http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 45, 2015

Pages 175–187

http://topology.nipissingu.ca/tp/

Topological entropy for automorphisms of totally disconnected locally compact groups

by

Anna Giordano Bruno

Electronically published on August 19, 2014

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 45 (2015)
Pages 175-187

E-Published on August 19, 2014

TOPOLOGICAL ENTROPY FOR AUTOMORPHISMS OF TOTALLY DISCONNECTED LOCALLY COMPACT GROUPS

ANNA GIORDANO BRUNO

Abstract. We give a "limit-free formula" simplifying the computation of the topological entropy for topological automorphisms of totally disconnected locally compact groups. This result allows us to extend to our setting several basic properties of the topological entropy known for compact groups.

1. Introduction

In this paper we consider the topological entropy for topological automorphisms of totally disconnected locally compact groups.

For a locally compact group G, let μ denote a left Haar measure on G. Moreover, let $\mathcal{B}(G)$ be the family of all open compact subgroups of G. A totally disconnected locally compact group G has $\mathcal{B}(G)$ as a local base at 1, as proved by van Dantzig in [14].

In [1] Adler, Konheim and McAndrew defined the topological entropy for continuous self-maps of compact spaces. Later on, in [3] Bowen introduced another notion of topological entropy for uniformly continuous self-maps of metric spaces. This definition was extended to uniformly continuous self-maps of uniform spaces by Hood in [11]. In particular, Hood's notion of topological entropy applies to continuous endomorphisms of totally disconnected locally compact groups; in this setting, as explained in [7], it can be introduced as follows.

2010 *Mathematics Subject Classification.* 37B40, 22D05, 22D40, 54H11, 54H20, 54C70.

Key words and phrases. Topological entropy, totally disconnected group, locally compact group.

Research supported by "Progetti di Eccellenza 2011/12" of Fondazione CARIPARO and by INdAM - Istituto Nazionale di Alta Matematica.

©2014 Topology Proceedings.

Let G be a totally disconnected locally compact group and $\phi : G \to G$ a continuous endomorphism. For every $U \in \mathcal{B}(G)$ and every positive integer n, the *n-th ϕ-cotrajectory* of U is

$$C_n(\phi, U) = U \cap \phi^{-1}(U) \cap \ldots \cap \phi^{-n+1}(U).$$

For every positive integer n, we have $C_n(\phi, U) \in \mathcal{B}(G)$; moreover, the index $[U : C_n(\phi, U)]$ is finite, as U is compact and $C_n(\phi, U)$ is open.

The *ϕ-cotrajectory* of U is

$$C(\phi, U) = \bigcap_{n=0}^{\infty} \phi^{-n}(U) = \bigcap_{n=1}^{\infty} C_n(\phi, U).$$

It is compact, but not open in general, and it is the greatest ϕ-invariant subgroup of G contained in U.

The *topological entropy of ϕ with respect to U* is

$$H_{top}(\phi, U) = \lim_{n\to\infty} \frac{\log[U : C_n(\phi, U)]}{n};$$

the limit in this definition is proved to exist in Lemma 2.2 below. The *topological entropy* of ϕ is

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{B}(G)\}.$$

In Section 2 we prove the main result of this paper, that is, the following limit-free formula for the computation of the topological entropy. Namely, if G is a totally disconnected locally compact group, $\phi : G \to G$ is a topological automorphism and $U \in \mathcal{B}(G)$, then

$$H_{top}(\phi, U) = \log[\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)]. \tag{1.1}$$

This result extends the limit-free formula proved in [6] in the compact case. That formula was inspired by its counterpart for the algebraic entropy, claimed by Yuzvinski in [18] and verified in [6]. Indeed, in the abelian case the algebraic entropy and the topological entropy are strictly related by Pontryagin duality (see [4, 5, 15]).

In the compact case the limit-free formula holds for continuous endomorphisms under some necessary assumptions. So, we leave the following open problem.

Problem 1.1. *Is it possible to extend the limit-free formula* (1.1) *to continuous endomorphisms of locally compact groups? Under which assumptions?*

An interesting application of the limit-free formula (1.1) is given in [2]. In fact, it allows for a comparison of the topological entropy with the scale function $s(-)$, a function introduced by Willis in [16, 17] for topological automorphisms $\phi : G \to G$ of totally disconnected locally compact groups G. By the so-called tidying procedure given in [17],

$$s(\phi) = \min\{[\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)] : U \in \mathcal{B}(G)\}.$$

Therefore, comparing this equality with (1.1), it is clear that $\log s(\phi) \leq h_{top}(\phi)$.

Furthermore, it is worth to mention that [2] contains a characterization of the topological automorphisms $\phi : G \to G$ of totally disconnected locally compact groups G for which the equality $\log s(\phi) = h_{top}(\phi)$ holds.

In Section 4, as consequences of the limit-free formula (1.1), we verify the basic properties of the topological entropy, namely, the so-called Invariance under conjugation, Monotonicity, Logarithmic Law, Continuity with respect to inverse limits and weak Addition Theorem. These properties are known for continuous endomorphisms of compact groups (see [1, 13]), and now we extend them to the case of topological automorphisms of totally disconnected locally compact groups.

The so-called Addition Theorem was proved for continuous endomorphisms $\phi : G \to G$ of compact groups in [13]. The Addition Theorem is a fundamental property of the topological entropy and states that, if N is a closed normal ϕ-invariant subgroup of G and $\overline{\phi} : G/N \to G/N$ is the continuous endomorphism induced by ϕ, then

$$h_{top}(\phi) = h_{top}(\phi \restriction_N) + h_{top}(\overline{\phi}).$$

We leave open the following problem, asking whether the Addition Theorem holds in the setting of this paper. One should start by studying the abelian case.

Problem 1.2. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism, N a closed normal subgroup of G such that $\phi(N) = N$, and $\overline{\phi} : G/N \to G/N$ the topological automorphism induced by ϕ. Is it true that $h_{top}(\phi) = h_{top}(\phi \restriction_N) + h_{top}(\overline{\phi})$?*

2. Limit-free computation of topological entropy

For a group G and a subgroup H of G, we denote by G/H the set of all left cosets of H in G. The index $[G : H]$ of H in G is the size of G/H. If K is another subgroup of G, then we denote by KH/H the family of all left cosets of H in G with representing elements in K, that is $KH/H = \{kH : k \in K\}$. Moreover, generalizing the usual notion of index, we write $[KH : H]$ for the size of KH/H. If $H \subseteq K$, then we write simply K/H and $[K : H]$, as usual.

We use several times the following easy-to-check properties.

Fact 2.1. *Let G be a group and let H, K, L be subgroups of G, with $H \subseteq K$. Then:*

(a) $[G : H] = [G : K][K : H]$;
(b) $[LH : H] = [L : H \cap L]$;
(c) $[K : H] \geq [K \cap L : H \cap L]$.

If N is a normal subgroup of G, $q : G \to G/N$ is the canonical projection and $N \subseteq H$, then

(d) $[K : H] = [q(K) : q(H)]$.

The following is the counterpart of [9, Lemma 1.1] and [8, Lemma 2.2] for the topological entropy. Its proof follows the one of [6, Lemma 3.1].

Lemma 2.2. *Let G be a locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. For every positive integer n, let $c_n = [U : C_n(\phi, U)]$. Then:*

(a) *c_n divides c_{n+1} for every $n > 0$.*

For every $n > 0$ let $\alpha_n = \frac{c_{n+1}}{c_n} = [C_n(\phi, U) : C_{n+1}(\phi, U)]$. Then:

(b) *$\alpha_{n+1} \leq \alpha_n$ for every $n > 0$;*
(c) *the sequence $\{\alpha_n\}_{n>0}$ stabilizes (i.e., there exist integers $n_0 > 0$ and $\alpha > 0$ such that $\alpha_n = \alpha$ for every $n \geq n_0$);*
(d) *$H_{top}(\phi, U) = \log \alpha$.*

Proof. Let $n > 0$. Since there is no possibility of confusion, in this proof we denote $C_n(\phi, U)$ simply by C_n.

(a) Since $U \supseteq C_n \supseteq C_{n+1}$, it follows from Fact 2.1(a) that $[U : C_{n+1}] = [U : C_n][C_n : C_{n+1}]$. So, $\frac{c_{n+1}}{c_n} = [C_n : C_{n+1}]$, and in particular c_n divides c_{n+1}.

(b) Since $C_{n+1} = C_n \cap \phi^{-1}(U)$, using Fact 2.1(b), we have

$$\alpha_n = [C_n : C_{n+1}] = [C_n : C_n \cap \phi^{-n}(U)] = [C_n \cdot \phi^{-n}(U) : \phi^{-n}(U)].$$

Now $C_n \subseteq \phi^{-1}(C_{n-1})$, so

$$[C_n \cdot \phi^{-n}(U) : \phi^{-n}(U)] \leq [\phi^{-1}(C_{n-1}) \cdot \phi^{-n}(U) : \phi^{-n}(U)].$$

The map $G/\phi^{-n}(U) \to G/\phi^{-n+1}(U)$ induced by ϕ is injective, therefore the family of cosets $(\phi^{-1}(C_{n-1}) \cdot \phi^{-n}(U))/\phi^{-n}(U)$ has the same size of its image $(C_{n-1} \cdot \phi^{-n+1}(U))/\phi^{-n+1}(U)$. Applying also Fact 2.1(b) and the fact that $C_n = C_{n-1} \cap \phi^{-n+1}(U)$, we have

$$\begin{aligned}[\phi^{-1}(C_{n-1}) \cdot \phi^{-n}(U) : \phi^{-n}(U)] &= [C_{n-1} \cdot \phi^{-n+1}(U) : \phi^{-n+1}(U)] \\ &= [C_{n-1} : C_{n-1} \cap \phi^{-n+1}(U)] \\ &= [C_{n-1} : C_n] = \alpha_{n-1}.\end{aligned}$$

This concludes the proof of (b).

(c) follows immediately from (b).

(d) By item (c), for $n_0 > 0$ we have $c_{n_0+n} = \alpha^n c_{n_0}$ for every $n \geq 0$. Then, by the definition of topological entropy,

$$H_{top}(\phi, U) = \lim_{n\to\infty} \frac{\log c_n}{n} = \lim_{n\to\infty} \frac{\log(\alpha^n c_{n_0})}{n} = \log \alpha.$$

This concludes the proof. □

Lemma 2.2 yields that the limit defining the topological entropy exists. It gives also a precise description of the value of the topological entropy $H_{top}(\phi, U)$. We can see in particular that this value is in $\log \mathbb{N}_+$, therefore $h_{top}(\phi) \in \log \mathbb{N}_+ \cup \{\infty\}$ for any totally disconnected locally compact group G and any topological automorphism $\phi : G \to G$. It is worth to add here that all the possible values in $\log \mathbb{N}_+ \cup \{\infty\}$ are realized as $h_{top}(\phi)$ for some topological automorphism ϕ of some totally disconnected compact group G (see [13]).

We are now in position to prove Theorem 2.4, that is, the main result of this paper, stated in (1.1) in the Introduction. The following folklore fact is needed in its proof.

Lemma 2.3. [6, Lemma 3.2] *Let G be a topological group and let T be a closed subset of G. Let $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \supseteq \ldots$ be a descending chain of closed subsets of G, where B_1 is compact. Then the intersection $B = \bigcap_{n=1}^{\infty} B_n$ is non-empty and $\bigcap_{n=1}^{\infty}(B_n T) = BT$.*

Theorem 2.4. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then*

$$H_{top}(\phi, U) = \log[\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)].$$

Proof. By Lemma 2.2 there exist integers $n_0 > 0$ and $\alpha > 0$ such that $\alpha_n = \alpha$ for every $n \geq n_0$, where $\alpha_n = [C_n(\phi, U) : C_{n+1}(\phi, U)]$ for every $n > 0$, and $H_{top}(\phi, U) = \log \alpha$. So, it suffices to prove that

$$[\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)] = \alpha. \tag{2.1}$$

Since $C(\phi^{-1}, U) = U \cap \phi(C(\phi^{-1}, U))$, by Fact 2.1(b), we have

$$\begin{aligned}[\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)] &= [\phi(C(\phi^{-1}, U)) : U \cap \phi(C(\phi^{-1}, U))] \\ &= [\phi(C(\phi, U)) \cdot U : U].\end{aligned} \tag{2.2}$$

Since both U and $\phi(U)$ are compact, $\phi(U) \cdot U$ is compact as well. Thus, as U is open, the family of cosets $(\phi(U) \cdot U)/U$ is finite. Consequently, the sequence $\{(\phi(C_n(\phi^{-1}, U)) \cdot U)/U\}_{n>0}$ is a descending chain of finite subsets of $(\phi(U) \cdot U)/U$ (note that $\phi(C_1(\phi^{-1}, U)) = \phi(U)$).

Hence, $\{(\phi(C_n(\phi^{-1},U))\cdot U)/U\}_{n>0}$ stabilizes, that is, there exists $n_1>0$ such that

$$\phi(C_n(\phi^{-1},U))\cdot U=\phi(C_{n_1}(\phi^{-1},U))\cdot U \text{ for every } n\geq n_1. \tag{2.3}$$

In other words,

$$\bigcap_{n=1}^{\infty}(\phi(C_n(\phi^{-1},U))\cdot U)=\phi(C_{n_1}(\phi^{-1},U))\cdot U. \tag{2.4}$$

Now Lemma 2.3 (with $B_1=\phi(U)$ compact, $B_n=C_n(\phi,U)$ for every $n>0$, and $T=U$) gives

$$\begin{aligned}\bigcap_{n=1}^{\infty}(\phi(C_n(\phi^{-1},U))\cdot U)&=\left(\bigcap_{n=1}^{\infty}\phi(C_n(\phi^{-1},U))\right)\cdot U\\&=\phi(C(\phi^{-1},U))\cdot U,\end{aligned} \tag{2.5}$$

where in the last equality we use that $\bigcap_{n=1}^{\infty}\phi(C_n(\phi^{-1},U))=\phi(C(\phi^{-1},U))$. Therefore, (2.3), (2.4) and (2.5) yield

$$\phi(C(\phi^{-1},U))\cdot U=\phi(C_{n_1}(\phi^{-1},U))\cdot U=\phi(C_n(\phi^{-1},U))\cdot U, \tag{2.6}$$

for every $n\geq n_1$.

Let $n\geq \max\{n_0,n_1\}$. Then (2.2) and (2.6), together with Fact 2.1(b), give

$$\begin{aligned}[\phi(C(\phi^{-1},U)):C(\phi^{-1},U)]&=[\phi(C(\phi^{-1},U))\cdot U:U]\\&=[\phi(C_n(\phi^{-1},U))\cdot U:U]\\&=[\phi(C_n(\phi^{-1},U)):\phi(C_n(\phi^{-1},U))\cap U]\\&=[\phi(C_n(\phi^{-1},U)):C_{n+1}(\phi^{-1},U)].\end{aligned} \tag{2.7}$$

Applying ϕ^{-n}, we obtain the equalities

$$\phi^{-n}(\phi(C_n(\phi^{-1},U)))=C_n(\phi,U) \text{ and } \phi^{-n}(C_{n+1}(\phi^{-1},U))=C_{n+1}(\phi,U).$$

Since ϕ^{-n} is an isomorphism, we derive that

$$[\phi(C_n(\phi^{-1},U)):C_{n+1}(\phi^{-1},U)]=[C_n(\phi,U):C_{n+1}(\phi,U)]=\alpha. \tag{2.8}$$

By (2.7) and (2.8), we can conclude that (2.1) holds. Hence, the thesis is proved. □

3. The topological entropy of the inverse automorphism

For a locally compact group G and a left Haar measure μ on G, the *modulus* is a group homomorphism $\Delta : \mathrm{Aut(G)} \to \mathbb{R}_+$ such that $\mu(\alpha E) = \Delta(\alpha)\mu(E)$ for every topological automorphism $\alpha \in \mathrm{Aut}(G)$ and every measurable subset E of G (see [10]). If G is compact, then $\Delta(\phi) = 1$ for every topological automorphism $\phi : G \to G$.

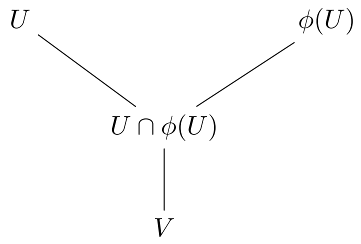
Lemma 3.1. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then*

(a) $\Delta(\phi) = \frac{\mu(\phi(U))}{\mu(U)} = \frac{[\phi(U):U\cap\phi(U)]}{[U:U\cap\phi(U)]}$.

(b) *If $V \in \mathcal{B}(G)$ and $V \subseteq U \cap \phi(U)$, then $[\phi(U) : V] = [U : V] \cdot \Delta(\phi)$.*

Proof. (a) Since U and $\phi(U)$ are compact, and $U \cap \phi(U)$ is open, it follows that $[\phi(U) : U \cap \phi(U)]$ and $[U : U \cap \phi(U)]$ are finite. Therefore, $\mu(\phi(U)) = [\phi(U) : U \cap \phi(U)] \cdot \mu(U \cap \phi(U))$ and $\mu(U) = [U : U \cap \phi(U)] \cdot \mu(U \cap \phi(U))$. Then apply the definition of modulus.

(b) The situation is described by the following diagram.



By Fact 2.1(a) we have

$$[U : V] = [U : U \cap \phi(U)][U \cap \phi(U) : V]$$

and

$$[\phi(U) : V] = [\phi(U) : U \cap \phi(U)][U \cap \phi(U) : V].$$

Therefore,

$$[\phi(U) : V] = [U : V]\frac{[\phi(U) : U \cap \phi(U)]}{[U : U \cap \phi(U)]}.$$

To conclude apply item (a). □

In the next proposition we compare the topological entropy of a topological automorphism ϕ and of its inverse ϕ^{-1}. These two values coincide precisely when the modulus of ϕ is 1. Indeed, it is known that in the compact case a topological automorphism has the same topological entropy as its inverse.

Proposition 3.2. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then*

$$H_{top}(\phi^{-1}, U) = H_{top}(\phi, U) - \log \Delta(\phi).$$

Hence,

$$h_{top}(\phi^{-1}) = h_{top}(\phi) - \log \Delta(\phi).$$

Proof. For every positive integer n let $c_n = [U : C_n(\phi, U)]$ and $c_n^* = [U : C_n(\phi^{-1}, U)]$. According to items (a)–(c) of Lemma 2.2 applied to ϕ and ϕ^{-1} respectively, $H_{top}(\phi, U) = \log \alpha$ and $H_{top}(\phi^{-1}, U) = \log \alpha^*$, where α is the value at which stabilizes the sequence $\alpha_n = \frac{c_{n+1}}{c_n}$ and α^* is the value at which stabilizes the sequence $\alpha_n^* = \frac{c_{n+1}^*}{c_n^*}$. For every $n > 0$, since $\phi^n(C_n(\phi, U)) = C_n(\phi^{-1}, U)$ and ϕ^n is an automorphism, by Lemma 3.1(b) we have that

$$\begin{aligned} c_n &= [U : C_n(\phi, U)] = [\phi^n(U) : \phi^n(C_n(\phi, U))] = \\ &= [\phi^n(U) : C_n(\phi^{-1}, U)) = [U : C_n(\phi^{-1}, U)] \cdot \Delta(\phi^n) = c_n^* \cdot \Delta(\phi^n). \end{aligned}$$

Therefore, since Δ is a homomorphism,

$$\alpha = \frac{c_{n+1}}{c_n} = \frac{c_{n+1}^*}{c_n^*} \cdot \frac{\Delta(\phi^{n+1})}{\Delta(\phi^n)} = \alpha^* \Delta(\phi).$$

Then, $\log \alpha = \log \alpha^* + \log \Delta(\phi)$. This gives $\log \alpha^* = \log \alpha - \log \Delta(\phi)$, that is, the first assertion of the proposition.

The second equality in the thesis follows from the first one taking the supremum for U ranging in $\mathcal{B}(G)$. □

The following is an immediate consequence of Theorem 2.4 and Proposition 3.2.

Corollary 3.3. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then*

$$H_{top}(\phi, U) = \log[\phi^{-1}(C(\phi, U)) : C(\phi, U)] + \log \Delta(\phi).$$

Proof. Theorem 2.4 applied to ϕ^{-1} gives

$$H_{top}(\phi^{-1}, U) = \log[\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)].$$

Moreover, Proposition 3.2 yields $H_{top}(\phi, U) = H_{top}(\phi^{-1}, U) + \log \Delta(\phi)$, hence the thesis follows. □

4. Properties of topological entropy

In this section we consider the basic properties of the topological entropy. When the group is compact it is known that they hold for continuous endomorphisms. Thanks to the limit-free formula proved in Theorem 2.4, it is possible to extend them to the case of topological automorphisms of totally disconnected locally compact groups.

We start proving that the function $H_{top}(\phi, -)$ is antimonotone. It is already known from [7].

Lemma 4.1. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a continuous endomorphism. For any $U, V \in \mathcal{B}(G)$, if $U \subseteq V$, then $H_{top}(\phi, V) \leq H_{top}(\phi, U)$.*

Proof. Since $U \subseteq V$, we have that $C_n(\phi, U) \subseteq C_n(\phi, V)$ for every $n > 0$. Then, by Fact 2.1(a), for every $n > 0$

$$[V : C_n(\phi, V)] \leq [V : C_n(\phi, U)] = [V : U][U : C_n(\phi, U)].$$

Applying the definition of topological entropy, that is, taking the logarithm, dividing by n and letting $n \to \infty$, we find that $H_{top}(\phi, V) \leq H_{top}(\phi, U)$. □

By the definition of topological entropy we immediately derive from Lemma 4.1 that, in order to compute the topological entropy of a continuous endomorphism, it suffices to take the supremum when U ranges in a local base at 1 of G:

Corollary 4.2. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a continuous endomorphism and $\mathcal{B} \subseteq \mathcal{B}(G)$ a local base at 1 of G. Then $h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{B}\}$.*

We start proving that the topological entropy is invariant under conjugation.

Proposition 4.3. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Let H be another totally disconnected locally compact group and $\xi : G \to H$ a topological isomorphism. Then $\xi(C(\phi, U)) = C(\xi\phi\xi^{-1}, \xi(U))$ for every $U \in \mathcal{B}(G)$, and so $h_{top}(\phi) = h_{top}(\xi\phi\xi^{-1})$.*

Proof. Let $\psi = \xi\phi\xi^{-1}$. It is clear that, for $U \in \mathcal{B}(G)$,

$$\xi(C(\phi, U)) = C(\psi, \xi(U)). \tag{4.1}$$

Since $\xi : G \to H$ is a topological isomorphism, it induces a bijection $\mathcal{B}(G) \to \mathcal{B}(H)$. So, by Theorem 2.4 it suffices to check that

$$[\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)] = [\psi(C(\psi^{-1}, \xi(U))) : C(\psi^{-1}, \xi(U))]. \tag{4.2}$$

From (4.1) applied to ϕ^{-1} and ψ^{-1}, we have $C(\psi^{-1},\xi(U))=\xi(C(\phi^{-1},U))$. Using also the fact that ξ is an isomorphism, we obtain that

$$[\psi(C(\psi^{-1},\xi(U))) : C(\psi^{-1},\xi(U))] = [\psi(\xi(C(\phi^{-1},U))) : \xi(C(\phi^{-1},U))] = \\ = [\xi(\phi(C(\phi^{-1},U))) : \xi(C(\phi^{-1},U))] = [\phi(C(\phi^{-1},U)) : C(\phi^{-1},U)].$$

Then (4.2) is verified and this concludes the proof. □

In the next proposition we prove the monotonicity of the topological entropy under taking subgroups and quotients.

Proposition 4.4. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and N a closed normal subgroup of G such that $\phi(N) = N$. Let $q : G \to G/N$ be the canonical projection and $\overline{\phi} : G/N \to G/N$ the topological automorphism induced by ϕ. Then:*

(a) *$C(\phi\restriction_N, U\cap N) = C(\phi,U)\cap N$ for every $U \in \mathcal{B}(G)$, and $h_{top}(\phi) \geq h_{top}(\phi\restriction_N)$;*
(b) *$C(\overline{\phi}, q(U)) = q(C(\phi,U))$ for every $U \in \mathcal{B}(G)$ such that $N \subseteq U$, and $h_{top}(\phi) \geq h_{top}(\overline{\phi})$.*

Proof. (a) The first assertion is clear. Let $V \in \mathcal{B}(N)$. Then $V = U \cap N$ for some $U \in \mathcal{B}(G)$. We verify that

$$H_{top}(\phi\restriction_N, V) \leq H_{top}(\phi, U),$$

which implies the wanted inequality. By the first assertion in item (a), $C((\phi\restriction_N)^{-1}, V) = C(\phi^{-1}, U) \cap N$, so Fact 2.1(c) gives

$$[\phi\restriction_N (C((\phi\restriction_N)^{-1}, V)) : C((\phi\restriction_N)^{-1}, V)] = \\ [\phi(C(\phi^{-1},U)\cap N) : C(\phi^{-1},U)\cap N] \leq [\phi(C(\phi^{-1},U)) : C(\phi^{-1},U)].$$

Taking the logarithms, this inequality implies $H_{top}(\phi\restriction_N, V) \leq H_{top}(\phi, U)$, by Theorem 2.4.

(b) The proof of the first assertion is straightforward. Let $q(U) \in \mathcal{B}(G/N)$, where $U \in \mathcal{B}(G)$; we can assume without loss of generality that $N \subseteq U$. Since $N \subseteq C(\phi^{-1}, U)$, by the first assertion in item (b) and Fact 2.1(d), we have

$$[\overline{\phi}(C(\overline{\phi}^{-1}, q(U))) : C(\overline{\phi}^{-1}, q(U))] = [\overline{\phi}(q(C(\phi^{-1},U))) : q(C(\phi^{-1},U))] = \\ = [q(\phi(C(\phi^{-1},U))) : q(C(\phi^{-1},U))] = [\phi(C(\phi^{-1},U)) : C(\phi^{-1},U)].$$

Theorem 2.4 applies to this equality and gives $H_{top}(\overline{\phi}, q(U)) = H_{top}(\phi, U)$. Hence, $h_{top}(\overline{\phi}) \leq h_{top}(\phi)$. □

We go on proving the so-called Logarithmic Law for the topological entropy. Note that in the compact case $h_{top}(\phi) = h_{top}(\phi^{-1})$ and so the Logarithmic Law extends to all integers. This is not possible in general in the locally compact case, as follows immediately from Proposition 3.2, where the modulus of ϕ is involved.

Proposition 4.5. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $k > 0$. Then $h_{top}(\phi^k) = k \cdot h_{top}(\phi)$.*

Proof. Let $U \in \mathcal{B}(G)$ and denote $C = C(\phi^{-1}, U)$. We start noting that

$$C = C(\phi^{-k}, V), \text{ where } V = C_k(\phi^{-1}, U).$$

Moreover, $\phi^{n+1}(C) \supseteq \phi^n(C)$ and $[\phi^{n+1}(C) : \phi^n(C)] = [\phi(C) : C]$ for every $n \geq 0$, since ϕ^{-n} is an automorphism. Then Fact 2.1(a) gives

$$\begin{aligned}[\phi^k(C) : C] &= [\phi^k(C) : \phi^{k-1}(C)] \cdot [\phi^{k-1}(C) : \phi^{k-2}(C)] \cdot \ldots \cdot [\phi(C) : C] \\ &= [\phi(C) : C]^k.\end{aligned}$$

Therefore,

$$[\phi^k(C(\phi^{-k}, V)) : C(\phi^{-k}, V)] = [\phi^k(C) : C] = [\phi(C) : C]^k;$$

so, Theorem 2.4 implies

$$H_{top}(\phi^k, V) = k \cdot H_{top}(\phi, U).$$

This immediately gives $h_{top}(\phi^k) \geq k \cdot h_{top}(\phi)$. By Lemma 4.1, since $V \subseteq U$, we have also

$$H_{top}(\phi^k, U) \leq H_{top}(\phi^k, V) = k \cdot H_{top}(\phi, U).$$

Thus, the missing converse inequality $h_{top}(\phi^k) \leq k \cdot h_{top}(\phi)$ holds as well, and this concludes the proof. □

We verify now the continuity of the topological entropy with respect to inverse limits.

Proposition 4.6. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Assume that $\{N_i : i \in I\}$ is a directed system of closed normal subgroups of G with $\phi(N_i) = N_i$ and $\bigcap_{i \in I} N_i = \{1\}$, and let $\overline{\phi}_i : G/N_i \to G/N_i$ be the continuous endomorphism induced by ϕ for every $i \in I$. Then $G \cong \varprojlim G/N_i$ and $h_{top}(\phi) = \sup_{i \in I} h_{top}(\overline{\phi}_i)$.*

Proof. By Proposition 4.4(b) the inequality $h_{top}(\phi) \geq \sup_{i \in I} h_{top}(\overline{\phi}_i)$ holds. So let $U \in \mathcal{B}(G)$. There exists $i \in I$ such that $N_i \subseteq U$. Let $q_i : G \to G/N_i$ be the canonical projection. Since $N_i \subseteq C(\phi^{-1}, U)$, by Proposition 4.4(b) and Fact 2.1(d) we have

$$\begin{aligned}[\overline{\phi}_i(C(\overline{\phi}_i^{-1}, q_i(U))) : C(\overline{\phi}_i^{-1}, q_i(U))] &= [\overline{\phi}_i(q_i(C(\phi^{-1}, U))) : q_i(C(\phi^{-1}, U))] \\ &= [q_i(\phi(C(\phi^{-1}, U))) : q_i(C(\phi^{-1}, U))] = [\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)].\end{aligned}$$

Theorem 2.4 applies to this equality and gives $H_{top}(\overline{\phi}_i, q_i(U)) = H_{top}(\phi, U)$. Therefore, $H_{top}(\phi, U) \leq h_{top}(\overline{\phi}_i)$, and hence $h_{top}(\phi) \leq \sup_{i\in I} h_{top}(\overline{\phi}_i)$. □

The following property is the weak Addition Theorem for the topological entropy. It is a particular case of the Addition Theorem (see Problem 1.2).

Theorem 4.7. *Let G and H be totally disconnected locally compact groups, $\phi : G \to G$ and $\psi : H \to H$ topological automorphisms. Then*

$$h_{top}(\phi \times \psi) = h_{top}(\phi) + h_{top}(\psi).$$

Proof. Let $\eta = \phi \times \psi$. The family $\mathcal{B} = \mathcal{B}(G) \times \mathcal{B}(H)$ is a local base at 1 of $G \times H$. Then, in view of Corollary 4.2, it suffices to take $U \in \mathcal{B}(G)$ and $V \in \mathcal{B}(H)$, and to verify that

$$H_{top}(\eta, U \times V) = H_{top}(\phi, U) + H_{top}(\psi, V).$$

This equality holds true, since $C(\eta^{-1}, U \times V) = C(\phi^{-1}, U) \times C(\psi^{-1}, V)$, and so

$$[\eta(C(\eta^{-1}, U \times V)) : C(\eta^{-1}, U \times V)] = [\phi(C(\phi^{-1}, U)) : C(\phi^{-1}, U)] \cdot [\psi(C(\psi^{-1}, V)) : C(\psi^{-1}, V)],$$

Now, Theorem 2.4 concludes the proof. □

References

[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.

[2] F. Berlai, D. Dikranjan, A. Giordano Bruno, *Scale function vs Topological entropy*, Topology Appl. **160** (18) (2013), 2314–2334.

[3] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414.

[4] D. Dikranjan, A. Giordano Bruno, *The Bridge Theorem for totally disconnected LCA groups*, Topology Appl. **169** (2014), 21–32.

[5] ———, *The connection between topological and algebraic entropy*, Topology Appl. **159** (13) (2012), 2980–2989.

[6] ———, *Limit free computation of entropy*, Rend. Istit. Mat. Univ. Trieste **44** (2012), 297–312.

[7] ———, *Topological entropy and algebraic entropy for group endomorphisms*, Proceedings ICTA2011 Islamabad, Pakistan July 4-10 2011 Cambridge Scientific Publishers (2012), 133–214.

[8] D. Dikranjan, A. Giordano Bruno, L. Salce, *Adjoint algebraic entropy*, J. Algebra **324** (2010), 442–463.

[9] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, *Algebraic entropy of endomorphisms of abelian groups*, Trans. Amer. Math. Soc. **361** (2009), 3401–3434.

[10] E. Hewitt, K. A. Ross, *Abstract harmonic analysis I*, Springer-Verlag, Berlin-Heidelberg-New York, 1963.

[11] B. M. Hood, *Topological entropy and uniform spaces*, J. London Math. Soc. **8** (2) (1974), 633–641.

[12] L. S. Pontryagin, *Topological Groups*, Gordon and Breach, New York, 1966.

[13] L. N. Stoyanov, *Uniqueness of topological entropy for endomorphisms on compact groups*, Boll. Un. Mat. Ital. B (7) **1** (3) (1987), 829–847.

[14] D. van Dantzig, *Studien over topologische Algebra*, Dissertation, Amsterdam 1931.

[15] M. D. Weiss, *Algebraic and other entropies of group endomorphisms*, Math. Systems Theory **8** (3) (1974/75), 243–248.

[16] G. A. Willis, *The structure of totally disconnected locally compact groups*, Math. Ann. **300** (2) (1994), 341–363.

[17] ______, *Further properties of the scale function on a totally disconnected group*, J. Algebra **237** (1) (2001), 142–164.

[18] S. Yuzvinski, *Metric properties of endomorphisms of compact groups*, Izv. Acad. Nauk SSSR, Ser. Mat. **29** (1965), 1295–1328 (in Russian). English Translation: Amer. Math. Soc. Transl. (2) **66** (1968), 63–98.

Dipartimento di Matematica e Informatica; Università di Udine; via delle Scienze, 206 - Udine, Italy

E-mail address: `anna.giordanobruno@uniud.it`