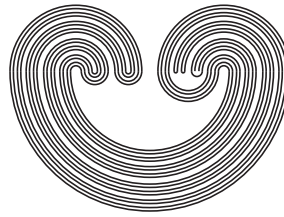


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by

W. R. BRIAN

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Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

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W. R. BRIAN

ABSTRACT. We say that two topological spaces are similar when each admits a continuous bijection onto the other. We will explore the similarity relation for spaces that can be represented as ω -length trees, namely the completely ultrametrizable spaces. We will prove that the separable, perfect, completely ultrametrizable spaces (i.e., the perfect zero-dimensional Polish spaces) come in exactly three similarity classes. The non-separable, perfect, completely ultrametrizable spaces are less tame, but we will show that, under the assumption of the Continuum Hypothesis, those of size \mathfrak{c} come in exactly four similarity classes.

1. INTRODUCTION

This paper began as a short note about zero-dimensional Polish spaces and continuous bijections between them. The germ from which the paper grew is given below as Theorem 3.2: if X is a zero-dimensional Polish space and is not σ -compact, then there is a continuous bijection $X \rightarrow \mathcal{N}$ (here, as elsewhere, we use \mathcal{N} to denote the Baire space ω^ω). This result, while new, has the same flavor as several classical results. For instance, it is known that every Polish space is the continuous image of \mathcal{N} , and that every perfect Polish space is the image of \mathcal{N} under a continuous bijection. Our result shows that these properties are not unique to \mathcal{N} .

Every zero-dimensional Polish space can be represented as a tree, in a sense to be made precise in Section 2. The aforementioned results have proofs with a partly combinatorial flavor, with these trees playing a prominent part. One aim of this paper is to determine how well these proof techniques can be extended beyond the realm of zero-dimensional Polish spaces.

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The class of spaces we study here is the class of completely ultrametrizable spaces, which are precisely the spaces representable as ω -length trees. The zero-dimensional Polish spaces are simply the spaces corresponding to countable trees.

It turns out that many of the techniques useful for the countable trees of Polish spaces work just as well for uncountable trees (although not for arbitrarily large trees: as we will see in Section 5, the situation is more complicated for trees of size \aleph_ω and larger). Certain aspects of these proofs become more intricate in the non-separable case. Indeed, it is with the uncountable trees that we begin to run into questions of consistency and independence. Everything proved below about Polish spaces is proved in *ZFC*, but we very quickly begin to require extra hypotheses for our results about spaces arising from uncountable trees.

A **condensation** is a continuous bijection. If there is a condensation $X \rightarrow Y$, we also say that Y is a condensation of X and that X condenses onto Y . Up to a relabeling of the underlying set, a topological space Y is a condensation of X if and only if X can be obtained by refining the topology of Y . We say that X and Y are **similar** or have the same **similarity type** if and only if each is a condensation of the other. This definition is essentially due to Sierpiński (see [19], pp. 151-152), although he uses the term “ γ -type” instead. The similarity relation is an equivalence relation on topological spaces, and the equivalence classes are naturally partially ordered by the relation

$$[X] \leq [Y] \Leftrightarrow \text{there is a condensation } Y \rightarrow X.$$

Intuitively, this can be viewed as an ordering of topologies with respect to their complexity. We will call this partial ordering of similarity classes the **condensation relation**.

The theme of this paper is to study the similarity and condensation relations in the class of completely ultrametrizable spaces. Because we focus on bijective maps, we will restrict our attention to spaces of the same size: we will mostly consider spaces of size \mathfrak{c} .

We will show in Section 3 that the perfect zero-dimensional Polish spaces divide into exactly three similarity classes, and that these classes are linearly ordered by the condensation relation. While only two theorems of this section (Theorems 3.2 and 3.7) are new, this section serves as a motivation, and in some sense a prototype, for what follows.

Section 5 is devoted to a combinatorial result about trees. Sections 4 and 6 explore the similarity and condensation relations for non-separable completely ultrametrizable spaces. We will show in Section 4 that *CH* implies that there are exactly four similarity types of perfect completely ultrametrizable spaces, and that these four types are linearly ordered by the condensation relation. Section 6 investigates what can be proved in *ZFC*.

2. NOTATION AND PRELIMINARIES

In what follows, we will use the Greek letters κ and λ for cardinal numbers, as is customary, but we will also use them for the discrete topological spaces of size κ and λ , respectively. Which is intended should always be clear from context. We will use the symbols \aleph_0, \aleph_1 , etc., and \mathfrak{c} as cardinals. The symbols ω, ω_1 , etc., and \mathfrak{c} will denote both the corresponding ordinals and the discrete topological spaces of a given cardinality. An ordinal will always carry the discrete topology (except in one proof below, where the exception is made explicit).

A space is **σ -compact** if it is a countable union of compact spaces. A space is **zero-dimensional** if it has a basis consisting of clopen sets. A space is **perfect** if it has no isolated points. A space is **completely ultrametrizable** if there is a complete ultrametric that generates its topology. In this paper we deal with completely ultrametrizable spaces, rather than complete ultrametric spaces, as a way of emphasizing that our results are topological and do not depend on a particular metric. A space is **Polish** if it is separable and completely metrizable. The **weight** of a topological space is the least size of a basis for that space.

In the context of this paper, a **tree** is a connected, nonempty, infinite graph with no simple closed paths (an infinite tree in the sense of graph theory), together with a distinguished node called the **root**. Any two nodes of a given tree are connected by exactly one path. If T is a tree and $s, t \in T$, we say that t **extends** s if the unique path from the root to t goes through s . In any given tree, we denote the extension relation by \leq .

The extension relation allows us to think of trees as partial orders, and it also allows us to think of certain partial orders as trees. For example, the set $2^{<\omega}$ of finite 0-1 sequences is a tree, where the empty sequence is the root and $s \leq t$ if and only if t extends s as a sequence. In what follows, $\kappa^{<\omega}$ denotes the set of finite sequences in κ with its natural structure as a tree.

Two nodes of a tree T are **incomparable** if neither one extends the other. A tree is **pruned** if each of its elements has a proper extension, and is **perfect** if each of its elements has two incomparable proper extensions. In what follows, a “tree” will always mean a pruned, nonempty tree.

A **subtree** of a tree T is any subset S of T such that $s \in S$ whenever $t \in S$ and t extends s ; equivalently, a subtree of T is a subset of T that is also a tree (with the same root).

If T is a tree and s is a node of T , then

$$T_s = \{t \in T : t \leq s \text{ or } s \leq t\}$$

is the set of all nodes of T that compare with s under the extension relation. If $s \lesssim t$ and there is no r such that $s \lesssim r \lesssim t$, then t is a **child** of s . Notice that if $t \in T$ then t is an extension of s if and only if there is some sequence $\langle s_i : i \leq n \rangle$ of nodes such that $s = s_0$, $t = s_n$, and each s_{i+1} is a child of s_i .

In every tree T , there is a unique path from the root to a given node. This naturally divides T into levels. We say that a node s is at level n , denoted $\text{lev}(s) = n$, if the unique path from the root to s has $n + 1$ elements. Thus the root is the unique node at level 0, the children of the root are all at level 1, etc. We write $\text{Lev}_n(T)$ for $\{s \in T : \text{lev}(s) = n\}$.

A **branch** of a tree T is an infinite sequence x of nodes in T such that $x(0)$ is the root and $x(n+1)$ is a child of $x(n)$ for every n . $\llbracket T \rrbracket$ is the set of all branches of T . $\llbracket T \rrbracket$ has a natural topology defined by taking $\{\llbracket T_s \rrbracket : s \in T\}$ to be a basis. In other words, each node of T represents a basic open subset of $\llbracket T \rrbracket$, namely the set of all branches in $\llbracket T \rrbracket$ that pass through that node. Clearly $\llbracket T_t \rrbracket \subseteq \llbracket T_s \rrbracket$ if and only if t extends s , and $\llbracket T_s \rrbracket \cap \llbracket T_t \rrbracket = \emptyset$ if and only if s and t are incomparable. From this it is not hard to see that $\llbracket T \rrbracket$ is Hausdorff, and also that each $\llbracket T_s \rrbracket$ is a clopen subset of T , making $\llbracket T \rrbracket$ zero-dimensional.

Proposition 2.1. *If T is a (perfect) tree, then $\llbracket T \rrbracket$ is a (perfect) completely ultrametrizable space. If X is a (perfect) completely ultrametrizable space, then there is a (perfect) tree T such that $\llbracket T \rrbracket \cong X$.*

Proof. A thorough treatment of this well-known result can be found in [8]. We sketch a proof here for completeness.

If T is a tree and $x, y \in \llbracket T \rrbracket$, define $d(x, y) = \inf \{ \frac{1}{2^n} : x(n) = y(n) \}$. One can check that this is a complete ultrametric that generates the topology of $\llbracket T \rrbracket$. Conversely, if d is a fixed ultrametric on X , then $\{ B_{\frac{1}{2^n}}(x) : x \in X \}$ is, for every n , a partition of X into clopen sets. Taking $T = \{ B_{\frac{1}{2^n}}(x) : x \in X \text{ and } n < \omega \}$, ordered by inclusion, one obtains a partial order that gives rise to a tree T . If d is also complete, it is straightforward to check that $\llbracket T \rrbracket \cong X$. \square

In fact, many equivalent characterizations beyond Proposition 2.1 can be given for this class of spaces. Completely ultrametrizable spaces are precisely the completely metrizable spaces with large inductive dimension 0 (see [5]); they are precisely the metrizable, zero-dimensional, Čech-complete spaces (see [12], Corollary 5); they are precisely the inverse limits over ω of discrete topological spaces (this is implicit in the tree representation; also see [11]). A characterization in terms of domain theory is given in [20].

We will say that a space X is **represented** by a tree T whenever $X \cong \llbracket T \rrbracket$. Thus Proposition 2.1 can be rephrased by saying that the (perfect) completely ultrametrizable spaces are precisely those representable by (perfect) trees. It is worth pointing out that the countable trees correspond exactly to the separable spaces, and the spaces representable by (perfect) countable trees are precisely the (perfect) zero-dimensional Polish spaces. For a proof of this and countless examples of how trees can be used to prove interesting results about Polish spaces, see [9].

The following proposition outlines some basic facts about trees. Here and throughout, we use \mathcal{N} to denote the Baire space ω^ω and we use \mathcal{C} to denote the Cantor space 2^ω . The three spaces \mathcal{C} , $\omega \times \mathcal{C}$, and \mathcal{N} will play a special role in the analysis of Section 3.

Proposition 2.2.

- (1) For all cardinals κ , $\llbracket \kappa^{<\omega} \rrbracket \cong \kappa^\omega$. In particular, $\llbracket 2^{<\omega} \rrbracket \cong \mathcal{C}$ and $\llbracket \omega^{<\omega} \rrbracket \cong \mathcal{N}$.
- (2) The map $T \mapsto \llbracket T \rrbracket$ is a bijection between possibly empty (perfect) subtrees of $\kappa^{<\omega}$ and closed (perfect) subsets of κ^ω . Its inverse map is given by

$$F \mapsto T_F = \{x \upharpoonright n : x \in F, n \in \omega\}.$$

- (3) If S is a subtree of T , then $\llbracket S \rrbracket$ is a closed subset of $\llbracket T \rrbracket$. Conversely, if C is a closed subset of $\llbracket T \rrbracket$ then there is a subtree S of T such that $C = \llbracket S \rrbracket$.

Proof. A version of this lemma for countable trees is found in Chapter 2 of [9]. The extension to uncountable trees is straightforward. \square

The space κ^ω , which can be viewed as a generalization of the Baire space and is sometimes denoted $B(\kappa)$, is well studied. For example, a non-separable version of the theory of Borel and analytic sets has been developed in which κ^ω plays the role of \mathcal{N} ; see [6] for details.

The following definitions and lemmas are an adaptation of material from [9], pp. 36-37.

Let T be a tree and let X be a topological space. A **T -scheme** on X is a family $(B_s)_{s \in T}$ of subsets of X such that

- $B_t \subseteq B_s$ whenever t is an extension of s .
- $B_s \cap B_t = \emptyset$ whenever s and t are incompatible.

If d is a metric on X then $(B_s)_{s \in T}$ has **vanishing diameter** (with respect to d) if $\lim_{n \rightarrow \infty} \text{diam}(B_{x(n)}) = 0$ whenever $x \in \llbracket T \rrbracket$. If X is a metric space and $(B_s)_{s \in T}$ is a T -scheme with vanishing diameter, then let $D = \{x \in \llbracket T \rrbracket : \bigcap_{n \in \omega} A_{x(n)} \neq \emptyset\}$ and define $f : D \rightarrow X$ by $\{f(x)\} = \bigcap_{n < \omega} B_{x(n)}$. We call f the **associated map**.

Lemma 2.3. *Let $(B_s)_{s \in T}$ be a T -scheme with vanishing diameter on a metric space (X, d) . If $f : D \rightarrow X$ is the associated map, then*

- (1) *f is injective and continuous.*
- (2) *if $B_s = \bigcup \{B_t : t \text{ is a child of } s\}$ for all $s \in T$, then f is surjective.*

Proof. Part (1) is straightforward. For part (2), pick any $x \in X$. By the definition of a T -scheme, there is, for every n , at most one $s \in T$ with $\text{lev}(s) = n$ such that $x \in B_s$. By the hypothesis of (2), together with a simple induction, there is, for every n , exactly one $s \in T$ with $\text{lev}(s) = n$ such that $x \in B_s$. Moreover, it is clear that the sequence enumerating these nodes forms a branch y of T . But then $\bigcap_{n \in \omega} B_{y(n)} = \{x\}$, i.e., $f(y) = x$. \square

3. MOTIVATION: POLISH SPACES

In this section we will see precisely how the similarity relation behaves for perfect zero-dimensional Polish spaces. We will prove that these spaces come in exactly three similarity types: the non- σ -compact spaces, the σ -compact but not compact spaces, and the (unique) compact space \mathcal{C} . To avoid trivialities, we assume throughout this section that all our spaces are infinite. The following facts about Polish spaces will be useful:

Proposition 3.1.

- (1) *\mathcal{N} is, up to homeomorphism, the unique zero-dimensional Polish space in which no nonempty open set is compact.*
- (2) *\mathcal{N} is not σ -compact. In fact, a Polish space fails to be σ -compact if and only if it contains a closed set homeomorphic to \mathcal{N} .*
- (3) *A nonempty Polish space is perfect if and only if it is a condensation of \mathcal{N} .*
- (4) *\mathcal{C} is the unique compact, zero-dimensional, perfect Polish space. Every compact Polish space is a continuous image of \mathcal{C} .*
- (5) *Every open subset of \mathcal{C} is homeomorphic either to \mathcal{C} or to $\omega \times \mathcal{C}$.*

Proof. (1)-(4) can be found in [9]: they are, respectively, Theorem 7.7, Theorem 7.10, Exercise 7.15, and Theorems 4.18 and 7.4. (5) could be considered an exercise, but we will give a proof here.

Let X be an open subset of $[[2^{<\omega}]] \cong \mathcal{C}$. Let

$$A = \{s \in 2^{<\omega} : [[2_s^{<\omega}]] \subseteq X \text{ and if } t \preceq s \text{ then } [[2_t^{<\omega}]] \not\subseteq X\}.$$

Clearly $[[2_s^{<\omega}]] \cong \mathcal{C}$ for every $s \in T$. Furthermore, it is easily checked that $X = \bigcup \{[[2_s^{<\omega}]] : s \in A\}$, and that this is a disjoint union. If A is finite then X is a copy of \mathcal{C} , and if not then X is a copy of $\omega \times \mathcal{C}$. \square

Theorem 3.2. *If X is a non- σ -compact, zero-dimensional Polish space, then \mathcal{N} is a condensation of X .*

Proof. This theorem is a special case of Theorem 6.8, whose proof will be postponed until Section 6. \square

Given Lemma 3.1(3), the previous theorem implies that all non- σ -compact perfect zero-dimensional Polish spaces have the same similarity type. We will now look at the σ -compact spaces and prove that these too are all equivalent up to similarity.

Lemma 3.3. *Let X be any perfect Polish space, and let $K \subseteq X$ be zero-dimensional and compact. Then there is a $C \subseteq X$ such that $C \supseteq K$ and $C \cong \mathcal{C}$.*

Proof. If K has no isolated points then there is nothing to prove. By Proposition 3.1(4). Let $\{x_n : n \in \omega\}$ be the set of isolated points of K (replace ω with N if K has finitely many isolated points). Let

$$\varepsilon_n = \min \left\{ \frac{1}{2^n}, \frac{1}{2} \cdot d(x_n, K \setminus \{x_n\}) \right\}$$

for each $n \in \omega$. Since X is perfect, we can find for each n a copy C_n of the Cantor set such that $x_n \in C_n \subseteq B(x_n, \varepsilon_n)$. It is straightforward to check that $C = K \cup \bigcup_{n \in \omega} C_n$ is compact (use the fact that compactness is equivalent to sequential compactness for metric spaces). Since C is also perfect and metrizable, $C \cong \mathcal{C}$ by Proposition 3.1(4). \square

Proposition 3.4. *Every zero-dimensional, σ -compact, perfect Polish space is a condensation of $\omega \times \mathcal{C}$.*

Proof. Because \mathcal{C} is a compact Hausdorff space, the only condensation of \mathcal{C} onto a Hausdorff space is the identity map. A condensation of $\omega \times \mathcal{C}$ can be viewed as a countable union of such maps, so a Hausdorff space X is a condensation of $\omega \times \mathcal{C}$ if and only if it can be partitioned into countably many copies of \mathcal{C} .

Let X be zero-dimensional, σ -compact, perfect, and Polish. If X is also compact then $X \cong \mathcal{C}$. By Proposition 3.1(4), $\mathcal{C} \cong (\omega + 1) \times \mathcal{C}$, where $\omega + 1$ is given its usual order topology, so \mathcal{C} can be partitioned into countably many copies of \mathcal{C} .

Assume that X is not compact. Let $X = \bigcup_{n \in \omega} C_n$, with each C_n compact. By Lemma 3.3, we may assume that each C_n is a Cantor set. Replacing C_n with $\bigcup_{m \leq n} C_m$ if necessary, we may also assume that $C_0 \subseteq C_1 \subseteq C_2 \subseteq \dots$. By deleting any C_n with $C_n = C_{n-1}$, we may also assume that all the C_n are distinct. Since X is not compact, this deletion still leaves infinitely many sets. In summary, we may write X as a strictly increasing union of infinitely many Cantor sets.

Let $D_0 = C_0$ and let $D_n = C_n \setminus C_{n-1}$ for all $n > 0$. By Proposition 3.1(5), every D_n is either a copy of \mathcal{C} or a copy of $\omega \times \mathcal{C}$. Thus we have found a way to partition X into countably many copies of \mathcal{C} , and this finishes the proof. \square

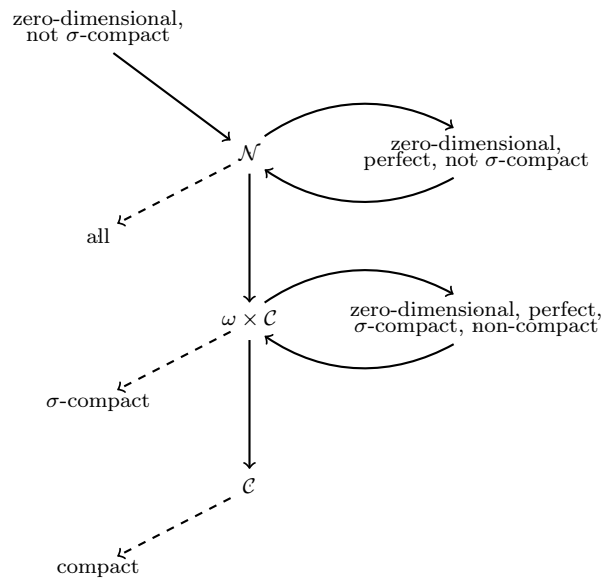
Proposition 3.5. *If X is a perfect, zero-dimensional Polish space, then \mathcal{C} is a condensation of X .*

Proof. See [11] or [16]. Also, a more general version of this result is proved as Theorem 4.1 below. \square

Corollary 3.6. *If X is a perfect, zero-dimensional, non-compact Polish space, then $\omega \times \mathcal{C}$ is a condensation of X .*

Proof. If X is not compact then one may partition X into countably many disjoint clopen subsets. For example, if T represents X then T is not finitely branching, so $\text{Lev}_n(T)$ is infinite for some n , and $\{\llbracket T_s \rrbracket : s \in \text{Lev}_n(T)\}$ gives the required partition. Applying Proposition 3.5 to each partition element individually finishes the proof. \square

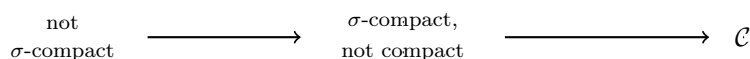
Proposition 3.4 and Corollary 3.6 together show that all σ -compact perfect zero-dimensional Polish spaces other than \mathcal{C} are similar. We can summarize the results of this section, together with the well known classical results listed in Proposition 3.1, in the following diagram. Solid arrows indicate the existence of a condensation, and dotted arrows indicate the existence of a continuous surjection. All spaces referred to in the diagram are nonempty and Polish.



Neither of the downward pointing solid arrows can be reversed. For the bottom arrow, this is because $\omega \times \mathcal{C}$ is not compact, but a continuous image of \mathcal{C} is. Similarly, the top arrow cannot be reversed because every continuous image of $\omega \times \mathcal{C}$ is σ -compact, and \mathcal{N} is not by Proposition 3.1(2). We have now proved the following theorem:

Theorem 3.7. *There are precisely three similarity classes of perfect zero-dimensional Polish spaces. They are the class of non- σ -compact spaces, the class of σ -compact spaces other than \mathcal{C} , and the class containing only the space \mathcal{C} . Furthermore, these classes are naturally totally ordered by the condensation relation.*

In other words, if we consider only the perfect zero-dimensional spaces then the above diagram collapses down to the following rather simple diagram:



We will show in Section 4 that, if the Continuum Hypothesis is assumed, then this diagram extends in the simplest possible way to the analogous class of non-separable spaces.

4. THE PICTURE UNDER CH

In this section we will extend the results of Section 3 to non-separable spaces using the Continuum Hypothesis. Since we are studying (continuous) bijections, it is convenient to fix the cardinality of our spaces. Henceforth, a **tree space** will be a completely ultrametrizable space of size \mathfrak{c} . The name is justified by Proposition 2.1, which we use frequently below.

Under CH , the simplest imaginable picture emerges: there are exactly four similarity types of perfect completely ultrametrizable spaces, and these are ordered by the condensation relation. Not every result of this section uses CH , and we will explicitly label all those that do.

First, we note that Proposition 3.5 admits a generalization to uncountable trees:

Theorem 4.1. *Suppose \mathcal{C} can be partitioned into κ homeomorphic copies of \mathcal{C} for every $\kappa \leq \lambda$. If T is a perfect tree with λ nodes, then \mathcal{C} is a condensation of $\llbracket T \rrbracket$.*

Proof. If \mathcal{C} can be partitioned into κ homeomorphic copies of \mathcal{C} (with κ infinite), then it can be partitioned into κ copies of \mathcal{C} all with diameter less than any prescribed $\varepsilon > 0$. This is because we can first partition \mathcal{C} into finitely many copies of \mathcal{C} , each smaller than ε , and then each of these (or one of these) can be further partitioned into κ copies of \mathcal{C} .

Let T be a perfect tree with λ nodes. By Lemma 5.2 below, we may assume that, for every $s \in T$, either $\llbracket T_s \rrbracket$ is compact or s has infinitely many children.

We will prove the existence of a condensation $\llbracket T \rrbracket \rightarrow \mathcal{C}$ by finding an appropriate T -scheme $(B_s)_{s \in T}$ in \mathcal{C} . The T -scheme will be defined by recursion for those $s \in T$ with $\llbracket T_s \rrbracket$ non-compact, and directly when $\llbracket T_s \rrbracket$ is compact. Moreover, each B_s will be homeomorphic to \mathcal{C} .

Fix a metric d for \mathcal{C} . To begin, let $B_\emptyset = \mathcal{C}$ (where \emptyset represents the root of T). Assume now that B_s has been defined for some $s \in T$, that $B_s \cong \mathcal{C}$, and that B_t has not yet been defined for any child t of s .

If $\llbracket T_s \rrbracket$ is compact, then, because T is a perfect tree, $\llbracket T_s \rrbracket \cong \mathcal{C} \cong B_s$. Fix a homeomorphism $g : \llbracket T_s \rrbracket \rightarrow B_s$, and, for every $t \in T$ that properly extends s , set $B_t = g(\llbracket T_t \rrbracket)$.

If $\llbracket T_s \rrbracket$ is not compact, it follows from our choice of T that s has infinitely many children. Label these $\{s \frown \alpha : \alpha < \kappa\}$, where $\kappa \leq \lambda$. There is a partition $\{C_\alpha : \alpha < \kappa\}$ of B_s such that $C_\alpha \cong \mathcal{C}$ and $\text{diam}(C_\alpha) < \frac{1}{\text{lev}(s)+1}$ for each α . Set $B_{s \frown \alpha} = C_\alpha$ for all α .

This defines, by recursion, a T -scheme in \mathcal{C} . It is easily checked that $(B_s)_{s \in T}$ has vanishing diameter. It is obvious from our construction that $B_\emptyset = \mathcal{C}$ and $B_s = \bigcup \{B_t : t \text{ is a child of } s\}$ for every $s \in T$. Furthermore, $\bigcap_{n \in \omega} B_{x(n)} \neq \emptyset$ for every $x \in \llbracket T \rrbracket$ because this is a nested intersection of Cantor sets. It now follows from Lemma 2.3 that the associated map of $(B_s)_{s \in T}$ is a continuous bijection. \square

Corollary 4.2. *Assuming CH: If X is a perfect tree space then \mathcal{C} is a condensation of X .*

Proof. \mathcal{C} can be partitioned into \aleph_0 copies of \mathcal{C} (because $\mathcal{C} \cong (\omega + 1) \times \mathcal{C}$) and into \mathfrak{c} copies of \mathcal{C} (because $\mathcal{C} \cong \mathcal{C} \times \mathcal{C}$). Thus the result follows from the previous theorem. \square

Arnie Miller has proved it consistent with *ZFC* that there is a partition of \mathcal{C} into \aleph_2 closed sets when $\mathfrak{c} = \aleph_3$ (see [13], Theorem 4). On the other hand, Miller also proved it consistent with $\mathfrak{c} = \aleph_3$ to have no partitions of \mathcal{C} into \aleph_2 closed sets (see [14], Theorem 3.7). Actually, Miller proves this for Borel sets, and the special case of closed sets follows from the earlier Shelah-Fremlin Theorem (see [3]). Noting a space X is a condensation of $\kappa \times \mathcal{C}$ if and only if it can be partitioned into κ copies of \mathcal{C} , it is consistent that \mathcal{C} is not a condensation of every perfect completely ultrametrizable space. In other words, Proposition 3.5 becomes a consistency result when we leave the realm of separable spaces.

Theorem 4.3. *Assuming CH: If X is a non-separable tree space, then ω_1^ω is a condensation of X . If, in addition, X is perfect, then X is a condensation of ω_1^ω .*

Proof. Let X be a non-separable tree space, and let T be a tree representing X . To prove the first assertion of the theorem, we will show that $\omega_1 \times \mathcal{C}$ is a condensation of X and that ω_1^ω is a condensation of $\omega_1 \times \mathcal{C}$. Since a composition of condensations is a condensation, this is enough.

Note that $|T| = \aleph_1$ because X is non-separable. Therefore there is some n such that $|\text{Lev}_n(T)| = \aleph_1$. By Corollary 4.2, there is for every $s \in \text{Lev}_n(T)$ a condensation $f_s : \llbracket T_s \rrbracket \rightarrow \{s\} \times \mathcal{C}$. Then $\bigcup_{s \in \text{Lev}_n(T)} f_s$ is a condensation from $\llbracket T \rrbracket$ to $\text{Lev}_n(T) \times \mathcal{C}$, where $\text{Lev}_n(T)$ is given the discrete topology. Thus $\omega_1 \times \mathcal{C}$ is a condensation of X .

By Exercise 7.2.G in [2], $\omega_1^\omega \cong \omega_1^\omega \times \mathcal{C}$. Since $|\omega_1^\omega| = \aleph_1$ under CH, this shows that ω_1^ω can be partitioned into \aleph_1 copies of \mathcal{C} . This is the same as saying that ω_1^ω is a condensation of $\omega_1 \times \mathcal{C}$. This completes the proof of the first assertion.

For the second assertion, first note that every perfect Polish space is a condensation of ω_1^ω . This follows from Proposition 3.1(3) and Proposition 4.6 below. Using this fact, we will now prove the second assertion of the theorem using a scheme argument.

Let X be a perfect, non-separable, completely ultrametrizable space and let T be a tree that represents X . By Lemma 5.2, we may assume that every node s of T has exactly $|T_s|$ children. We will build a $\omega_1^{<\omega}$ -scheme $(B_s)_{s \in \omega_1^{<\omega}}$ in $\llbracket T \rrbracket$ by recursion.

Set $B_\emptyset = \llbracket T \rrbracket$. Assume now that B_s has been defined and is equal to $\llbracket T_t \rrbracket$ for some node $t \in T$. If $\llbracket T_t \rrbracket$ is Polish, then $\llbracket T_t \rrbracket$ is a condensation of $\omega_1^\omega \cong \llbracket (\omega_1^{<\omega})_s \rrbracket$ (as above, every perfect Polish space is a condensation of ω_1^ω). Let $g : \llbracket (\omega_1^{<\omega})_s \rrbracket \rightarrow \llbracket T_t \rrbracket$ be a condensation, and define $B_r = g(\llbracket (\omega_1^{<\omega})_r \rrbracket)$ for every extension r of s . If $\llbracket T_t \rrbracket$ is not Polish, then t has \aleph_1 children in T by our choice of T . Enumerating these as $\{t_\alpha : \alpha < \omega_1\}$, we let $B_{s \smallfrown \alpha} = \llbracket T_{t_\alpha} \rrbracket$. This recursion defines a $\omega_1^{<\omega}$ -scheme $(B_s)_{s \in \omega_1^{<\omega}}$.

Let $x \in \llbracket \omega_1^{<\omega} \rrbracket$. If there is some n such that $B_{x(n)}$ is Polish, then $B_{x(n)}$ is defined by some embedding $g : B_{x(n)} \rightarrow X$ for all $m \geq n$. Because g is an embedding, $\lim_{m \rightarrow \infty} \text{diam}(B_{x(n)}) = 0$ and $\bigcap_{n \in \omega} B_{x(n)} = g(x)$. If $B_{x(n)}$ is never Polish, then (by an easy induction) $B_{x(n)} = \llbracket T_{y(n)} \rrbracket$ for some $y \in \llbracket T \rrbracket$ and every n . Since $\{\llbracket T_{y(n)} \rrbracket : n < \omega\}$ is a local basis for y , $\lim_{n \rightarrow \infty} \text{diam}(B_{x(n)}) = 0$ in this case too; also, clearly, $\bigcap_{n \in \omega} B_{x(n)} = y$. Thus $(B_s)_{s \in \omega_1^{<\omega}}$ has vanishing diameter, and $\bigcap_{n \in \omega} B_{x(n)} \neq \emptyset$ for every $x \in \llbracket \omega_1^{<\omega} \rrbracket$. Furthermore, it is clear from our construction that $B_s = \bigcup \{B_t : t \text{ is a child of } s\}$ for every $s \in \omega_1^{<\omega}$. It follows from Lemma 2.3 that the associated map of $(B_s)_{s \in \omega_1^{<\omega}}$ is a condensation. \square

Corollary 4.4. *Assuming CH: Any two perfect, non-separable tree spaces are similar.*

Proof. By the previous theorem, there are condensations $f : \omega_1^\omega \rightarrow Y$ and $g : X \rightarrow \omega_1^\omega$; but then $f \circ g$ is a condensation $X \rightarrow Y$. Similarly, there is a condensation $Y \rightarrow X$. \square

We now see that there are, under CH, at most four similarity types of perfect completely metrizable spaces. The next two propositions show, respectively, that there are exactly four types, and that these types are linearly ordered by the condensation relation. Each proposition actually proves a stronger statement that does not depend on CH.

Proposition 4.5. *If $|S| < |T|$, then there is no condensation $\llbracket S \rrbracket \rightarrow \llbracket T \rrbracket$. In particular, there is no condensation $\mathcal{N} \rightarrow \omega_1^\omega$.*

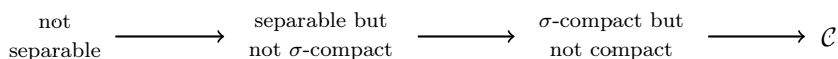
Proof. It is obvious that a condensation can never increase the cellularity of a space, so it suffices to show that $\llbracket T \rrbracket$ has cellularity greater than $\llbracket S \rrbracket$. Suppose $|S| < |T|$. $|S|^{+\omega} = \kappa$ is a regular uncountable cardinal. As $\kappa \leq |T|$, there is some $n < \omega$ with $|\text{Lev}_n(T)| \geq \kappa$. Thus the cellularity of $\llbracket T \rrbracket$ is greater than the weight (hence the cellularity) of $\llbracket S \rrbracket$. \square

Proposition 4.6. *If $2 \leq \kappa \leq \mathfrak{c}$, then κ^ω is a condensation of \mathfrak{c}^ω .*

Proof. If $2 \leq \kappa \leq \mathfrak{c}$ then $|\kappa^\omega| = \mathfrak{c}$. Viewing \mathfrak{c} as a discrete topological space, any bijection $f : \mathfrak{c} \rightarrow \kappa^\omega$ is also a condensation. Then $f^\omega : \mathfrak{c}^\omega \rightarrow (\kappa^\omega)^\omega$ is also a condensation (here f^ω denotes the map $(\alpha_0, \alpha_1, \dots) \mapsto (f(\alpha_0), f(\alpha_1), \dots)$). Since $(\kappa^\omega)^\omega \cong \kappa^\omega$, this finishes the proof. \square

The following theorem and diagram summarize the results of this section together with the results of Section 3.

Theorem 4.7. *Assuming the Continuum Hypothesis, there are precisely four similarity classes of perfect tree spaces. They are the class of non-separable spaces, the class of separable non- σ -compact spaces, the class of σ -compact spaces other than \mathcal{C} , and the class containing only the space \mathcal{C} . Furthermore, these classes are naturally totally ordered by the condensation relation.*



Given this theorem, one naturally asks whether it is always the case that the similarity types of perfect completely ultrametrizable spaces are totally ordered by the condensation relation. This is not so, and in fact a consistent counterexample is easy to find:

Proposition 4.8. *If $\text{cov}(\text{meager}) \neq \aleph_1$ (for example, if $MA + \neg CH$ holds), then \mathcal{N} and $\omega_1 \times \mathcal{C}$ are incomparable under the condensation relation.*

Proof. There is no condensation $\mathcal{N} \rightarrow \omega_1 \times \mathcal{C}$ by Proposition 4.5. If there were a condensation $\omega_1 \times \mathcal{C} \rightarrow \mathcal{N}$, then \mathcal{N} would be partitioned into ω_1 homeomorphic copies of \mathcal{C} . Since any compact subset of \mathcal{N} is nowhere dense by Proposition 3.1(1), this means that \mathcal{N} is covered by \aleph_1 meager sets. This contradicts $\text{cov}(\text{meager}) \neq \aleph_1$. \square

Question 4.9. *In general, what is the maximum (minimum) number of similarity types of perfect completely ultrametrizable spaces for a given value of \mathfrak{c} ? For $\mathfrak{c} = \aleph_2$?*

The condensation relation on similarity types is not a well order in general. It was shown by Sierpiński in [18] that there are uncountable increasing and decreasing transfinite sequences of similarity types, even if we restrict our attention just to countable metric spaces. However, it is not clear how to generalize Sierpiński’s arguments to a consistency result for perfect spaces. This consideration, together with Theorem 4.7, leads to the following question.

Question 4.10. *Is the condensation relation always (in every model of ZFC) well-founded on the similarity classes of perfect completely ultrametrizable spaces? Is it consistent with large values of \mathfrak{c} that these similarity classes are well ordered by the condensation relation?*

5. WELL-BEHAVED TREE REPRESENTATIONS

The main result of this section is a representation lemma for completely ultrametrizable spaces arising from trees of cardinality less than \aleph_ω . Roughly speaking, it states that every such space arises from a particularly nice-looking tree.

Let X be a completely ultrametrizable space of weight κ . For $x \in X$, we say $x \in \text{Ker}(X)$ if and only if every neighborhood of x contains a closed copy of κ^ω . If T is a tree representing X and $s \in T$, we say $s \in \text{Ker}(T)$ if and only if $\llbracket T_s \rrbracket \cap \text{Ker}(X) \neq \emptyset$.

The kernel of a tree T can be defined directly without reference to the topology of $\llbracket T \rrbracket$. We leave the details of this to the interested reader. For an example of this in the special case that T is countable, see [4] or [10].

Lemma 5.1. *If T is a tree and $\varepsilon > 0$, then every open subset of $\llbracket T \rrbracket$ can be written as a disjoint union of clopen sets of diameter at most ε (with respect to any fixed metric on $\llbracket T \rrbracket$).*

Proof. Fix a metric on $\llbracket T \rrbracket$ and an open $U \subseteq \llbracket T \rrbracket$. Let us say that $s \in T$ is “nice” if $\llbracket T_s \rrbracket \subseteq U$ and $\text{diam}(\llbracket T_s \rrbracket) \leq \varepsilon$. Let A be the set of all $s \in T$ such that s is nice but does not properly extend any other nice node. It is straightforward to verify that $\{\llbracket T_s \rrbracket : s \in A\}$ is the required set of clopen sets. \square

Theorem 5.2. *Let X be a completely ultrametrizable space with weight $\kappa < \aleph_\omega$. There is a tree T representing X such that*

- (1) *for all $s \in T$ with infinitely many children, if t is an extension of s then s has at least as many children as t .*
- (2) *if $s \in \text{Ker}(T)$ then s has κ children.*

Proof. Let S be any tree representing X , and let \mathcal{B} denote the corresponding basis for X ; i.e., $\mathcal{B} = \{\llbracket S_s \rrbracket : s \in S\}$.

For each non-compact open subset U of X , let $W(U)$ be the size of the largest partition of U into clopen subsets. Every open set has such a partition by Lemma 5.1, and it has a largest such because the weight of X is less than \aleph_ω . If $\varepsilon > 0$, then U has a partition into $W(U)$ elements of \mathcal{B} of diameter less than ε . This is possible because, if U is partitioned into $W(U)$ open sets, then each of these can be further partitioned into elements of \mathcal{B} that are smaller than ε by Lemma 5.1.

To establish the first claim, we build a tree T by recursion. Let X be the root of T . Assume $U \in T$ has been defined as some clopen subset of X . If U is compact, let \mathcal{U} be a partition of U into finitely many clopen sets, each at most half the diameter of U . If U is not compact, fix some partition \mathcal{U} of U into $W(U)$ clopen sets, each at most half the diameter of U . Then let \mathcal{U} be the set of children of U in T . This defines a tree T by recursion, and it is easy to see that the map $x \mapsto \bigcap_{n \in \omega} x(n)$ is a homeomorphism $\llbracket T \rrbracket \rightarrow X$.

If $U, V \in T$ with $U \subseteq V$, then clearly $W(U) \leq W(V)$ (if \mathcal{U} is a partition of U , then $\mathcal{U} \cup \{V \setminus U\}$ is a partition of V that is at least as large). This proves (1).

If $U \in T$, clearly $U \cong \llbracket T_U \rrbracket$. Using (1), the weight of U is $W(U)$ whenever U is not compact. If $U \in \text{Ker}(T)$ then U contains a closed copy of κ^ω . Thus the weight $W(U)$ of U must be at least κ , which means it must be exactly κ . This proves (2). \square

Note: the phrase “with infinitely many children” can be removed from (1) without making the above theorem false. Doing so complicates the proof without adding to the usefulness of the theorem, so we leave this improvement as an exercise. The following example shows that the cardinality restriction in the statement of Theorem 5.2 is necessary:

Example 5.3. For each n , let X_n denote the ultrametric space ω_n^ω , and let ∞ be a point not in any X_n . Let $X = \{\infty\} \cup \bigcup_{n \in \omega} X_n$, and let the basic open neighborhoods of ∞ be of the form $\{\infty\} \cup \bigcup_{n \geq N} X_n$. It is easy to show that X is a completely ultrametrizable space of weight \aleph_ω . It is also easy to see that any open cover of X has a subcover that is strictly smaller than \aleph_ω . If T is any tree representing X , it follows from this fact that the root of T cannot have \aleph_ω children (otherwise we could get an open cover of X of size \aleph_ω with no proper subcover). Let $|\text{Lev}_1(T)| = \kappa < \aleph_\omega$. Assuming the conclusion (1) of Theorem 5.2, every node of T has at most κ children, in which case $|T| = \kappa$. This contradicts the fact that the weight of X is \aleph_ω .

6. CONDENSATIONS TO AND FROM κ^ω

Most of Section 4 was concerned with what can be proved from *CH*. This section will attempt the more difficult question of what can be proved in *ZFC*. The two main results are (1) \mathcal{N} is a condensation of ω_1^ω (2) if $|T| = \kappa$ and $\text{Ker}(T) \neq \emptyset$, then κ^ω is a condensation of $\llbracket T \rrbracket$.

If \mathcal{P} is a partition of X such that every element of \mathcal{P} is Borel, we say that \mathcal{P} is an **element-Borel partition** of X . This is not to be confused with the well studied notion of a Borel partition. A partition \mathcal{P} is a Borel partition whenever the equivalence relation induced by \mathcal{P} is Borel in X^2 . In general, every Borel partition is element-Borel but not every element-Borel partition is Borel. Silver shows in [17] that every Borel equivalence relation in \mathcal{N}^2 produces a partition on \mathcal{N} that is either countable or of size \mathfrak{c} . On the other hand, Hausdorff shows in [7] that \mathcal{N} always admits an element-Borel partition of size \aleph_1 . Thus there are element-Borel partitions that fail to be Borel.

Lemma 6.1. *Let κ be any cardinal. There is an element-Borel partition of \mathcal{N} of size κ if and only if \mathcal{N} is a condensation of $\kappa \times \mathcal{N}$.*

Proof. Suppose $f : \kappa \times \mathcal{N} \rightarrow \mathcal{N}$ is a condensation. For each $\alpha \in \kappa$, the image of $\{\alpha\} \times \mathcal{N}$ is Borel in \mathcal{N} . This follows from a theorem of Lusin and Souslin, stating that the image of a Borel subset of a Polish space under an injective Borel function is always Borel (see [9], Theorem 15.1). Thus $\{f(\{\alpha\} \times \mathcal{N}) : \alpha \in \kappa\}$ is an element-Borel partition of \mathcal{N} of size κ .

Now suppose there is an element-Borel partition \mathcal{A} of \mathcal{N} of size κ . For any Borel set A , there is a continuous bijection $f_A : F_A \rightarrow A$, where F_A is some closed subset of \mathcal{N} (this too is a theorem of Lusin and Souslin; see [9], Theorem 13.7). Then $\mathcal{A}' = \{A \times \mathcal{N} : A \in \mathcal{A}\}$ is an element-Borel partition of $\mathcal{N} \times \mathcal{N} \cong \mathcal{N}$. Furthermore, $f_A \times \text{id} : F_A \times \mathcal{N} \rightarrow A \times \mathcal{N}$ is a continuous bijection, and $F_A \times \mathcal{N} \cong \mathcal{N}$ by an application of Proposition 3.1(1). Taken together, these maps give a condensation $\kappa \times \mathcal{N} \rightarrow \mathcal{N}$. □

Theorem 6.2. *Suppose \mathcal{N} admits an element-Borel partition of size κ , where κ is an infinite cardinal. Then \mathcal{N} is a condensation of κ^ω .*

The author would like to thank Arnie Miller for the following simple proof of this theorem.

Proof of Theorem 6.2. Suppose \mathcal{N} admits an element-Borel partition of size κ . By Lemma 6.1, there is a continuous bijection $f : \kappa \times \mathcal{N} \rightarrow \mathcal{N}$. Let $f^\omega : (\kappa \times \mathcal{N})^\omega \rightarrow \mathcal{N}^\omega$ denote the map $(x_0, x_1, \dots) \mapsto (f(x_0), f(x_1), \dots)$. This function is continuous because it is coordinate-wise continuous. Moreover, $\mathcal{N}^\omega \cong \mathcal{N}$ and $(\kappa \times \mathcal{N})^\omega \cong \kappa^\omega$ by Exercise 7.2.G in [2]. \square

Corollary 6.3. *\mathcal{N} is a condensation of ω_1^ω .*

Proof. In [7], Hausdorff shows how to write \mathbb{R} as the increasing union $\bigcup_{\alpha < \omega_1} X_\alpha$ of G_δ sets. This gives rise to an element-Borel partition of \mathbb{R} (and its co-countable subset \mathcal{N}) of size \aleph_1 . \square

Recalling that \mathcal{N} is homeomorphic to a co-countable subset of \mathcal{C} , \mathcal{N} can be partitioned into κ Borel sets if and only if \mathcal{C} can. Thus the results of Miller discussed in Section 4 show that for $\aleph_1 < \kappa < \mathfrak{c}$, it is not determined by *ZFC* whether there is a κ -sized element-Borel partition of \mathcal{N} . By a result of Burgess, any such partition fails to be analytic in \mathcal{N}^2 (see [1] or Chapter 32 of [15]). Miller's consistency results and Theorem 6.2 still leave the following open:

Question 6.4. *If $\kappa < \lambda < \mathfrak{c}$, is it always true that κ^ω is a condensation of λ^ω ?*

The second result of this section, Theorem 6.8, states that, among all weight- κ completely ultrametrizable spaces with nonempty kernel, κ^ω is minimal with respect to the condensation relation.

Let S and T be trees. A map $\phi : S \rightarrow T$ is called **monotone** if $s \leq t$ implies $\phi(s) \leq \phi(t)$. If $\phi : S \rightarrow T$ is a monotone map, then ϕ induces an S -scheme in $\llbracket T \rrbracket$, namely $(\llbracket T_{\phi(s)} \rrbracket)_{s \in T}$. If $f : \llbracket S \rrbracket \rightarrow \llbracket T \rrbracket$ is the associated map of this S -scheme, we also say that f is the associated map of ϕ . The following lemma is straightforward:

Lemma 6.5. *If $\phi : S \rightarrow T$ is an embedding of S into T that fixes the root, then the associated map of ϕ is an embedding of $\llbracket S \rrbracket$ into $\llbracket T \rrbracket$.*

Lemma 6.6. *If T is a tree and $\kappa \geq |T|$, then there is an embedding $g : \llbracket T \rrbracket \rightarrow \kappa^\omega$ such that $\kappa^\omega \setminus g(\llbracket T \rrbracket) \cong \kappa^\omega$.*

Proof. Let T be a tree and let $\kappa \geq |T|$. Since κ is infinite, we may partition κ into two sets A and B , each of which has size κ . Clearly, T is isomorphic to a subtree T' of $A^{<\omega}$. By Lemma 6.5, there is an embedding

$g : \llbracket T \rrbracket \rightarrow \kappa^\omega$ that takes $\llbracket T \rrbracket$ to $\llbracket T' \rrbracket$. In particular, $\kappa^\omega \setminus g(\llbracket T \rrbracket) = \kappa^\omega \setminus \llbracket T' \rrbracket$ is open in κ^ω . Furthermore, $\kappa^\omega \setminus g(\llbracket T \rrbracket)$ is nonempty because it contains $\llbracket B^{<\omega} \rrbracket$.

To complete the proof, we note that every open subset of κ^ω is homeomorphic to κ^ω (see Exercise 7.G.2 in [2]). □

Lemma 6.7. *Let κ be an infinite cardinal and let d be any metric on κ^ω compatible with its topology. If $\varepsilon > 0$, then there is a partition $\{B_\alpha : \alpha < \kappa\}$ of κ^ω into κ clopen sets such that $B_\alpha \cong \kappa^\omega$ and $\text{diam}(B_\alpha) < \varepsilon$ for every $\alpha < \kappa$.*

Proof. This follows from Lemma 5.1 and the observation that every clopen subset of κ^ω is homeomorphic to κ^ω (by Exercise 7.G.2 in [2]). □

Note that the following theorem does not assume T to be perfect.

Theorem 6.8. *Let T be a tree with $\text{Ker}(T) \neq \emptyset$ and let $\kappa = |T|$. Then κ^ω is a condensation of $\llbracket T \rrbracket$.*

Proof. Let T be a tree with $\text{Ker}(T) \neq \emptyset$ and let $\kappa = |T|$. By Theorem 5.2, we may assume that if $s \in \text{Ker}(T)$ then s has κ children also in $\text{Ker}(T)$.

We will prove that X condenses onto κ^ω by constructing an appropriate T -scheme $(B_s)_{s \in T}$ in κ^ω . More explicitly, we will construct a T -scheme $(B_s)_{s \in T}$ such that:

- (1) $(B_s)_{s \in T}$ has vanishing diameter.
- (2) $B_\emptyset = \kappa^\omega$ and, for every $s \in T$, $B_s = \bigcup \{B_t : t \text{ is a child of } s\}$.
- (3) for every $x \in \llbracket T \rrbracket$, $\bigcap_{n \in \omega} B_{x(n)} \neq \emptyset$.

It is clear from Lemma 2.3 that any such T -scheme induces a continuous bijection $f : \llbracket T \rrbracket \rightarrow \kappa^\omega$, namely its associated map.

We will define B_s by recursion for $s \in \text{Ker}(T)$ and by a slightly more direct method for $s \notin \text{Ker}(T)$. Moreover, we will do so in such a way that B_s is homeomorphic to κ^ω if and only if $s \in \text{Ker}(T)$. Simultaneously, we will define a family $(d_s)_{s \in \text{Ker}(T)}$ of metrics such that d_s is a complete metric compatible with the topology of B_s .

We view the elements of T as sequences, with \emptyset (the empty sequence) as the root. To begin, take $B_\emptyset = \kappa^\omega$. Note that $\emptyset \in \text{Ker}(T)$ because $\text{Ker}(T)$ is a nonempty subtree of T . Let d_\emptyset be any compatible complete metric on κ^ω .

Assume now that B_s and d_s have been defined for some $s \in \text{Ker}(T)$, and assume that $B_s \cong \kappa^\omega$.

First, consider $R = \{r : r \text{ is a child of } s \text{ and } r \notin \text{Ker}(T)\}$. For $r \in R$, we will now define simultaneously not only B_r , but also B_t for every extension t of r . Note that $K = \llbracket \bigcup_{r \in R} T_r \rrbracket$ is represented by a tree of size at most κ (namely $\bigcup_{r \in R} T_r$). By Lemma 6.6, there is an embedding $g : K \rightarrow B_s$ such that $B_s \setminus g(K) \cong \kappa^\omega$. For all nodes t extending an element of R (i.e., all t such that $\llbracket T_t \rrbracket \subseteq K$), we set $B_t = g(\llbracket T_t \rrbracket)$.

Next, consider $S = \{t : t \text{ is a child of } s \text{ and } t \in \text{Ker}(T)\}$. As $s \in \text{Ker}(T)$, our choice of T implies $|S| = \kappa$. As we have seen, $\tilde{B}_s = B_s \setminus \bigcup_{t \in R} B_t = B_s \setminus g(K)$ is homeomorphic to κ^ω . By hypothesis, d_s is a complete metric on B_s compatible with its topology. We may assume without loss of generality that $d_r \leq d_s$ for all $r \subseteq s$ because, given any such metric d_s , $d'_s = \sum_{r \subseteq s} d_r$ is also a complete metric compatible with B_s . The restriction of d_s to \tilde{B}_s is a (not necessarily complete) metric for B_s . By Lemma 6.7 there is a family $\{B_t : t \in S\}$ of pairwise disjoint clopen (in \tilde{B}_s) subsets of \tilde{B}_s such that $\bigcup_{t \in S} B_t = \tilde{B}_s$, $B_t \cong \kappa^\omega$ for each $t \in S$, and $\text{diam}_{d_s}(B_t) < \frac{1}{\text{lev}(t)}$ for every $t \in S$.

This defines a T -scheme in κ^ω . If $f : D \rightarrow \kappa^\omega$ is the associated map then, by Lemma 2.3, f is a continuous bijection provided $(B_s)_{s \in T}$ satisfies the three conditions listed above.

First we check (1): that $(B_s)_{s \in T}$ has vanishing diameter. Let $x \in \llbracket T \rrbracket$. If $x \notin \llbracket \text{Ker}(T) \rrbracket$ then there is some n such that $x(n) \notin \text{Ker}(T)$; let n be minimal such that this is the case. In our construction we defined $B_{x(n)}$ by an explicit embedding $g : \llbracket T_{x(n)} \rrbracket \rightarrow B_{x(n-1)}$. Because g is an embedding, we may find for every $\varepsilon > 0$ some basic open $\llbracket T_{x(m)} \rrbracket$, $m \geq n$, such that $g(x) \in g(\llbracket T_{x(m)} \rrbracket) = B_{x(m)} \subseteq \text{Ball}_\varepsilon^{d_{x(n-1)}}(g(x)) \subseteq \text{Ball}_\varepsilon^{d_\emptyset}(g(x))$. Thus $\lim_{m \rightarrow \infty} \text{diam}_{d_\emptyset}(B_{x(m)}) = 0$ when $x \notin \text{Ker}(T)$. Suppose next that $x \in \text{Ker}(T)$. At every stage of our construction, we ensured that $\text{diam}_{d_\emptyset}(B_{x(n+1)}) < \text{diam}_{d_{x(n)}}(B_{x(n+1)}) < \frac{1}{n+1}$, so in this case we still have $\lim_{n \rightarrow \infty} \text{diam}_{d_\emptyset}(B_{x(n)}) = 0$.

Condition (2), that $B_\emptyset = \kappa^\omega$ and that, for every $s \in T$, $B_s = \bigcup\{B_t : t \text{ is a child of } s\}$, is clear from our construction.

Finally, we check (3): that $\bigcap_{n \in \omega} B_{x(n)} \neq \emptyset$ for every $x \in \llbracket T \rrbracket$. Suppose that $x(n) \notin \text{Ker}(T)$ for some n , and consider the minimal such n . Then $B_{x(m)}$, $m \geq n$, is defined by means of an explicit embedding $g : \llbracket T_{x(n)} \rrbracket \rightarrow \kappa^\omega$, and in this case it is clear that $\bigcap_{m \in \omega} B_{x(m)} = \bigcap_{m \geq n} B_{x(m)} = \bigcap_{n \in \omega} g(\llbracket T_{x(m)} \rrbracket) = \{g(x)\} \neq \emptyset$. Suppose instead that $x(n) \in \text{Ker}(\llbracket T \rrbracket)$ for all n . For each n , let $x_n \in B_{x(n)}$. Since d_\emptyset is a complete metric on κ^ω and $\lim_{n \rightarrow \infty} \text{diam}_{d_\emptyset}(B_{x(n)}) = 0$, there is a unique $z \in \kappa^\omega$ such that $x_n \rightarrow z$. However, the same argument works in any particular $B_{x(m)}$. That is, note that $d_{x(m)}$ is a complete metric on $B_{x(m)}$, and, by construction, $\lim_{n \rightarrow \infty} \text{diam}_{d_{x(m)}}(B_{x(n)}) = 0$. Thus there is a unique $z \in B_{x(m)}$ such that $x_n \rightarrow z$. Clearly this is the same z as before, since $\langle x_n : n < \omega \rangle$ cannot converge to more than one point. This shows $z \in B_{x(m)}$. As m was arbitrary, $z \in \bigcap_{n \in \omega} B_{x(n)}$ and $\bigcap_{n \in \omega} B_{x(n)} \neq \emptyset$. \square

Ideally, one would like to reverse Theorem 6.8 and show that $\llbracket T \rrbracket$ is a condensation of $|T|^\omega$ whenever $\text{Ker}(T) \neq \emptyset$. However, this could be difficult. Consider, for example, the space $X = \kappa^\omega \oplus \mathcal{N}$. This space

is representable by a size- κ tree with a nonempty kernel. However, if $f : \kappa^\omega \rightarrow X$ were a condensation, then $f^{-1}(\mathcal{N})$ would be a clopen subset of κ^ω and hence homeomorphic to κ^ω . Thus, in this case, $f \upharpoonright f^{-1}(\mathcal{N})$ would be a condensation $\kappa^\omega \rightarrow \mathcal{N}$. This does not show that it is impossible to reverse Theorem 6.8 in *ZFC*, but it does show that doing so is at least as difficult as answering Question 6.4.

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DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, 6823 ST. CHARLES AVE.,
NEW ORLEANS, LA 70118

E-mail address: `wbrian.math@gmail.com`