})$ is an irreducible atoroidal anannular 3-manifold with




(1) $H_4$ (2) Bouquet of 3 circles (3) $B_{1,3}$

FIGURE 7. Modification of $H_4$ to $B_{1,3}$.

incompressible boundary. Therefore, Thurston's hyperbolization theorem [10] implies that $E(L'_{\mu+m-n})$ admits a complete hyperbolic structure of finite volume in its interior. Consequently, we obtain an infinite sequence $L'_{\mu+n-1}, L'_{\mu+n}, \ldots$ of partially peripheral hyperbolic links in $M$, which are distinguished by the number of components. Hence, the conclusion follows. □

*Proof of Theorem 1.3.* Let $c \geq 1$ and $s \geq 3$ be integers. The spatial graph $H_{cs+1}$ can be deformed by sliding $cs-1$ edges along another edge to a bouquet of $cs$ circles centered at a vertex $v$. By a further deformation, we obtain a graph $B_{c,s}$ with $s+1$ vertices consisting of a star graph centered at $v$ with $s$ leaves $w_1, \ldots, w_s$ and $s$ copies of a bouquet of $c$ circles centered at each $w_i$. We say $v$ is the *central vertex* of $B_{c,s}$. For example, Figure 7 illustrates the case $(c,s) = (1,3)$.

It is enough by Proposition 3.1 to consider the case $\chi < 0$. Let $M$ be a closed connected orientable 3-manifold. Suppose that $M$ is obtained by $\pm 1$-Dehn surgery on a link $L$ in $S^3$, and that $\mathcal{T}$ is an unknotting tunnel system for $L$ consisting of $n-2 \geq 0$ unknotting tunnels. Let $s \geq \max\{3, -n/\chi\}$ be any integer. Then $1 - s\chi > n$. By the argument similar to that presented in the proof of Proposition 3.1, a hyperbolic spatial graph $H'_{1-s\chi}$ in $S^3$ of Euler characteristic $1+s\chi$ is obtained so as to contain $\Gamma = L \cup \mathcal{T}$ as a subgraph. By performing the Dehn surgery, we obtain a spatial graph $H^M_{1-s\chi}$ in $M$ isomorphic to $H'_{1-s\chi}$ whose exterior is homeomorphic to $E(H_{1-s\chi})$. Deform $H^M_{1-s\chi}$ by sliding some edges to a spatial $\theta_{1-s\chi}$-curve, and deform it further to a spatial graph $B^M_{-\chi,s}$ in $M$ as we deformed $H_{1-s\chi}$ to $B_{-\chi,s}$. Denote by $v$ the vertex of $B^M_{-\chi,s}$ corresponding to the central vertex of $B_{-\chi,s}$. The vertex connected sum of $B^M_{-\chi,s}$ and $B_{1,s}$ at $v$ and the central vertex of $B_{1,s}$ yields a graph $G_s$ in $M$ with $s$ components each of which has Euler characteristic $\chi$. Then the glued surface $F$ is a separating sphere which meets each component of $G_s$ in a point. Therefore, the vertex connected sum realizes the $P_1$-sum of $E(B^M_{-\chi,s})$ and $E(B_{1,s})$. Moreover, $s \geq 3$ implies that the Euler characteristic of $F \cap E(G_s)$ is negative. Therefore, it follows from Theorem 1.1, Proposition 2.2 and Thurston's hyperbolization theorem [10] that $G_s$ is hyperbolic. Consequently, we obtain an infinite sequence $G_{\max\{3,-n/\chi\}}, G_{\max\{3,-n/\chi\}+1}, \ldots$ of partially peripheral hyperbolic spatial graphs in $M$, which are distinguished by the number of components. Hence, the conclusion follows. □

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