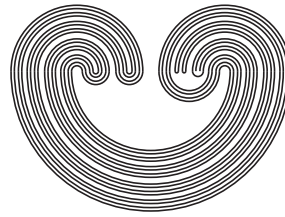


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## ON GRAEV TYPE ULTRA-METRICS

by

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## ON GRAEV TYPE ULTRA-METRICS

MENACHEM SHLOSSBERG

**ABSTRACT.** We study Graev ultra-metrics which were introduced by Gao [3]. We show that the free non-archimedean balanced topological group defined over an ultra-metric space is metrizable by a Graev ultra-metric. We prove that the Graev ultra-metric has a maximal property. Using this property, among others, we show that the Graev ultra-metric associated with an ultra-metric space  $(X, d)$  with diameter  $\leq 1$  coincides with the ultra-metric  $\hat{d}$  of Savchenko and Zarichnyi [12].

### 1. INTRODUCTION AND PRELIMINARIES

A uniform space is *non-archimedean* if it has a base of equivalence relations. A metric  $d$  is called *ultra-metric* if it satisfies the strong triangle inequality. Clearly, the metric uniformity of every ultra-metric space  $(X, d)$  is non-archimedean. By Graev's Extension Theorem (see [4]), for every metric  $d$  on  $X \cup \{e\}$  there exists a metric  $\delta$  on the free group  $F(X)$  with the following properties:

- (1)  $\delta$  extends  $d$ .
- (2)  $\delta$  is a two sided invariant metric on  $F(X)$ .
- (3)  $\delta$  is maximal among all invariant metrics on  $F(X)$  extending  $d$ .

Gao [3] has recently presented the notion of *Graev ultra-metric*, a natural ultra-metric modification to Graev's classical construction. We study this relatively new concept, after reviewing it in this section. In Section 2 we show that Graev ultra-metrics satisfy a maximal property (Theorem 2.2). Recall that according to [5] any continuous map from a Tychonoff

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space  $X$  to a topological group  $G$  can be uniquely extended to a continuous homomorphism from the (Markov) free topological group  $F(X)$  into  $G$ . Free topological groups were studied by researchers in different contexts. See for example, [1, 14, 10, 15, 13, 6, 9, 11, 8]. In Section 3 we show that the uniform free non-archimedean balanced topological group defined over an ultra-metric space is metrizable by a Graev ultra-metric (Theorem 3.6). In Section 4 we compare between seemingly different ultra-metrics that are defined on the free group  $F(X)$  (Theorem 4.6). We start with relevant notations and definitions from [3]. Considering a nonempty set  $X$  we define  $\overline{X} = X \cup X^{-1} \cup \{e\}$  where  $X^{-1} = \{x^{-1} : x \in X\}$  is a disjoint copy of  $X$  and  $e \notin X \cup X^{-1}$ . We agree that  $(x^{-1})^{-1} = x$  for every  $x \in X$  and also that  $e^{-1} = e$ . Let  $W(X)$  be the set of words over the alphabet  $\overline{X}$ .

We call a word  $w \in W(X)$  *irreducible* if either one of the following conditions holds:

- $w = e$
- $w = x_0 \cdots x_n$  does not contain the letter  $e$  or a sequence of two adjacent letters of the form  $xx^{-1}$  where  $x \in X \cup X^{-1}$ .

The length of a word  $w$  is denoted by  $lh(w)$ .  $w'$  is the *reduced word* for  $w \in W(X)$ . It is the irreducible word obtained from  $w$  by applying repeatedly the following algorithm: replace any appearance of  $xx^{-1}$  by  $e$  and eliminate  $e$  from any occurrence of the form  $w_1ew_2$ , where at least one of  $w_1$  and  $w_2$  is nonempty. A word  $w \in W(X)$  is *trivial* if  $w' = e$ . Now, as a set the free group  $F(X)$  is simply the collection of all irreducible words. The group operation is concatenation of words followed by word reduction. Note that the identity element of  $F(X)$  is  $e$  and not the empty word.

**Definition 1.1.** [3, Definition 2.1] Let  $d$  be an ultra-metric on  $\overline{X}$  for which the following conditions hold for every  $x, y \in X$ :

- (1)  $d(x^{-1}, y^{-1}) = d(x, y)$ .
- (2)  $d(x, e) = d(x^{-1}, e)$ .
- (3)  $d(x^{-1}, y) = d(x, y^{-1})$ .

For  $w = x_0 \cdots x_n, v = y_0 \cdots y_n \in W(X)$  put

$$\rho_u(w, v) = \max\{d(x_i, y_i) : 0 \leq i \leq n\}.$$

The *Graev ultra-metric*  $\delta_u$  on  $F(X)$  is defined as follows:

$\delta_u(w, v) = \inf\{\rho_u(w^*, v^*) : w^*, v^* \in W(X), lh(w^*) = lh(v^*), (w^*)' = w, (v^*)' = v\}$ ,  
for every  $w, v \in F(X)$ .

The following concepts have a lot of significance in studying Graev ultra-metrics.

**Definition 1.2.** [2, 3] Let  $m, n \in \mathbb{N}$  and  $m \leq n$ . A bijection  $\theta$  on  $\{m, \dots, n\}$  is a *match* if

- (1)  $\theta \circ \theta = id$  and
- (2) there are no  $m \leq i, j \leq n$  such that  $i < j < \theta(i) < \theta(j)$ .

For any match  $\theta$  on  $\{0, \dots, n\}$  and  $w = x_0 \cdots x_n \in W(X)$  define

$$x_i^\theta = \begin{cases} x_i, & \text{if } \theta(i) > i \\ e, & \text{if } \theta(i) = i \\ x_{\theta(i)}^{-1}, & \text{if } \theta(i) < i \end{cases}$$

and  $w^\theta = x_0^\theta \cdots x_n^\theta$ .

**Theorem 1.3.** (1) [3, Theorem 2.3] *For any*

$$w \in F(X), \delta_u(w, e) = \min\{\rho_u(w, w^\theta) : \theta \text{ is a match}\}.$$

- (2) [3, Theorem 2.4] *Let  $(X, d)$  be an ultra-metric space. Then the Graev ultra-metric  $\delta_u$  is a two-sided invariant ultra-metric on  $F(X)$  extending  $d$ . Furthermore,  $F(X)$  is a topological group in the topology induced by  $\delta_u$ . If  $X$  is separable, so is  $F(X)$ .*

## 2. A MAXIMAL PROPERTY OF GRAEV ULTRA-METRICS

Recall that given a metric  $d$  on  $X \cup \{e\}$ , its associated Graev metric is the maximal among all invariant metrics on  $F(X)$  extending  $d$ . This fact leads for a natural question:

*Question 2.1.* Is Graev ultra-metric maximal in any sense?

The following theorem provides a positive answer.

**Theorem 2.2.** *Let  $d$  be an ultra-metric on  $\overline{X}$  for which the following conditions hold for every  $x, y \in X$ :*

- (1)  $d(x^{-1}, y^{-1}) = d(x, y)$ .
- (2)  $d(x, e) = d(x^{-1}, e)$ .
- (3)  $d(x^{-1}, y) = d(x, y^{-1})$ .

*Then:*

- (a) *The Graev ultra-metric  $\delta_u$  is maximal among all invariant ultra-metrics on  $F(X)$  that extend the metric  $d$  defined on  $\overline{X}$ .*
- (b) *If in addition  $d(x^{-1}, y) = d(x, y^{-1}) = \max\{d(x, e), d(y, e)\}$  then  $\delta_u$  is maximal among all invariant ultra-metrics on  $F(X)$  that extend the metric  $d$  defined on  $X \cup \{e\}$ .*

*Proof.* We prove (a) using the following claim.

**Claim 1:** Let  $R$  be an invariant ultra-metric on  $F(X)$  that extends the metric  $d$  defined on  $\overline{X}$  and  $w = x_0 \cdots x_n \in F(X)$ . Then for every match  $\theta$  on  $\{0, \dots, lh(w) - 1\}$  we have

$$\rho_u(w, w^\theta) \geq R(w, e).$$

*Proof.* We prove the claim by induction on  $lh(w)$ . If  $lh(w) = 1$  then the only match is the identity. In this case by definition  $w^\theta = e$  and also  $w \in \overline{X}$  so

$$\rho_u(w, w^\theta) = \rho_u(w, e) = d(w, e) = R(w, e).$$

If  $lh(w) = 2$  then  $w = x_0x_1$  where  $x_0, x_1 \in \overline{X}$  and there are only two matches to consider: the identity map and a transposition.

If  $\theta = Id$  then

$$\begin{aligned} \rho_u(w, w^\theta) &= \max\{d(x_0, e), (x_1, e)\} = \\ &= \max\{R(x_0, e), R(x_1, e)\} = \max\{R(x_0x_1, x_1), R(x_1, e)\} \geq R(w, e). \end{aligned}$$

If  $\theta$  is a transposition we have

$$\rho_u(w, w^\theta) = d(x_1, x_0^{-1}) = R(x_1, x_0^{-1}) = R(w, e).$$

We can now assume that  $lh(w) \geq 3$  and also that the assertion is true for every word  $t$  with  $lh(t) < lh(w)$ . Let  $\theta$  be a match on  $\{0, \dots, lh(w) - 1\}$  (where  $w = x_0 \cdots x_n$  and  $lh(w) = n + 1$ ).

First case:  $\theta(0) \neq n$ . In this case there exists  $j \geq 1$  such that  $\theta(j) = n$ . For every  $j \leq i \leq n$  we have  $j \leq \theta(i) \leq n$ . Indeed, otherwise  $j > \theta(i)$ . Now,  $\theta(j) = n$ ,  $\theta(n) = j$  so we conclude that  $i \neq j$  and  $i \neq n$ . Therefore,  $\theta(i) < j < i < n$  and we obtain that

$$\theta(i) < j < i < \theta(j),$$

contradicting the definition of a match. This implies that  $\theta$  induces two matches:  $\theta_1$  on  $\{0, \dots, j-1\}$  and  $\theta_2$  on  $\{j, \dots, n\}$ . Let  $g_1 = x_0 \cdots x_{j-1}$ ,  $g_2 = x_j \cdots x_n$ .

Clearly  $w = g_1g_2$  and using the induction hypothesis we obtain that

$$\begin{aligned} \rho_u(w, w^\theta) &= \max\{\rho_u(g_1, g_1^{\theta_1}), \rho_u(g_2, g_2^{\theta_2})\} \geq \max\{R(g_1, e), R(g_2, e)\} = \\ &= \max\{R(g_1g_2, g_2), R(g_2, e)\} \geq R(g_1g_2, e) = R(w, e). \end{aligned}$$

Second case:  $\theta(0) = n$  where  $n \geq 2$ . Then,

$$\begin{aligned} R(x_0 \cdots x_n, e) &= R(x_1 \cdots x_{n-1}, x_0^{-1}x_n^{-1}) \leq \\ &\leq \max\{R(x_1 \cdots x_{n-1}, e), R(x_0^{-1}x_n^{-1}, e)\} = \\ &= \max\{R(x_0x_n, e), R(x_1 \cdots x_{n-1}, e)\}. \end{aligned}$$

Letting  $g_1 = x_0x_n$ ,  $g_2 = x_1 \cdots x_{n-1}$ , we have

$$R(w, e) \leq \max\{R(g_1, e), R(g_2, e)\}.$$

Now,  $\theta$  induces two matches on  $\{0, n\}$  and on  $\{1, \dots, n-1\}$  which we denote by  $\theta_1, \theta_2$  respectively. From the inductive step and also from the fact that the assertion is true for words of length 2 we have:  $R(g_1, e) = R(x_1 x_n, e) \leq \rho_u(g_1, g_1^{\theta_1})$  and also  $R(g_2, e) \leq \rho_u(g_2, g_2^{\theta_2})$ . On the one hand,

$$\begin{aligned} \rho_u(w, w^\theta) &= \max\{\rho_u(x_0, x_0^\theta), \rho_u(x_n, x_n^\theta), \rho_u(g_2, g_2^{\theta_2})\} = \\ &= \max\{\rho_u(x_0, x_0), \rho_u(x_n, x_0^{-1}), \rho_u(g_2, g_2^{\theta_2})\}. \end{aligned}$$

On the other hand,  $\rho_u(g_1, g_1^{\theta_1}) = \max\{\rho_u(x_0, x_0), \rho_u(x_n, x_0^{-1})\}$ .

Hence,

$$\begin{aligned} \rho_u(w, w^\theta) &= \max\{\rho_u(g_i, g_i^{\theta_i}) : 1 \leq i \leq 2\} \geq \\ &\geq \max\{R(g_1, e), R(g_2, e)\} \geq R(w, e). \end{aligned}$$

□

To prove (a) let  $R$  be an invariant ultra-metric on  $F(X)$  which extends the metric  $d$  defined on  $\overline{X}$ . By the invariance of both  $\delta_u$  and  $R$  it suffices to show that  $\delta_u(w, e) \geq R(w, e) \forall w \in F(X)$ . The proof now follows from Theorem 1.3 (1) and Claim 1. The proof of (b) is quite similar. It follows from the obvious analogue of Claim 1. We mention few necessary changes and observations in the proof. Note that this time  $R$  is an invariant ultra-metric on  $F(X)$  which extends the metric  $d$  defined on  $X \cup \{e\}$ . We have  $d(x, e) = R(x, e) \forall x \in \overline{X}$ . This is due to the invariance of  $R$  and the equality  $d(x, e) = d(x^{-1}, e)$ . This allows us to use the same arguments, as in the proof of Claim 1, to prove the cases  $lh(w) = 1$  and  $lh(w) = 2$  where  $\theta = id$ . For the case  $lh(w) = 2$  where  $\theta$  is a transposition note that we do not necessarily have  $d(x_1, x_0^{-1}) = R(x_1, x_0^{-1})$ . However, by the additional assumption we do have

$$d(x_1, x_0^{-1}) \geq R(x_1, x_0^{-1}).$$

Indeed,

$$\begin{aligned} d(x_1, x_0^{-1}) &= \max\{d(x_1, e), d(x_0, e)\} \\ &= \max\{R(x_1, e), R(x_0, e)\} = \max\{R(x_1, e), R(x_0^{-1}, e)\} \\ &\geq R(x_1, x_0^{-1}). \end{aligned}$$

So, the assertion is true for  $lh(w) = 2$ . The inductive step is left unchanged. □

### 3. UNIFORM FREE NON-ARCHIMEDEAN BALANCED GROUPS

**Definition 3.1.** A topological group is:

- (1) *non-archimedean* if it has a base at the identity consisting of open subgroups.
- (2) *balanced* if its left and right uniformities coincide.

In [7] we proved that the free non-archimedean balanced group of an ultra-metrizable uniform space is metrizable. Moreover, we claimed that this group is metrizable by a Graev type ultra-metric. In this section we prove the last assertion in full details (see Theorem 3.6). For the reader's convenience we review the definition of this topological group and some of its properties (see [7] for more details). For a topological group  $G$  denote by  $N_e(G)$  the set of all neighborhoods at the identity element  $e$ .

**Definition 3.2.** Let  $(X, \mathcal{U})$  be a non-archimedean uniform space. The *uniform free non-archimedean balanced topological group of  $(X, \mathcal{U})$*  is denoted by  $F_{\mathcal{N}, \mathcal{A}}^b$  and defined as follows:  $F_{\mathcal{N}, \mathcal{A}}^b$  is a non-archimedean balanced topological group for which there exists a universal uniform map  $i : X \rightarrow F_{\mathcal{N}, \mathcal{A}}^b$  satisfying the following universal property. For every uniformly continuous map  $\varphi : (X, \mathcal{U}) \rightarrow G$  into a balanced non-archimedean topological group  $G$  there exists a unique continuous homomorphism  $\Phi : F_{\mathcal{N}, \mathcal{A}}^b \rightarrow G$  for which the following diagram commutes:

$$\begin{array}{ccc} (X, \mathcal{U}) & \xrightarrow{i} & F_{\mathcal{N}, \mathcal{A}}^b \\ & \searrow \varphi & \downarrow \Phi \\ & & G \end{array}$$

Let  $(X, \mathcal{U})$  be a non-archimedean uniform space,  $Eq(\mathcal{U})$  be the set of equivalence relations from  $\mathcal{U}$ . Define two functions from  $X^2$  to  $F(X) : j_2$  is the mapping  $(x, y) \mapsto x^{-1}y$  and  $j_2^*$  is the mapping  $(x, y) \mapsto xy^{-1}$ .

**Definition 3.3.** [7, Definition 4.9]

- (1) Following [10], for every  $\psi \in \mathcal{U}^{F(X)}$  let

$$V_\psi := \bigcup_{w \in F(X)} w(j_2(\psi(w)) \cup j_2^*(\psi(w)))w^{-1}.$$

- (2) As a particular case in which every  $\psi$  is a constant function we obtain the set

$$\tilde{\varepsilon} := \bigcup_{w \in F(X)} w(j_2(\varepsilon) \cup j_2^*(\varepsilon))w^{-1}.$$

*Remark 3.4.* [7, Remark 4.10] Note that if  $\varepsilon \in Eq(\mathcal{U})$  then  $(j_2(\varepsilon))^{-1} = j_2(\varepsilon)$ ,  $(j_2^*(\varepsilon))^{-1} = j_2^*(\varepsilon)$  and

$$\tilde{\varepsilon} = \bigcup_{w \in F(X)} w(j_2(\varepsilon) \cup j_2^*(\varepsilon))w^{-1} = \bigcup_{w \in F(X)} wj_2(\varepsilon)w^{-1}.$$

Indeed, this follows from the equality  $wts^{-1}w^{-1} = (ws)s^{-1}t(ws)^{-1}$ . Note also that the subgroup  $[\tilde{\varepsilon}]$  generated by  $\varepsilon$  is normal in  $F(X)$ .

**Theorem 3.5.** [7, Theorem 4.13.2] *Let  $(X, \mathcal{U})$  be non-archimedean and let  $\mathcal{B} \subseteq \text{Eq}(\mathcal{U})$  be a base of  $\mathcal{U}$ .*

*Then:*

- (1) *the family (of normal subgroups)  $\{[\tilde{\varepsilon}] : \varepsilon \in \mathcal{B}\}$  is a base of  $N_e(F_{\mathcal{N}\mathcal{A}}^b)$ .*
- (2) *the topology of  $F_{\mathcal{N}\mathcal{A}}^b$  is the weak topology generated by the system of homomorphisms  $\{\overline{f_\varepsilon} : F(X) \rightarrow F(X/\varepsilon)\}_{\varepsilon \in \mathcal{B}}$  on discrete groups  $F(X/\varepsilon)$ .*

It follows from Theorem 3.5 that  $F_{\mathcal{N}\mathcal{A}}^b$  is metrizable if the uniform space  $(X, \mathcal{U})$  is metrizable. In fact, in this case  $F_{\mathcal{N}\mathcal{A}}^b$  is metrizable by a Graev type ultra-metric as the following theorem suggests.

**Theorem 3.6.** *Let  $(X, d)$  be an ultra-metric space.*

- (1) *Fix  $x_0 \in X$  and extend the definition of  $d$  from  $X$  to  $X' := X \cup \{e\}$  by letting  $d(x, e) = \max\{d(x, x_0), 1\}$ . Next, extend it to  $\overline{X} := X \cup X^{-1} \cup \{e\}$  by defining for every  $x, y \in X \cup \{e\}$  :*
  - (a)  $d(x^{-1}, y^{-1}) = d(x, y)$
  - (b)  $d(x^{-1}, y) = d(x, y^{-1}) = \max\{d(x, e), d(y, e)\}$

*Then for  $r < 1$  we have  $B_{\delta_u}(e, r) = [\tilde{\mathcal{R}}]$  where  $\delta_u$  is the Graev ultra-metric associated with  $d$  and*

$$\mathcal{R} := \{(x, y) \in X \times X : d(x, y) < r\}.$$

- (2)  $F_{\mathcal{N}\mathcal{A}}^b(X, d)$  *is metrizable by the Graev ultra-metric associated with  $(X, d)$ .*

*Proof.* (1) : We first show that  $[\tilde{\mathcal{R}}] \subseteq B_{\delta_u}(e, r)$ . Since the open ball  $B_{\delta_u}(e, r)$  is a normal subgroup of  $F(X)$  it suffices to show by Remark 3.4 that  $j_2(\mathcal{R}) \subseteq B_{\delta_u}(e, r)$ . Assuming that  $d(x, y) < r$  we have  $\delta_u(x^{-1}y, e) = \delta_u(x, y) = d(x, y) < r$ . This implies that  $x^{-1}y \in B_{\delta_u}(e, r)$  and therefore  $j_2(\mathcal{R}) \subseteq B_{\delta_u}(e, r)$ .

We now show that  $B_{\delta_u}(e, r) \subseteq [\tilde{\mathcal{R}}]$ . Let  $e \neq w \in B_{\delta_u}(e, r)$ , then by the definition of  $\delta_u$  there exist words

$$w^* = x_0 \cdots x_n, \quad v = y_0 \cdots y_n \in W(X)$$

such that  $w = (w^*)', v' = e$  and  $d(x_i, y_i) < r \forall i$ . We prove using induction on  $lh(w^*) = lh(v)$ , that  $w \in [\tilde{\mathcal{R}}]$ . For  $lh(w^*) = 1$  the assertion holds trivially. For  $lh(w^*) = 2$  assume that  $d(x_0, y_0) < r, d(x_1, y_1) < r$  and  $y_1 = y_0^{-1}$ . Then  $d(x_1, y_1) = d(x_1^{-1}, y_0)$  and since  $d(x_0, y_0) < r$  we obtain, using the strong triangle inequality, that  $d(x_0^{-1}, x_1) = d(x_0, x_1^{-1}) < r$ . Since  $r < 1$  and  $x_0 \neq x_1^{-1}$  it follows that  $(x_0^{-1}, x_1) \in X \times X$  or  $(x_0^{-1}, x_1) \in X^{-1} \times X^{-1}$ . In the first case  $(x_0^{-1}, x_1) \in \mathcal{R}$  and thus  $w = x_0 x_1 \in j_2(\mathcal{R}) \subseteq [\tilde{\mathcal{R}}]$ . In the second case  $(x_0, x_1^{-1}) \in \mathcal{R}$  and thus  $w = x_0 x_1 \in j_2^*(\mathcal{R}) \subseteq [\tilde{\mathcal{R}}]$ . Now assume the assertion is true for  $k < lh(w^*)$  and that  $lh(w^*) \geq 3$ .



First case:  $y_0 \neq y_n^{-1}$ . There exists  $n > m$  such that  $y_0 \cdots y_m = y_{m+1} \cdots y_n = e$ . By the induction hypothesis

$$x_0 \cdots x_m, x_{m+1} \cdots x_n \in [\tilde{\mathcal{R}}].$$

Since  $[\tilde{\mathcal{R}}]$  is a subgroup we have  $w \in [\tilde{\mathcal{R}}]$ .

Second case:  $y_0 = y_n^{-1}$ . In this case  $y_1 \cdots y_{n-1} = e$  and by the induction hypothesis  $x_1 \cdots x_{n-1} \in [\tilde{\mathcal{R}}]$ . Since  $[\tilde{\mathcal{R}}]$  is normal,  $x_n^{-1}x_1 \cdots x_{n-1}x_n \in [\tilde{\mathcal{R}}]$ . Since  $y_0y_n = e$  it follows from the induction hypothesis (for  $lh(w^*) = 2$ ) that  $x_0x_n \in [\tilde{\mathcal{R}}]$ . Finally, since  $[\tilde{\mathcal{R}}]$  is a subgroup,  $(x_0x_n)x_n^{-1}x_1 \cdots x_{n-1}x_n = w \in [\tilde{\mathcal{R}}]$ . This completes the proof of (1).

(2) : Immediately follows from (1) and Theorem 3.5.1 □

#### 4. COMPARISON BETWEEN GRAEV TYPE ULTRA-METRICS

In [12] Savchenko and Zarichnyi introduced an ultra-metrization  $\hat{d}$  of the free group over an ultra-metric space  $(X, d)$  with  $diam(X) \leq 1$ . They used this ultra-metrization to study a functor on the category of ultra-metric spaces of diameter  $\leq 1$  and nonexpanding maps.

Let  $(X, d)$  be an ultra-metric space with diameter  $\leq 1$ . Extend  $d$  to an ultra-metric on  $\overline{X}$  by defining

$$d(x^{-1}, y^{-1}) = d(x, y), \quad d(x^{-1}, y) = d(x, y^{-1}) = d(x, e) = d(x^{-1}, e) = 1$$

for every  $x, y \in X$ . Consider its associated Graev ultra-metric  $\delta_u$ . Our aim is to show that  $\delta_u = \hat{d}$  (Theorem 4.6). We first provide the definition of  $\hat{d}$  from [12].

Let  $\alpha : F(X) \rightarrow \mathbb{Z}$  be the continuous homomorphism extending the constant map  $X \rightarrow \{1\} \subseteq \mathbb{Z}$ . For every  $r > 0$  let  $F_r$  be the partition of  $X$  formed by the open balls with radius  $r$  and  $q_r : X \rightarrow X/F_r$  is the quotient map. Let  $F(q_r) : F(X) \rightarrow F(X/F_r)$  be the extension of  $q_r : X \rightarrow X/F_r \hookrightarrow F(X/F_r)$ .

**Definition 4.1.** ([12, page 726]) The function  $\hat{d} : F(X) \times F(X) \rightarrow \mathbb{R}$  is defined as follows:

$$\hat{d}(v, w) = \begin{cases} 1, & \text{if } \alpha(v) \neq \alpha(w) \\ \inf\{r > 0 \mid F(q_r)(v) = F(q_r)(w)\}, & \text{if } \alpha(v) = \alpha(w) \end{cases}$$

for  $v, w \in F(X)$ .

**Theorem 4.2.** [12, Theorem 3.1] *The function  $\hat{d}$  is an invariant continuous ultra-metric on the topological group  $F(X)$ .*

**Lemma 4.3.** *For every  $v, w \in F(X)$  we have  $\delta_u(v, w) \geq \hat{d}(v, w)$ .*

*Proof.* By Theorem 4.2 and Theorem 2.2.b it suffices to prove that  $\hat{d}$  extends the ultra-metric  $d$  defined on  $X \cup \{e\}$ . For every  $x \in X$ ,  $\alpha(x) = 1 \neq 0 = \alpha(e)$ . Thus for every  $x \in X$  we have  $\hat{d}(x, e) = d(x, e) = 1$ . Let  $x, y \in X$ . We have to show that  $\hat{d}(x, y) = d(x, y)$ .

Clearly  $\alpha(x) = \alpha(y) = 1$ . Therefore,

$$\hat{d}(x, y) = \inf\{r > 0 \mid F(q_r)(x) = F(q_r)(y)\} = \inf\{r > 0 \mid q_r(x) = q_r(y)\}.$$

Denote  $d(x, y) = s$ . It follows that  $q_s(x) \neq q_s(y)$  and for every  $r > s$ ,  $q_r(x) = q_r(y)$ . This implies that

$$\inf\{r > 0 \mid q_r(x) = q_r(y)\} = s = d(x, y).$$

Hence  $\hat{d}(x, y) = d(x, y)$ , which completes the proof.  $\square$

**Lemma 4.4.** [2, Lemma 3.5] *For any trivial word  $w = x_0 \cdots x_n$  there is a match  $\theta$  such that for any  $i \leq n$ ,  $x_{\theta(i)} = x_i^{-1}$ .*

**Lemma 4.5.** *For every  $v, w \in F(X)$  we have  $\delta_u(v, w) \leq \hat{d}(v, w)$ .*

*Proof.* According to Theorems 1.3.2 and 4.2 both  $\hat{d}$  and  $\delta_u$  are invariant ultra-metrics. Therefore it suffices to show that

$$\forall e \neq v \in F(X), \delta_u(v, e) \leq \hat{d}(v, e).$$

Let  $v = x_0 \cdots x_n \in F(X)$ . Clearly  $\delta_u(v, e) \leq 1$ . Thus we may assume that  $\alpha(v) = \alpha(e)$ . Assume that  $s > 0$  satisfies  $F(q_s)(v) = F(q_s)(e)$ . We are going to show that there exists a match  $\theta$  such that  $\rho_u(v, v^\theta) < s$ . Using the definition of  $\hat{d}$  and Theorem 1.3.1 this will imply that  $\delta_u(v, e) \leq \hat{d}(v, e)$ . For every  $0 \leq i \leq n$  let  $\bar{x}_i = F(q_s)(x_i)$ . The equality  $F(q_s)(v) = F(q_s)(e)$  suggests that  $\bar{x}_0 \cdots \bar{x}_n \in W(X/F_s)$  is a trivial word. By Lemma 4.4 there exists a match  $\theta$  such that for any  $i \leq n$ ,  $\overline{x_{\theta(i)}} = \bar{x}_i^{-1}$ . Observe that  $\theta$  does not have fixed points. Indeed if  $j$  is a fixed point of  $\theta$  then from the equalities  $\overline{x_{\theta(j)}} = \bar{x}_j^{-1}$  and  $\overline{x_{\theta(j)}} = \bar{x}_j$  we obtain that  $\bar{x}_j$  is the identity element of  $F(X/F_s)$ . This contradicts the fact that  $x_j$  is not the identity element of  $F(X)$  and that  $F(X/F_s)$  is algebraically free over  $X/F_s$ .

For every  $0 \leq i \leq n$  we conclude from the equality  $\overline{x_{\theta(i)}} = \bar{x}_i^{-1}$  that  $d(x_i^{-1}, x_{\theta(i)}) < s$ .

Since  $\theta$  does not have fixed points we obtain that

$$\rho_u(v, v^\theta) = \max\{d(x_i^{-1}, x_{\theta(i)}) : \theta(i) < i\} < s.$$

This completes the proof.  $\square$

We finally obtain:

**Theorem 4.6.**  $\delta_u = \hat{d}$ .

*Proof.* Use Lemma 4.3 and Lemma 4.5.  $\square$

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