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ON ONE-LOCAL RETRACT IN QUASI-METRIC SPACES

OLIVIER OLELA OTAFUDU

Abstract. We study a concept of 1-local retract in quasi-metric spaces. In this article, we generalize further known results about 1-local retract subsets from metric setting to quasi-metric point of view. In particular we show that any commuting family of nonexpansive self-mappings in a nonempty T_0-quasi-metric space (X,d) for which $\mathcal{A}_q(X)$ is compact and normal has a common fixed point and the common fixed point set is a 1-local retract of (X,d).

1. Introduction

A subset A of a metric space (X,m) is said to be a 1-local retract of (X,m) if for every family $\{B_i; i\in I\}$ of closed balls centered in A with nonempty intersection, then $A\cap((B_i)_{i\in I})\neq\emptyset$ (see [3], compare [4]). In [4], Khamsi showed that any commutative family of nonexpansive self-mapping defined on a metric space with compact and normal convexity structure has a common fixed point. In this article, we study the concept of 1-local retract in asymmetric setting. Among other things in this paper we consider subspaces 1-local retracts of a nonempty T_0-quasi-metric space and also present some fixed point theorems. In particular we prove that a nonexpansive self-mapping nonempty T_0-quasi-metric space (X,d) for which the set of all q-admissible subsets of X is compact and normal has at least one fixed point.

The concept of 1-local retract is due to Pouzet [4, p.4] and it has been investigated in detail by Khamsi and others (see for instance [3] and [4]). Our investigations are done in parallel with the well-known metric theory of 1-local retract (see [4]) and they confirm the surprising fact that many classical results about 1-local retract in metric spaces do not make essential use of the symmetry of the metric and thus still hold in quasi-metric spaces.

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2. Basic definitions and results

We begin by recalling the most important definitions that we shall use in this article.

Definition 2.1. Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then, d is called a *quasi-pseudometric* on X if

(a) $d(x, x) = 0$ whenever $x \in X$,

(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

We shall say that d is a T_0-*quasi-metric* provided that d also satisfies the following condition: For each $x, y \in X$,

$d(x, y) = 0 = d(y, x)$ implies that $x = y$.

Remark 2.2. Let d be a quasi-pseudometric on a set X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate quasi-pseudometric of* d. As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric*. Note that for any (T_0-)quasi-pseudometric d, $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric).

Let (X, d) be a quasi-pseudometric space. For each $x \in X$ and $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ denotes the *open* ϵ*-ball* at x. The collection of all "open" balls yields a base for a topology $\tau(d)$. It is called the *topology induced by* d on X. Similarly we set for each $x \in X$ and $\epsilon \geq 0$, $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$. Note that this latter set is $\tau(d^t)$-closed, but not $\tau(d)$-closed in general.

If $a, b \in \mathbb{R}$, we shall put $a \dot{-} b = \max\{a - b, 0\}$. Note that $u(x, y) = x \dot{-} y$ with $x, y \in \mathbb{R}$ defines a T_0 quasi-metric on the set $\mathbb{R}$ of the reals.

A map $f : (X, d) \to (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called an *isometry* provided that $e(f(x), f(y)) = d(x, y)$ whenever $x, y \in X$.

Two quasi-pseudometric spaces (X, d) and (Y, e) will be called *isometric* provided that there exists a bijective isometry $f : (X, d) \to (Y, e)$.

A map $f : (X, d) \to (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called *nonexpansive* provided that $e(f(x), f(y)) \leq d(x, y)$ whenever $x, y \in X$.

Let (X, d) be a T_0-quasi-metric space. For a nonempty bounded subspace A of X, we set:

$$r_x(A)_d := \sup\{d(x, y) : y \in A\}, \text{where} \quad x \in X$$

and

$$r_x(A)_{d^{-1}} := \sup\{d^{-1}(x, y) : y \in A\}, \text{where} \quad x \in X.$$

Moreover, let

$$r_x(A) := r_x(A)_d \vee r_x(A)_{d^{-1}}, \text{where} \quad x \in X$$

and

$$r(A) := \inf\{r_x(A) : x \in A\}.$$

Also, we set

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}.$$

Furthermore

$$C(A) := \{x \in A : r_x(A) = r(A)\}.$$

Finally,

$$\operatorname{cov}(A)_d := \cap\{C_d(x, r) : A \subseteq C_d(x, r), x \in X, r \geq 0\}$$

and

$$\operatorname{cov}(A)_{d^{-1}} := \cap\{C_{d^{-1}}(x, s) : A \subseteq C_{d^{-1}}(x, s), x \in X, s \geq 0\}$$

and

$$\operatorname{bicov}(A) := \operatorname{cov}(A)_d \cap \operatorname{cov}(A)_{d^{-1}}.$$

Note that the values of $\operatorname{diam}(A)$,$r_x(A)$, $r(A)$ and $C(A)$ do not change when defined for the space (X, d^s) instead of (X, d).

Remark 2.3. ([7, Remark 4.1.1]) Let (X, d) be a T_0-quasi-metric space. Let A be a nonempty bounded subset in X. Then

$$\operatorname{cov}(A)_{d^s} = \cap\{C_{d^s}(x, r) : A \subseteq C_{d^s}(x, r), x \in X, r \geq 0\}.$$

Obviously we have $\operatorname{bicov}(A) \subseteq \operatorname{cov}(A)_{d^s}$.

Example 2.4. ([6, Example 1]) Let $X = [0, 1] \times [\frac{1}{4}, \frac{3}{4}]$ be equipped with the T_0-quasi-metric defined by $D((\alpha, \beta), (\alpha', \beta')) = (\alpha \dot{-} \alpha') \vee (\beta \dot{-} \beta')$ whenever $(\alpha, \beta), (\alpha', \beta') \in X$.

Consider $A = \{(0, \frac{1}{2}), (1, \frac{1}{2})\} \subseteq X$. Then $\operatorname{bicov}(A)$ is equal to the line segment from $x = (0, \frac{1}{2})$ to $y = (1, \frac{1}{2})$. This follows from the fact that for each $\epsilon \in [0, \frac{1}{4}]$, $y \in C_D(x, \epsilon) = [0, 1] \times [\frac{1}{2} - \epsilon, \frac{3}{4}]$ and $x \in C_{D^{-1}}(y, \epsilon) = [0, 1] \times [\frac{1}{4}, \frac{1}{2} + \epsilon]$ and that segment is a subset of any set of the form $C_D(a, r) \cap C_{D^{-1}}(b, s)$ for which $\{x, y\} \subseteq C_D(a, r) \cap C_{D^{-1}}(b, s)$. Indeed assume that z belongs to this segment. Then $D(z, y) = 0 = D(x, z)$ and therefore $z \in C_D(a, r) \cap C_{D^{-1}}(b, s)$ by the triangle inequality.

On the other hand $\operatorname{cov}(A)_{d^s} = X$, since $\{x, y\} \subseteq C_{D^s}(z, \epsilon)$ with $z \in X$ implies that $\epsilon \leq \frac{1}{2}$. Indeed assume that $z = (a, b)$. Then $a \leq D^s((a, b), (0, \frac{1}{2})) \leq \epsilon$ and $1 - a \leq D^s((a, b), (a, \frac{1}{2})) \leq \epsilon$. Thus $\epsilon \geq \max\{a, 1 - a\} \geq \frac{1}{2}$. It follows that $X \subseteq C_{D^s}(z, \epsilon)$, because the interval $[\frac{1}{4}, \frac{3}{4}]$ has length $\frac{1}{2}$. Therefore $\operatorname{cov}_{D^s}(A) = X$.

The following lemma should be compared with [3, Lemma 4.1].

Proposition 2.5. (Compare [7, Lemma 4.1.1]) *Let A be a nonempty bounded subspace of a T_0-quasi-metric space (X,d). Then:*

(1) *$bicov(A) = \bigcap_{x\in A}(C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}}))$.*
(2) *$r_x(bicov(A)) = r_x(A)$, whenever $x \in X$.*
(3) *$r(bicov(A)) \leq r(A)$.*
(4) *$diam(bicov(A)) = diam(A)$.*

Proof. (1) Let $x\in X$. For $y\in A$, we have $d(x,y) \leq \sup\{d(x,y) : y\in X\}$. Then $d(x,y) \leq r_x(A)_d$ which implies $y \in C_d(x, r_x(A)_d)$. Hence $A \subseteq C_d(x, r_x(A)_d)$ whenever $x \in X$. It must therefore be the case that $\text{cov}(A)_d \subseteq C_d(x, r_x(A)_d)$ whenever $x \in X$.

Similarly one can show that $A \subseteq C_{d^{-1}}(x, r_x(A)_{d^{-1}})$ whenever $x \in X$. We then have $\text{cov}(A)_{d^{-1}} \subseteq C_{d^{-1}}(x, r_x(A)_{d^{-1}})$ whenever $x \in X$.

Then

$$\text{bicov}(A) \subseteq \bigcap_{x\in X}(C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}})). \tag{2.1}$$

On the other hand, suppose that $A \subseteq C_d(x,r)$ and $A \subseteq C_{d^{-1}}(x,s)$ for some $x \in X$ and $r,s \geq 0$. For any $y \in A$, we have $d(x,y) \leq r$ and $d^{-1}(x,y) \leq s$ which implies $r_y(A)_d \leq r$ and $r_y(A)_{d^{-1}} \leq s$. Thus $C_d(x, r_x(A)_d) \subseteq C_d(x,r)$. Hence $C_d(x, r_x(A)_d) \subseteq \text{cov}(A)_d$ whenever $x \in X$.

Similarly one can show that $C_d(x, r_x(A)_{d^{-1}}) \subseteq \text{cov}(A)_{d^{-1}}$ whenever $x \in X$.

Hence $C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}}) \subseteq \text{bicov}(A)$ whenever $x \in X$.

Furthermore

$$\bigcap_{x\in X}(C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}})) \subseteq \text{bicov}(A). \tag{2.2}$$

Combination of (2.1) and (2.2) yields

$$\text{bicov}(A) = \bigcap_{x\in X}(C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}})).$$

(2) By (1) we have that

$$r_x(\text{bicov}(A)) = \sup\{d(x,y) : y \in \bigcap_{x\in X}(C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}})).$$

In particular, $y \in \text{bicov}(A)$ implies that $y \in C_d(x, r_x(A)_d)$ and $y \in C_{d^{-1}}(x, r_x(A)_{d^{-1}})$ whenever $x \in X$.

Hence $d(x,y) \leq r_x(A)_d$ and $d^{-1}(x,y) \leq r_x(A)_{d^{-1}}$, which implies

$$r_x(\text{bicov}(A))_d \leq r_x(A)_d \leq r_x(A)$$

and

$$r_x(\text{bicov}(A))_{d^{-1}} \leq r_x(A)_{d^{-1}} \leq r_x(A).$$

Altogether we have $r_x(\text{bicov}(A)) = r_x(\text{bicov}(A))_d \vee r_x(\text{bicov}(A))_{d^{-1}} \leq r_x(A)$. The reverse inequality is obvious since $A \subseteq \text{bicov}(A)$.

(3) This is immediate from the definition of r and property (2).

The proof of (4) can be completed similarly as in the proof of [3, Lemma 4.1]. Moreover, $\operatorname{diam}(A)$ is a concept from symmetric topology. $\square$

Let us mention that it has been proved in [7] that if (X, d) is a q-hyperconvex space in Proposition 2.5, the assertion (3) is an equality.

We next define a q-admissible subset of a T_0-quasi-metric space similarly to [3, Definition 4.2].

Definition 2.6. ([7, Definition 4.1.1]) Let (X, d) be a T_0-quasi-metric space. A nonempty bounded subset D of X is *q-admissible* if $D = \operatorname{bicov}(D)$.

The collection of all q-admissible subsets of a T_0-quasi-metric space (X, d) will be denoted by $\mathcal{A}_q(X)$.

Remark 2.7. ([7, Remark 4.1.2]) Let (X, d) be a T_0-quasi-metric space.
(a) Note that a subset of X is q-admissible if and only if it can be written as the intersection of a family of sets of the form $C_d(x, r) \cap C_{d^{-1}}(x, s)$ with $r, s \geq 0$ and $x \in X$. For this reason, the family $\mathcal{A}_q(X)$ is closed under nonempty intersection of nonempty families.
(b) For any $A \in \mathcal{A}_q(X)$ let $\delta = diam(A)$. We have that

$$C(A) = \bigcap\nolimits_{a \in A} (C_d(a, \frac{\delta}{2}) \cap C_{d^{-1}}(a, \frac{\delta}{2})) \cap A \in \mathcal{A}_q(X).$$

Moreover, $\operatorname{diam}(C(A)) \leq \operatorname{diam}(A)/2$. So we have $A = C(A)$ if and only if $A \in \mathcal{A}_q(X)$ and $\operatorname{diam}(A) = 0$, i.e. A is reduced to one point.

3. Compactness and normal structure

The following definitions can be found in [3].

Definition 3.1. ([3, Definition 5.1]) Consider a T_0-quasi-metric space (X, d). Then, $\mathcal{A}_q(X)$ is said to be *compact* if every descending chain of nonempty members of $\mathcal{A}_q(X)$ has nonempty intersection.

Definition 3.2. ([3, Definition 5.2]) Consider a T_0-quasi-metric space (X, d). Then, $\mathcal{A}_q(X)$ is said to be *normal*(or *have normal structure*) if $r(D) < diam(D)$ whenever $D \in \mathcal{A}_q(X)$ and $diam(D) > 0$.

The normality of $\mathcal{A}_q(X)$ is equivalent to saying that: If $D \in \mathcal{A}_q(X)$ and has more than one point then there exists two numbers r, s with $r < diam(D)$ and $s < diam(D)$ and a point $z \in D$ such that $D \subseteq C_d(z, r) \cap C_{d^{-1}}(z, s)$.

Lemma 3.3. (Compare [3, Lemma 5.1]) *Suppose (X, d) is a nonempty bounded T_0-quasi-metric space for which $\mathcal{A}_q(X)$ is compact and normal. Let $A \subseteq X$ which is a 1-local retract of X. Then $r(bicov(A)) = r(A)$ for each $A \in \mathcal{A}_q(A)$.*

The following theorem gives a quasi-metric variant of [3, Theorem 5.1] (compare also [6, Theorem 1]).

Theorem 3.4. *Suppose (X, d) is a nonempty bounded T_0-quasi-metric space for which $\mathcal{A}_q(X)$ is compact and normal. Then every nonexpansive $T : (X, d) \to (X, d)$ has at least one fixed point.*

Proof. Consider the following set $\mathcal{F} = \{D \in \mathcal{A}_q(X) : D \neq \emptyset \text{ and } T : D \to D\}$. We have that $X \in \mathcal{F}$, therefore $\mathcal{F} \neq \emptyset$. One can partially order $\mathcal{F}$ by $A \leq B$ if and only if $B \subseteq A$ whenever $A, B \in \mathcal{F}$. We first observe that if $\mathcal{C}$ is a chain in $\mathcal{F}$ then $\bigcap \mathcal{C} \in \mathcal{A}_q(X)$. To show that $\bigcap \mathcal{C} \in \mathcal{F}$, it suffices to prove that $\bigcap \mathcal{C} \neq \emptyset$. Let $\mathcal{C} \in \mathcal{F}$, we say that $\mathcal{C} = \{C_\alpha\}_{\alpha \in A}$ where C_α can be written as the intersection of balls in X, say $C_\alpha = \bigcap_{i_\alpha \in I_\alpha} C_d(x_{i_\alpha}, r_{i_\alpha}) \cap C_{d^{-1}}(x_{i_\alpha}, s_{i_\alpha})$. Then, by compactness of $\mathcal{A}_q(X)$ we have

$$\bigcap_{\alpha \in A} C_\alpha = \bigcap_{\alpha \in A} \Big(\bigcap_{i_\alpha \in I_\alpha} C_d(x_{i_\alpha}, r_{i_\alpha}) \cap C_{d^{-1}}(x_{i_\alpha}, s_{i_\alpha})\Big) \neq \emptyset.$$

Moreover, $\bigcap_{\alpha \in A} C_\alpha \subseteq C_\alpha$ which means that $\mathcal{F}$ is bounded above by $\bigcap_{\alpha \in A} C_\alpha$ then by Zorn's lemma $\mathcal{F}$ has a maximal element. Let D be a maximal element of $\mathcal{F}$, then $D \neq \emptyset$ and $T : D \to D$.

From Lemma 2.5 (1) we have

$$\text{bicov}(T(D)) = \bigcap_{x \in X} (C_d(x, r_x(T(D))_d) \cap C_{d^{-1}}(x, r_x(T(D))_{d^{-1}})).$$

Furthermore, since $r_x(T(D))_d \leq r_x(D)_d$ and $r_x(T(D))_{d^{-1}} \leq r_x(D)_{d^{-1}}$, it follows that

$$\text{bicov}(T(D)) \subseteq \bigcap_{x \in X} (C_d(x, r_x(D)_d) \cap C_{d^{-1}}(x, r_x(D)_{d^{-1}})) = D$$

and $T : \text{bicov}(T(D)) \to \text{bicov}(T(D))$. Then by minimality of D, we have $D = \text{bicov}(T(D))$.

Suppose that $\text{diam}(D) > 0$, then there exists r, s with $r < \text{diam}(D)$ and $s < \text{diam}(D)$ and $x \in D$ such that $D \subseteq C_d(x, r) \cap C_{d^{-1}}(x, s)$. Then the set

$$C = \{x \in D : D \subseteq C_d(x, r) \cap C_{d^{-1}}(x, s)\} \neq \emptyset.$$

Moreover, we have

$$C = \Big(\bigcap_{x \in D} C_d(x, r) \cap C_{d^{-1}}(x, s)\Big) \cap D,$$

which implies that $C \in \mathcal{A}_q(X)$. Consider $z \in C$, then if $x \in D$ we have

$$d(T(x), T(z)) \leq d(x, z) \leq r$$

and

$$d(T(z), T(x)) \leq d(z, x) \leq s.$$

Therefore, $T(x) \in C_d(T(z), r)$ and $T(x) \in C_{d^{-1}}(T(z), s)$ whenever $x \in D$. Hence $T(D) \subseteq C_d(T(z), r)$ which implies that

$$\mathrm{cov}(T(D))_d \subseteq C_d(T(z), r).$$

Similarly we have

$$\mathrm{cov}(T(D))_{d^{-1}} \subseteq C_{d^{-1}}(T(z), s).$$

Therefore

$$\mathrm{bicov}(T(D)) \subseteq C_d(x, r) \cap C_{d^{-1}}(x, s).$$

Since $D = \mathrm{bicov}(T(D))$, we that $D \subseteq C_d(x, r) \cap C_{d^{-1}}(x, s)$ which implies that $T(z) \in C$. So $T : C \to C$.

Moreover, if $z, w \in C$ then $d(z, w) \leq r$ and $d^{-1}(z, w) \leq s$, so $\mathrm{diam}(C) \leq r < \mathrm{diam}(D)$ and $\mathrm{diam}(C) \leq s < \mathrm{diam}(D)$. This shows that C is a proper subset of D.

Since $C \in \mathcal{A}_q(X)$ and $T : C \to C$ this contradicts the minimality of D. We then conclude that $\mathrm{diam}(D) = 0$, therefore D is reduced to one point which must be a fixed point of T. □

4. One-local retract in quasi-metric space

We next define 1-local retract subset of a T_0-quasi-metric space and it can be compared with [4, p.7] or [3, p.103] in metric setting.

Definition 4.1. Let (X, d) be a T_0-quasi-metric space. A subset A of X is said to be a 1-*local retract* of (X, d) if for each family

$$\{C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)\}_{i \in I}$$

of balls, where $r_i, s_i \geq 0$ and $x_i \in A$ whenever $i \in I$ for which

$$\bigcap_{i \in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \neq \emptyset$$

it is the case that

$$A \cap \bigcap_{i \in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \neq \emptyset.$$

Proposition 4.2. *Let (X, d) be a quasi-pseudometric space and A subset of X.*

(a) If A is 1-local retract of (X, d), then A is 1-local retract of (X, d^{-1}).

(b) If A is 1-local retract of (X, d), then A is 1-local retract of the metric space (X, d^s).

Proof. (a) The statement immediately follow from the definition.

(b) Suppose that A is 1-local retract of (X, d).
Let $\{C_{d^s}(x_i, r_i)\}_{i\in I}$ a family of balls with $x_i \in A$, $r_i \geq 0$ whenever $i \in I$ such that

$$\bigcap_{i\in I} C_{d^s}(x_i, r_i) \neq \emptyset.$$

It follows that $\emptyset \neq A \cap \bigcap_{i\in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) = A \cap \bigcap_{i\in I} C_{d^s}(x_i, r_i)$. Hence A is 1-local retract of (X, d^s). □

Theorem 4.3. *Suppose (X, d) is a nonempty bounded T_0-quasi-metric space for which $\mathcal{A}_q(X)$ is compact and normal and let $T : (X, d) \to (X, d)$ be a nonexpansive map. Then the fixed point set $Fix(T)$ of T is a nonempty 1-local retract of (X, d). Moreover, $\mathcal{A}_q(Fix(T))$ is compact and normal.*

Proof. Note first that $\mathrm{Fix}(T) \neq \emptyset$ by Theorem 3.4. To show that $\mathrm{Fix}(T)$ is a 1-local retract of (X, d), let us consider a family of balls

$$\{C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)\}_{i\in I}$$

where $x_i \in \mathrm{Fix}(T)$ and $r_i, s_i \geq 0$ whenever $i \in I$ such that

$$S = \bigcap_{i\in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \neq \emptyset.$$

By nonexpansivity of T, we have that $T : S \to S$ and since S is q-admissible implies that $\mathcal{A}_q(S)$ is compact and normal.

Furthermore, again by Theorem 3.4, $T : S \to S$ has a fixed point, then $\mathrm{Fix}(T) \cap S \neq \emptyset$. Therefore, the fixed point set $\mathrm{Fix}(T)$ is a 1-local retract of (X, d). Moreover, the definition of a 1-local retract assures that $\mathcal{A}_q(\mathrm{Fix}(T))$ is compact.

We have to show that $\mathcal{A}_q(\mathrm{Fix}(T))$ is normal. Let $A \in \mathcal{A}_q(\mathrm{Fix}(T))$. From Proposition 2.5 (4) we have

$$\mathrm{diam}(\mathrm{bicov}(A)) = \mathrm{diam}(A)$$

and

$$r(\mathrm{bicov}(A)) = r(A)$$

by Lemma 3.3. Moreover, the normality of $\mathcal{A}_q(X)$ implies that

$$r(\mathrm{bicov}(A)) < \mathrm{diam}(\mathrm{bicov}(A)).$$

Then, it follows that $r(A) < \mathrm{diam}(A)$. □

Theorem 4.4. *Let (X, d) be a nonempty bounded T_0-quasi-metric space such that $\mathcal{A}_q(X)$ is compact and normal. Then any commuting family of nonexpansive maps $(T_i)_{i\in\{1,\cdots,n\}}$, with $T_i : (X, d) \to (X, d)$, has a nonempty common fixed point set. Moreover, the common fixed point set $\bigcap_{i=1}^{n} Fix(T_i)$ is a 1-local retract of (X, d).*

Proof. We first show that $\bigcap_{i=1}^{n}\mathrm{Fix}(T_i) \neq \emptyset$. For any $i \in \{1, \cdots, n\}$, we have $\mathrm{Fix}(T_i) \neq \emptyset$ by Theorem 4.3. Hence there is $x \in X$ such that $T_i(x) = x$ whenever $i \in \{1, \cdots, n\}$.

Since T_1 and T_2 commute. We show that $T_2(\mathrm{Fix}(T_1)) \subseteq \mathrm{Fix}(T_1)$: Indeed, if for some $x \in X$, we have $x = T_1(x)$, then $T_2(x) = T_2(T_1(x)) = T_1(T_2(x))$. So $T_2(x) \in \mathrm{Fix}(T_1)$.

We Conclude that $T_2 : \mathrm{Fix}(T_1) \to \mathrm{Fix}(T_1)$ has a fixed point $y \in \mathrm{Fix}(T_1)$, which is a fixed point of T_1 and T_2. Hence by indiction for each finite family $(T_i)_{i\in\{1,\cdots,n\}}$ of nonexpansive self-maps on X the set of common fixed point $\bigcap_{i=1}^{n}\mathrm{Fix}(T_i)$ is nonempty.

Now, we need to show that $\bigcap_{i=1}^{n}\mathrm{Fix}(T_i)$ is a 1-local retract of (X, d). Let a family of balls $\{C_d(x_j, r_j) \cap C_{d^{-1}}(x_j, s_j)\}_{j\in J}$ where $x_j \in \bigcap_{i=1}^{n}\mathrm{Fix}(T_i)$ and $r_j, s_j \geq 0$ whenever $j \in J$ such that

$$U = \bigcap_{j\in J} C_d(x_j, r_j) \cap C_{d^{-1}}(x_j, s_j) \neq \emptyset.$$

For any $i \in \{1, \cdots, n\}$, we have that $T_i : U \to U$ is a nonexpansive map. Also, since U is q-admissible, $\mathcal{A}_q(U)$ is compact and normal. Therefore by Theorem 3.4, T_i has a fixed point in U, that is, $\mathrm{Fix}(T_i) \cap U \neq \emptyset$. Therefore $\bigcap_{i=1}^{n}\mathrm{Fix}(T_i) \cap U \neq \emptyset$. This proves that $\bigcap_{i=1}^{n}\mathrm{Fix}(T_i)$ is a 1-local retract of (X, d). □

Theorem 4.5. *Let (X, d) be a nonempty T_0-quasi-metric space such that $\mathcal{A}_q(X)$ is compact and normal. Suppose that $(H_i)_{i\in I}$ be a descending family of 1-local retracts of (X, d), where we assume that I is totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ holds if and only if $H_{i_2} \subseteq H_{i_1}$. Then $\bigcap_{i\in I} H_i$ is nonempty and is a 1-local retract of (X, d).*

Proof. We start by showing that $\bigcap_{i\in I} H_i \neq \emptyset$, since (X, d^s) is a nonempty metric space and $(H_i)_{i\in I}$ is a descending chain of 1-local retracts of (X, d^s) by Proposition 4.2. By the well-known result of Khamsi [4, Theorem 6], we conclude that $\bigcap_{i\in I} H_i \neq \emptyset$.

We next show that $H = \bigcap_{i\in I} H_i$ is 1-local retract. Let a family of balls $\{C_d(x_j, r_j) \cap C_{d^{-1}}(x_j, s_j)\}_{j\in J}$, where $r_j, s_j \geq 0$ and $x_j \in H$ whenever $j \in J$ for which

$$\bigcap_{j\in J} (C_d(x_j, r_j) \cap C_{d^{-1}}(x_j, s_j)) \neq \emptyset.$$

Fix $i \in I$, since H_i is 1-local retract of (X, d) and since $x_j \in H_i$ whenever $j \in J$, hence $\mathcal{D}_i = \bigcap_{j\in J} C_d(x_j, r_j) \cap C_{d^{-1}}(x_j, s_j) \cap H_i \neq \emptyset$. Therefore

$$\emptyset \neq \bigcap_{i\in I} \mathcal{D}_i = \bigcap_{i\in I} [\bigcap_{j\in J} (C_d(x_j, r_j) \cap C_{d^{-1}}(x_j, s_j)) \cap H_i] =$$
$$= \bigcap_{j\in J} (C_d(x_j, r_j) \cap C_{d^{-1}}(x_j, s_j)) \cap \bigcap_{i\in I} H_i,$$

since $(\mathcal{D}_i)_{i\in I}$ is descending. This proves that $H = \bigcap_{i\in I} H_i$ is 1-local retract of (X, d). □

Our next lemma is a consequence of Theorem 4.5 and an application of Zorn's lemma. It can be compared to [6, Lemma 1] and [1, Corollary 8].

Lemma 4.6. *If $(H_\alpha)_{\alpha\in S}$ is family of* 1*-local retract of subsets of a nonempty T_0-quasi-metric space (X,d) such that $\bigcap_{\alpha\in F} H_\alpha$ is* 1*-local retract of (X,d) whenever $F\subseteq S$ is finite, then the intersection $\bigcap_{\alpha\in S} H_\alpha$ is* 1*-local retract of (X,d).*

Theorem 4.7. *Let (X,d) be a nonempty T_0-quasi-metric space such that $\mathcal{A}_q(X)$ is compact and normal. Then any commuting family of nonexpansive maps $(T_i)_{i\in I}$, with $T_i:(X,d)\to(X,d)$, has a common fixed point. Furthermore, the common fixed point set $\bigcap_{i\in I} Fix(T_i)$ is a* 1*-local retract of (X,d).*

Proof. Indeed, by nonexpansivity of T_i whenever $i\in I$, we have

$$d(T_i(x),T_i(y))\leq d(x,y)\leq d^s(x,y)$$

and

$$d(T_i(y),T_i(x))\leq d(y,x)\leq d^s(x,y)$$

whenever $x,y\in X$. Hence $d^s(T_i(x),T_i(y))\leq d^s(x,y)$ whenever $x,y\in X$. Moreover, we have that (X,d^s) is a nonempty metric space and the map $T_i:(X,d^s)\to(X,d^s)$ is nonexpansive whenever $i\in I$. Furthermore, $(H_i)_{i\in I}$ is a descending chain of 1-local retracts of (X,d^s) by Proposition 4.2. Therefore, by Theorem 3.4 each T_i has a fixed point. Hence there is $x\in X$ such that $T_i(x)=x$. We now show, given any $j\in I$, we have that $T_j(\mathrm{Fix}(T_i))\subseteq \mathrm{Fix}(T_i)$. Then if for some $x\in X$, we have $x=T_i(x)$, then $T_j(x)=T_j(T_i(x))=T_i(T_j(x))$. Hence $T_j(x)\in\mathrm{Fix}(T_i)$.

Therefore, the map $T_j:\mathrm{Fix}(T_i)\to\mathrm{Fix}(T_i)$ has a fixed point by Theorem 3.4, which is the common fixed point of T_i and T_j. Moreover, the set of common fixed points of T_i and T_j is 1-local retract by Theorem 4.3. Thus by induction for each finite family $(T_i)_{i\in F}$ of nonexpansive self-maps on (X,d) the set of common fixed points is 1-local retract of (X,d).

By Lemma 4.6, we conclude that $\bigcap_{i\in I}\mathrm{Fix}(T_i)$ is a 1-local retract of (X,d) since $\bigcap_{i\in F}\mathrm{Fix}(T_i)$ is a 1-local retract of (X,d) whenever F is a nonempty finite subset I. □

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