http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 46, 2015

Pages 1–14

http://topology.nipissingu.ca/tp/

Completeness and Related Properties of the Graph Topology

by

L'ubica Holá and László Zsilinszky

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 46 (2015)
Pages 1-14

COMPLETENESS AND RELATED PROPERTIES OF THE GRAPH TOPOLOGY

ĽUBICA HOLÁ AND LÁSZLÓ ZSILINSZKY

Abstract. The graph topology τ_Γ is the topology on the space $C(X)$ of all continuous functions defined on a Tychonoff space X inherited from the Vietoris topology on $X \times \mathbb{R}$ after identifying continuous functions with their graphs. It is shown that all completeness properties between complete metrizability and hereditary Baireness coincide for the graph topology if and only if X is countably compact; however, the graph topology is α-favorable in the strong Choquet game, regardless of X. Analogous results are obtained for the fine topology on $C(X)$. Pseudocompleteness, along with properties related to 1st and 2nd countability of $(C(X), \tau_\Gamma)$, is also investigated.

1. Preliminaries

There has been a plethora of topologies studied on the space $C(X, Y)$ of continuous functions $f : X \to Y$; most of these topologies can be described as having base elements of the form

$$F_U = \{f \in C(X, Y) : \text{graph}(f) \subseteq U\},$$

where U ranges over a specific collection of open subsets of $X \times Y$. If X is T_1, the best known topologies of this kind are the topology of pointwise convergence τ_p, the compact-open topology τ_k, the topology of uniform convergence τ_u, and the fine topology τ_w [23]. We will be interested in

---

2010 *Mathematics Subject Classification.* Primary 54C35; Secondary 54E52, 54B20, 54C05.

Key words and phrases. Banach-Mazur game, countable compact, fine topology, graph topology, hereditarily Baire space, pseudocompact, pseudocomplete space, strong Choquet game, (strongly) α-favorable space.

The first author would like to thank the support of the grant APVV-0269-11 and Vega 2/0177/12.

the finest of these topologies, the so-called *graph topology* τ_Γ of Naimpally [22], which is generated by the sets F_U when U ranges over all the nonempty open subsets of $X \times Y$. The symbol $C_\Gamma(X,Y)$ will stand for the space $(C(X,Y),\tau_\Gamma)$, and we will write $C_\Gamma(X)$ instead of $C_\Gamma(X,\mathbb{R})$. Denote by $cl_\Gamma(A)$ the τ_Γ-closure of $A \subset C(X,Y)$. It is the purpose of this paper to investigate completeness and related topological properties of the graph topology. It will be shown (Theorem 2.1) that $C_\Gamma(X)$ is hereditarily Baire if and only if $C_\Gamma(X)$ is completely metrizable if and only if X is countably compact; so even the weakest closed hereditary completeness property—hereditary Baireness—of $C_\Gamma(X)$ imposes a strong restriction on X, however, there is another strong (non-closed hereditary) completeness property, that of strong α-favorability, which $C_\Gamma(X)$ always possesses regardless of X (Theorem 3.1). Analogous results are established for the fine topology on $C(X)$ (Theorem 2.2 and Theorem 3.2). Various (non-completeness) properties, from Arhangel'skii's p-spaces property to sequentiality, countable tightness, or the k-space property for $C_\Gamma(X)$ are shown to be equivalent to (complete) metrizability of $C_\Gamma(X)$ (Theorem 5.1). As a byproduct of these results, strongly α-favorable spaces can be constructed lacking all the discussed topological properties. Pseudocompleteness, along with properties related to the 2nd countability of the graph topology, is also investigated.

There are various ways of looking at the graph topology; for example, if X is T_1 and Y is T_2, then $C_\Gamma(X,Y)$ is the relative Vietoris topology [21] inherited from the hyperspace of nonempty closed subsets of $X \times Y$ after identifying elements of $C(X,Y)$ with their graphs. Indeed, since the F_U's form the upper Vietoris topology on $C(X,Y)$, we just need to show that a typical lower Vietoris open set is open in $C_\Gamma(X,Y)$: Let V and W be open subsets of X and Y, respectively, then

$$\{f \in C(X,Y) : \mathrm{graph}(f) \cap (V \times W) \neq \emptyset\} = \bigcup_{x \in V} F_{(X \times W) \cup ((X \setminus \{x\}) \times Y)} \in \tau_\Gamma.$$

Note that other hyperspace topologies also coincide with the graph topology on $C(X,Y)$, as was demonstrated in [1].

Another useful way of looking at τ_Γ was first observed by Eric K. van Douwen [7, Lemma 8.3.] for $C_\Gamma(X)$; since his proof works for any metrizable Y, we will state it in this more general form. First, we introduce some notation: denote by $C^+(X)$ ($LSC^+(X)$, respectively), the strictly positive real-valued continuous (lower semicontinuous, respectively) functions defined on the topological space X. Given a function $\varepsilon : X \to (0,\infty)$, a metric space (Y,d), and $f \in C(X,Y)$, define

$$B(f,\varepsilon) = \{g \in C(X,Y) : d(f(x),g(x)) < \varepsilon(x) \text{ for all } x \in X\}.$$

Proposition 1.1. *Let X be a topological space and let (Y, d) be a metric space. The collection*

$$\mathcal{B}_\Gamma = \{B(f, \varepsilon) : f \in C(X, Y), \varepsilon \in LSC^+(X)\}$$

is a base for $C_\Gamma(X, Y)$.

The previous result shows how close the graph topology is to the *fine topology* τ_w on $C(X, Y)$ with (Y, d) metric, in short, denoted as $C_w(X, Y)$; indeed, $C_w(X, Y)$ also has base elements of the form $B(f, \varepsilon)$, but with $\varepsilon \in C^+(X)$. The fine topology (also called the m-topology [7]) has been thoroughly investigated in the past; see [20], [19], [6], [14]. In particular, it is known that $C_w(X, Y)$ is sensitive to the metric d of the range space Y [20] which immediately shows a difference with the graph topology, as τ_Γ is clearly independent of the compatible metrics of Y (for completeness, when $Y = \mathbb{R}$, we will assume that $\mathbb{R}$ carries the Euclidean metric). Moreover, the following is not hard to see.

Proposition 1.2 ([7])**.** *Let X be a topological space and let Y be a metric space. The following are equivalent:*

1. $C_\Gamma(X, Y) = C_w(X, Y)$*;*
2. *X is a cb-space, i.e., for each $\varepsilon \in LSC^+(X)$ there is some $\varphi \in C^+(X)$ with $\varepsilon(x) \geq \varphi(x)$ for all $x \in X$.*

Note that a normal space is a *cb*-space if and only if it is countably paracompact [17] and there are non-normal, locally compact, countably paracompact, non-*cb*-spaces [18, p. 240].

It is also worth reviewing the relationships between the graph topology and the other discussed function space topologies (pointwise, compact-open, uniform).

Proposition 1.3. *Let X be a T_2 space. Then*

1. $\tau_p \subset \tau_k \subset \tau_u \subset \tau_w \subset \tau_\Gamma$*;*
2. $C_\Gamma(X) = C_u(X)$ *if and only if X is countably compact* [11]*;*
3. $C_\Gamma(X) = C_k(X)$ *if and only if X is compact* [22],[20]*;*
4. $C_\Gamma(X) = C_p(X)$ *if and only if X is finite* [20].

If X is regular and Y is T_1 (T_2, respectively), then $C_\Gamma(X, Y)$ is T_1 (T_2, respectively). Concerning regularity of $C_\Gamma(X, Y)$, we have the following example.

Example 1.4. *Let $X = \omega_1$ and $Y = \omega_1 + 1$, both with the order topology. Then $C_\Gamma(X, Y)$ is not regular.*

Proof. Let $f : X \to Y$ be the inclusion function $f(x) = x$ for all $x \in X$ and consider its τ_Γ-neighborhood $F_{\omega_1 \times \omega_1}$. Let $U \subseteq \omega_1 \times \omega_1$ be any open subset with $f \in F_U$. Since the graph of f is the diagonal of $\omega_1 \times \omega_1$, we can find an $\alpha < \omega_1$ so that $(x, y) \in U$ whenever $x, y > \alpha$. Define $g \in C(X, Y)$ via

$$g(x) = \begin{cases} x, & \text{if } x \neq \alpha + 1 \\ \omega_1, & \text{if } x = \alpha + 1. \end{cases}$$

It is not hard to see that $g \in cl_\Gamma(F_U) \setminus F_{\omega_1 \times \omega_1}$. □

As for an easy sufficient condition for regularity of $C_\Gamma(X, Y)$ we can assume that $X \times Y$ is T_4, since then the entire hyperspace of the nonempty closed subsets of $X \times Y$ with the Vietoris topology is regular [21]. If $X \times Y$ is non-normal, $C_\Gamma(X, Y)$ may be non-regular (see the previous example), but it also can be regular (since, if X is compact and Y is regular, then $C_\Gamma(X, Y)$ is regular [16]). Fortunately, most of our results concern $C_\Gamma(X)$, which is a topological group and hence a Tychonoff space if X is T_1.

2. Hereditary Baireness of the Graph and Fine Topologies

One can argue that, aside from compactness, the strongest closed hereditary property is complete metrizability and the weakest such property is hereditary Baireness. Recall that Z is *hereditary Baire* if and only if nonempty closed subspaces of Z are of the 2nd category in themselves if and only if nonempty closed subspaces of Z are Baire spaces; moreover, Z is a *Baire space* [12] if and only if nonempty open subsets are of the 2nd category in themselves if and only if a countable dense open collection in Z has a dense intersection.

Extending results of [13], [19], and [6], we will show that for the graph and fine topologies these properties coincide, and so does any other closed hereditary completeness property in between them. We just include two such well-studied properties, namely *Čech completeness* (being a G_δ-subspace in a compactification [8]) and *sieve completeness* (being a continuous open image of a Čech complete space [26]).

Theorem 2.1. *Let X be a Tychonoff space. The following are equivalent:*

1. $C_\Gamma(X)$ *is completely metrizable,*
2. $C_\Gamma(X)$ *is Čech complete,*
3. $C_\Gamma(X)$ *is sieve complete,*
4. $C_\Gamma(X)$ *is hereditarily Baire,*
5. X *is countably compact.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4) are well known.

(4)⇒(5) If X is not countably compact, there is a countable closed discrete set $\{x_n : n < \omega\}$. For each $n < \omega$, define

$$H_n = \{f \in C(X) :\ \forall k \geq n,\ f(x_k) = 0\},$$

and put $H = \bigcup_{n<\omega} H_n$. Then

- H is closed in $C_\Gamma(X)$: If $f \in C(X) \setminus H$, then $f(x_n) \neq 0$ for infinitely many n (w.l.o.g., all n). Define $U = X \times \mathbb{R} \setminus \{x_n : n < \omega\} \times \{0\}$; then $f \in F_U \subseteq C(X) \setminus H$.
- Each H_n is nowhere dense in H: It is easy to see that H_n is closed in $C_\Gamma(X)$; moreover, assume there is an $X \times \mathbb{R}$-open V such that $\emptyset \neq F_V \cap H \subseteq H_n$ and pick $f \in F_V \cap H_n$. Then $f(x_n) = 0$, so we can find $\varepsilon > 0$ and an X-open neighborhood U of x_n missing $\{x_k : k \neq n\}$ such that

$$U \times (-\varepsilon, \varepsilon) \subseteq V \text{ and } \overline{U} \subseteq f^{-1}((-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})).$$

Define a continuous function $g_0 : X \to [0, \frac{\varepsilon}{2}]$ so that $g_0(x_n) = \frac{\varepsilon}{2}$ and $g_0(x) = 0$ whenever $x \notin U$. Then $g = f + g_0 \in F_V \cap H \setminus H_n$, a contradiction.

(5)⇒(1) If X is countably compact, then each $\varepsilon \in LSC^+(X)$ has a (positive) minimum, so $C_\Gamma(X)$ coincides with the uniform topology on $C(X)$, which is completely metrizable by the sup-metric. □

Results about hereditary Baireness of function spaces are very rare and just partial ([10], [2]). We will show that it is possible to use the idea of Theorem 2.1 to completely characterize hereditary Baireness of the fine topology as well.

Theorem 2.2. *Let X be a Tychonoff space. The following are equivalent:*

(1) $C_w(X)$ *is completely metrizable,*
(2) $C_w(X)$ *is Čech complete,*
(3) $C_w(X)$ *is sieve complete,*
(4) $C_w(X)$ *is hereditarily Baire,*
(5) X *is pseudocompact.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4) are well known.

(4)⇒(5) If X is not pseudocompact, there is a countable collection $\{U_n : n < \omega\}$ of open sets such that $\{\overline{U_n} : n < \omega\}$ is discrete. For $n < \omega$, fix some $x_n \in U_n$ and define

$$H_n = \{f \in C(X) :\ \forall k \geq n\ f(U_k) = \{0\}\}.$$

CLAIM 1. H_n is nowhere dense in $H = \bigcup_{n<\omega} H_n$.
It is easy to see that H_n is closed in $C_w(X)$. Let $f \in H_n$, and consider $B(f, \varepsilon)$ for some $\varepsilon \in C^+(X)$. Then $V = U_n \cap \varepsilon^{-1}((\frac{2}{3}\varepsilon(x_n), \infty))$ is an open neighborhood of x_n. Let $g_0 : X \to [0, \frac{\varepsilon(x_n)}{2}]$ be a continuous function such

that $g_0(x_n) = \frac{\varepsilon(x_n)}{2}$ and $g_0(x) = 0$ whenever $x \notin V$; then $g = f + g_0 \in H_{n+1}$, since $g(x_n) = g_0(x_n) > 0$, and $g = f$ outside of V, so $g(U_k) = \{0\}$ whenever $k \geq n+1$. We have

- $B(g, \frac{\varepsilon}{4}) \subseteq B(f, \varepsilon)$: Let $h \in B(g, \frac{\varepsilon}{4})$. If $x \notin V$, then $|h(x) - f(x)| = |h(x) - g(x)| < \frac{\varepsilon(x)}{4} < \varepsilon(x)$, so $h \in B(f, \varepsilon)$. If $x \in V$, then $|h(x) - f(x)| \leq |h(x) - g(x)| + |g(x) - f(x)| < \frac{\varepsilon(x)}{4} + g_0(x) \leq \frac{\varepsilon(x)}{4} + \frac{\varepsilon(x_n)}{2} < \varepsilon(x)$, so $h \in B(f, \varepsilon)$ again.
- $B(g, \frac{\varepsilon}{4})$ is disjoint to H_n: If $h \in B(g, \frac{\varepsilon}{4})$, then $h(x_n) > g(x_n) - \frac{\varepsilon(x_n)}{4} = \frac{\varepsilon(x_n)}{2} - \frac{\varepsilon(x_n)}{4} = \frac{\varepsilon(x_n)}{4} > 0$, so $h \notin H_n$.

Claim 2. H is closed in $C_w(X)$.

Let $f \in C(X) \setminus H$. Then $f(U_n) \neq \{0\}$ for infinitely many n (w.l.o.g., all n). Let $u_n \in U_n$ be such that $f(u_n) \neq 0$ for all n. Given n, define the continuous function $\varepsilon_n : \overline{U_n} \to [\frac{1}{2}, \frac{|f(u_n)|}{2}]$ if $1 \leq |f(u_n)|$ or $\varepsilon_n : \overline{U_n} \to [\frac{|f(u_n)|}{2}, \frac{1}{2}]$ if $1 > |f(u_n)|$ so that $\varepsilon_n(u_n) = \frac{|f(u_n)|}{2}$ and $\varepsilon_n(\overline{U_n} \setminus U_n) = \{\frac{1}{2}\}$. Finally, define $\varepsilon \in C^+(X)$ as follows:

$$\varepsilon = \begin{cases} \varepsilon_n, & \text{on } \overline{U_n}, \text{ whenever } n < \omega, \\ \frac{1}{2}, & \text{on } X \setminus \bigcup_n \overline{U_n}. \end{cases}$$

If $h \in B(f, \varepsilon)$, then for each $n \in \omega$,

$$|h(u_n)| \geq |f(u_n)| - |f(u_n) - h(u_n)| > |f(u_n)| - \varepsilon_n(u_n) =$$
$$= |f(u_n)| - \frac{|f(u_n)|}{2} = \frac{|f(u_n)|}{2} > 0,$$

so $h \notin H$; thus, $f \in B(f, \varepsilon) \subseteq C(X) \setminus H$.

(5)⇒(1) If X is pseudocompact, then each $\varepsilon \in C^+(X)$ is bounded away from zero, so $C_w(X)$ coincides with the uniform topology on $C(X)$. □

3. Topological Games and the Graph and Fine Topologies

In the *strong Choquet game* $Ch(Z)$ (see [4], [15]), players α and β take turns in choosing objects in the topological space Z with an open base $\mathcal{B}$. β starts by picking (z_0, V_0) from $\mathcal{E} = \{(z, V) \in Z \times \mathcal{B} : z \in V\}$ and α responds by $U_0 \in \mathcal{B}$ with $z_0 \in U_0 \subseteq V_0$. The next choice of β is $(z_1, V_1) \in \mathcal{E}$ with $V_1 \subset U_0$ and again α picks U_1 with $z_1 \in U_1 \subseteq V_1$, etc. Player α wins the run $(z_0, V_0), U_0, \ldots, (z_n, V_n), U_n, \ldots$ provided $\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$; otherwise, β wins. The space Z is called *strongly α-favorable*, provided $Ch(Z)$ is α-favorable, i.e., when α has a winning tactic in $Ch(Z)$, which is a function $t : \mathcal{E} \to \mathcal{B}$ such that α wins every run of the game with $U_n = t(z_n, V_n)$ for all n. The *Banach-Mazur game* $BM(Z)$ (see [14] or the Choquet game in [15]) is played as the strong Choquet game, except

β's choice is only a nonempty open set contained in the previous choice of α. A space Z is *α-favorable*, provided α has a winning tactic in $BM(Z)$. β-favorability of $Ch(Z)$ and $BM(Z)$ can be defined analogously ([27], [14]).

The strong Choquet game is intimately related to the completeness properties considered in theorems 2.1 and 2.2; indeed, a Moore space is sieve complete if and only if it is strongly α-favorable [3, Corollary 3.2]; a metrizable space is completely metrizable if and only if it is strongly α-favorable [4] (Choquet's theorem). On the other hand, a Moore space is hereditarily Baire if and only if the strong Choquet game is not β-favorable (see [27, Corollary 3.3], or [5] for metrizable spaces). Note, however, that neither α-favorability nor β-unfavorability of the strong Choquet game is closed hereditary (the Michael line is one example; see [27] for more). It is clear that strong α-favorability implies α-favorability which, in turn, implies Baireness, since Z is a Baire space if and only if $BM(Z)$ is not β-favorable [15].

Theorem 3.1. *Let X be a topological space and let Y be completely metrizable. Then $C_\Gamma(X,Y)$ is strongly α-favorable.*

Proof. Let d be a compatible complete metric for Y. Define a tactic t for α in the strong Choquet game on $C_\Gamma(X,Y)$ as follows: given $U = (f, B(f,\varepsilon))$, put $t(U) = B(f, \frac{\min(1,\varepsilon)}{2})$.

CLAIM. t is a winning tactic for α.
Consider a run of the strong Choquet game compatible with t:

$$(f_0, B_0(f_0,\varepsilon_0)),\ t(f_0, B_0(f_0,\varepsilon_0)), \dots (f_n, B_n(f_n,\varepsilon_n)),\ t(f_n, B_n(f_n,\varepsilon_n)), \dots$$

Since d is complete, the sequence (f_n) uniformly converges to some $f \in C(X,Y)$. Also, $f_m \in t(B_n(f_n,\varepsilon_n)) \subseteq B_n(f_n, \frac{\varepsilon_n}{2})$ for each $m \geq n$, so $d(f_n(x), f_m(x)) < \frac{\varepsilon_n(x)}{2}$ for all $x \in X$. Fix some $x \in X$ and choose $m_0 \geq n$ with $d(f_{m_0}(x), f(x)) < \frac{\varepsilon_n(x)}{2}$. Then

$$d(f(x), f_n(x)) \leq d(f(x), f_{m_0}(x)) + d(f_{m_0}(x), f_n(x)) <$$
$$< \frac{\varepsilon_n(x)}{2} + \frac{\varepsilon_n(x)}{2} = \varepsilon_n(x),$$

so $f \in B_n(f_n, \varepsilon_n)$ for each n; thus, α wins. □

A similar argument also gives the following theorem.

Theorem 3.2. *Let X be a topological space and let Y be a complete metric space. Then $C_w(X,Y)$ is strongly α-favorable.*

Corollary 3.3. *Let X be a topological space.*

1. *If Y is completely metrizable, then $C_\Gamma(X,Y)$ is α-favorable, and hence a Baire space.*
2. *If Y is a complete metric space, then $C_w(X,Y)$ is α-favorable, and hence a Baire space.*

4. PSEUDOCOMPLETENESS OF THE GRAPH TOPOLOGY

If we compare theorems 2.1 and 2.2, we may fall under the impression that (closed hereditary) properties of the graph topology are to (closed hereditary) properties of the fine topology as countable compactness is to pseudocompactness, which is, indeed, a good guiding idea when investigating these topologies. Moreover, theorems 3.1 and 3.2 and Corollary 3.3 suggest an even closer relationship between these topologies for some completeness properties. It was surprising, therefore, to find that a property which is equivalent to α-favorability in, say, Moore spaces [25], namely that of pseudocompleteness, is relatively hard to come by for $C_\Gamma(X)$, although it is not that complicated for $C_w(X)$ [19, Theorem 3.2]. Recall that a space Z is *pseudocomplete* [24] if and only if Z is quasi-regular (i.e., each nonempty open set contains the closure of a nonempty open subset) and Z has a sequence $\{\mathcal{B}_n\}_n$ of π-bases such that if $B_n \in \mathcal{B}_n$ and $\overline{B_{n+1}} \subseteq B_n$ for all n, then $\bigcap_n B_n \neq \emptyset$.

Lemma 4.1. *Let X be a Tychonoff space. Given $B(f,\varepsilon), B(g,\phi) \in \mathcal{B}_\Gamma$, consider the properties:*

(i) *$cl_\Gamma(B(f,\varepsilon)) \subseteq B(g,\phi)$;*

(ii) *$[f(x)-\varepsilon(x), f(x)+\varepsilon(x)] \subseteq (g(x)-\phi(x), g(x)+\phi(x))$ for all $x \in X$.*

Then (i)⇒(ii) *in each of the following cases:*

1. *points of X are G_δ;*
2. *X is locally countably compact.*

Proof. Let $x_0 \in X$ and fix $t \in [f(x_0)-\varepsilon(x_0), f(x_0)+\varepsilon(x_0)]$.

(a) If $t = f(x_0)$, then $f \in \mathrm{cl}_\Gamma(B(f,\varepsilon)) \subseteq B(g,\phi)$ implies $|g(x_0)-t| = |g(x_0)-f(x_0)| < \phi(x_0)$, so $t \in (g(x_0)-\phi(x_0), g(x_0)+\phi(x_0))$.

(b) If $t > f(x_0)$, then $0 < \delta =: t - f(x_0) \leq \varepsilon(x_0)$. Put $U_0 = X$, and choose a strictly decreasing sequence $(U_n)_{n\geq 1}$ of open neighborhoods of x_0 such that, for each $n \geq 1$,

$$U_n \subseteq \varepsilon^{-1}((\frac{(2^n-1)\delta}{2^n}, \infty));$$

moreover, find a continuous $g_n : X \to [0, \frac{\delta}{2^n}]$ such that

$$g_n(x_0) = \frac{\delta}{2^n} \text{ and } g_n(X \setminus U_n) = \{0\}.$$

Inductively define the continuous functions $h_n = h_{n-1} + g_n$ where $h_0 = f$. It follows that

- $h_n = h_{n-1}$ on $X \setminus U_n$;
- $0 \le h_n(x) - f(x) \le \frac{(2^n-1)\delta}{2^n}$ on U_n
(proof by induction: $h_n(x) - f(x) = (h_n(x) - h_{n-1}(x)) + (h_{n-1}(x) - f(x)) \le g_n(x) + \frac{(2^{n-1}-1)\delta}{2^{n-1}} \le \frac{\delta}{2^n} + \frac{(2^{n-1}-1)\delta}{2^{n-1}} = \frac{(2^n-1)\delta}{2^n}$);
- $h_n(x_0) = f(x_0) + \frac{(2^n-1)\delta}{2^n}$
(proof by induction: $h_n(x_0) = h_{n-1}(x_0) + g_n(x_0) = f(x_0) + \frac{(2^{n-1}-1)\delta}{2^{n-1}} + \frac{\delta}{2^n} = f(x_0) + \frac{(2^n-1)\delta}{2^n}$).

Also, if $x \in X$ and $m > n \ge 1$, then

$$|h_m(x) - h_n(x)| = g_m(x) + \cdots + g_{n+1}(x) \le \frac{\delta}{2^m} + \cdots + \frac{\delta}{2^{n+1}} \le \frac{\delta}{2^n},$$

so (h_n) (uniformly) converges to some $h \in C(X)$.

Denote $D = \bigcap_{n \ge 1} U_n$ and take $x \in D$. Then $\varepsilon(x) \ge \delta$, so

$$h(x) = \lim_n h_n(x) \le \lim_n (f(x) + \frac{(2^n-1)\delta}{2^n}) = f(x) + \delta \le f(x) + \varepsilon(x);$$

also, $h(x_0) = \lim_n h_n(x_0) = f(x_0) + \delta = t$. Moreover, if $x \notin D$, then $x \in U_n \setminus U_{n+1}$ for some n. Then $h_m(x) = h_n(x)$ for all $m \ge n$; thus, $h(x) = h_n(x)$, and consequently,

$$(*) \qquad 0 \le h(x) - f(x) = h_n(x) - f(x) \le \frac{(2^n-1)\delta}{2^n} < \varepsilon(x), \text{ if } x \notin D.$$

It also follows that

$$(**) \qquad 0 \le h(x) - f(x) \le \min\{\delta, \varepsilon(x)\} \text{ for each } x \in X.$$

Claim. $h \in \mathrm{cl}_\Gamma(B(f,\varepsilon))$.
Take some $B(h,\xi) \in \mathcal{B}_\Gamma$ and consider two cases:

(1) If the points of X are G_δ, let (G_n) be a sequence of open sets with $\{x_0\} = \bigcap_n G_n$, and for each $n \ge 1$, choose

$$U_n = G_n \cap \varepsilon^{-1}((\frac{(2^n-1)\delta}{2^n}, \infty)).$$

Also find $m > 1$ with $\frac{\xi(x_0)}{m} \le \frac{\delta}{2}$, and define

$$V = \xi^{-1}((\frac{\xi(x_0)}{m}, \infty)) \cap \varepsilon^{-1}((\frac{\delta}{2}, \infty)),$$

which is an open neighborhood of x_0. Find a continuous function $k_0 : X \to [0, \frac{\xi(x_0)}{m}]$ so that

$$k_0(x_0) = \frac{\xi(x_0)}{m} \text{ and } k_0(X \setminus V) = \{0\}.$$

We will be done if we show that $k = h - k_0 \in B(f,\varepsilon) \cap B(h,\xi)$:

- $k \in B(h,\xi)$: If $x \in X \setminus V$, then $k(x) = h(x)$, so $|h(x) - k(x)| = 0 < \xi(x)$; if $x \in V$, then $|h(x) - k(x)| = k_0(x) \leq \frac{\xi(x_0)}{m} < \xi(x)$.
- $k \in B(f,\varepsilon)$: First, note that

$$k(x_0) - f(x_0) = (t - \frac{\xi(x_0)}{m}) - (t - \delta) = \delta - \frac{\xi(x_0)}{m} \leq \varepsilon(x_0) - \frac{\xi(x_0)}{m} < \varepsilon(x_0);$$

moreover, if $x \neq x_0$, using $(*)$ we have

$$k(x) - f(x) = (h(x) - k_0(x)) - f(x) = (h(x) - f(x)) - k_0(x) < \varepsilon(x).$$

On the other side,

$$k(x) - f(x) =$$

$$(h(x) - f(x)) - k_0(x) \begin{cases} \geq -k_0(x) \geq -\frac{\xi(x_0)}{m} \geq -\frac{\delta}{2} > -\varepsilon(x), & \text{if } x \in V \\ \geq 0 > -\varepsilon(x), & \text{if } x \notin V. \end{cases}$$

(2) If X is locally countably compact, let $(U_n)_{n\geq 1}$ be a sequence of open neighborhoods of x_0 with a countably compact closure such that whenever $n \geq 1$,

$$U_n \subseteq U_{n-1} \cap \varepsilon^{-1}((\frac{(2^n - 1)\delta}{2^n}, \infty)).$$

Since $\overline{U_1}$ is countably compact, ξ has a minimum on $\overline{U_1}$, so we can choose

$$0 < r < \min\{\delta, \min\{\xi(x) : x \in \overline{U_1}\}\}.$$

Define $k = h - \frac{r}{\delta}(h - f)$. The claim will be proved if we show that $k \in B(f,\varepsilon) \cap B(h,\xi)$:

- $k \in B(h,\xi)$, since, using $(**)$, we have

$$|h(x) - k(x)| = \frac{r}{\delta}|h(x) - f(x)| \begin{cases} \leq \frac{r}{\delta}\delta = r < \xi(x), & \text{if } x \in U_1, \\ = 0 < \xi(x), & \text{if } x \notin U_1. \end{cases}$$

- $k \in B(f,\varepsilon)$, since, by $(**)$, we have that for each $x \in X$,

$$|f(x) - k(x)| = (1 - \frac{r}{\delta})(h(x) - f(x)) \leq (1 - \frac{r}{\delta})\varepsilon(x) < \varepsilon(x).$$

It follows from the claim and (i) that $h \in B(g,\phi)$, so $|g(x_0) - t| = |g(x_0) - h(x_0)| < \phi(x_0)$; thus, $t \in (g(x_0) - \phi(x_0), g(x_0) + \phi(x_0))$.

(c) If $t < f(x_0)$, an argument analogous to (b) works; one just needs to define $h_n = h_{n-1} - g_n$ and $k = h + k_0$ in case (1) and $h_{n+1} = h_n - g_{n+1}$ and $k = h + \frac{r}{\delta}(h - f)$ in case (2). □

Theorem 4.2. *Let X be a Tychonoff space such that*

(1) *either the points of X are G_δ,*
(2) *or X is locally countably compact,*

(3) *or* X *is a cb-space.*

Then $C_\Gamma(X)$ *is pseudocomplete.*

Proof. (1) and (2). For each $n < \omega$, define a base for $C_\Gamma(X)$ as

$$\mathcal{B}_n = \{B(f,\varepsilon) \in \mathcal{B}_\Gamma : f \in C(X),\ \varepsilon(x) \leq \frac{1}{2^n} \text{ for all } x \in X\},$$

and choose a sequence $B(f_n, \varepsilon_n) \in \mathcal{B}_n$ with $cl_\Gamma(B(f_{n+1}, \varepsilon_{n+1})) \subseteq B(f_n, \varepsilon_n)$. Then, for each $x \in X$ and $m > n$,

$$|f_m(x) - f_{n+1}(x)| < \varepsilon_{n+1}(x) \leq \frac{1}{2^{n+1}},$$

so the sequence (f_n) (uniformly) converges to some $f \in C(X)$. Then, for each $x \in X$ and $n \geq 1$, $|f(x) - f_{n+1}(x)| \leq \varepsilon_{n+1}(x)$, so by Lemma 4.1, $|f(x) - f_n(x)| < \varepsilon_n(x)$; thus, $f \in \bigcap_n B(f_n, \varepsilon_n)$.

(3). By Proposition 1.2, $C_\Gamma(X) = C_w(X)$, and, by [19, Theorem 3.2], $C_w(X)$ is always pseudocomplete. □

5. PROPERTIES RELATED TO 1ST AND 2ND COUNTABILITY OF THE GRAPH TOPOLOGY

It is known that countable compactness of X is also equivalent to various non-completeness properties of $C_\Gamma(X)$, such as 1st countability or metrizability [13]. We will show that other properties can be added to this list. Recall that Z is *Fréchet* if and only if for each $A \subseteq Z$ and $z \in \overline{A}$ there is a sequence $a_n \in A$ converging to z; Z is *sequential* if and only if $A \subseteq Z$ is closed provided A contains the limits of all sequences from A; Z is a *k-space* if and only if $A \subseteq Z$ is closed provided $A \cap K$ is closed in K for each compact $K \subseteq Z$; Z is *countably tight* if and only if for each $A \subseteq Z$ and $z \in \overline{A}$ there is a countable $B \subseteq A$ with $z \in \overline{B}$; a Tychonoff space Z is a *p-space* if and only if there is a sequence $(\mathcal{U}_n)$ of families of open sets in a compactification of Z such that each $\mathcal{U}_n$ covers Z and $\bigcap_n \bigcup\{U \in \mathcal{U}_n : z \in U\} \subseteq Z$ whenever $z \in Z$; Z is a *q-space* if and only if each $z \in Z$ has a sequence (U_n) of open neighborhoods such that whenever $z_n \in U_n$, then (z_n) clusters. For more on these spaces see [8] and [9].

Theorem 5.1. *Let* X *be a Tychonoff space. The following are equivalent:*

(1) $C_\Gamma(X)$ *is metrizable,*
(2) $C_\Gamma(X)$ *is a p-space,*
(3) $C_\Gamma(X)$ *is a q-space,*
(4) $C_\Gamma(X)$ *is 1st countable,*
(5) $C_\Gamma(X)$ *is a Fréchet space,*
(6) $C_\Gamma(X)$ *is sequential,*
(7) $C_\Gamma(X)$ *is a k-space,*

(8) $C_\Gamma(X)$ *is countably tight,*
(9) X *is countably compact.*

Proof. The implications (9)⇒(1)⇒(2)⇒(3) are well known; for (3)⇒(4) note that $C_\Gamma(X)$ is a completely regular, submetrizable space (it contains the metrizable uniform topology), so the points of $C_\Gamma(X)$ are G_δ. The implication then follows from the fact that a regular q-space whose points are G_δ is 1st countable (see [9, Lemma 3.2]).

The implications (4)⇒(5)⇒(6)⇒(7) are well known; as for (7)⇒(8), observe that any k-space Z with G_δ singletons is countably tight, since otherwise there is some $A \subseteq Z$ so that $B = \bigcup\{\overline{C} : C \subseteq A \text{ countable}\}$ is not closed, so there exists some compact $K \subseteq Z$ such that $K \cap B$ is not closed in K. Since a compact space having G_δ singletons is 1st countable, there exists a sequence $z_n \in K \cap B$ converging to some $z \in K \setminus B$; also, there is a countable $C_n \subseteq A$ with $z_n \in \overline{C_n}$ for each n, so $z \in \overline{\bigcup_n C_n}$, which implies $z \in B$, a contradiction.

Finally, to prove (8)⇒(9), assume X is not countably compact. Let $D = \{x_n : n \geq 1\}$ be a countable closed discrete subset of X and let $\{U_n : n \geq 1\}$ be a pairwise disjoint sequence of open neighborhoods of the x_n's. Let f_0 be the identically zero function and define $L = \{g \in C(X) : D \subseteq g^{-1}((0,\infty))\}$.

Claim. $f_0 \in cl_\Gamma(L)$.
Indeed, consider some $B(f_0,\varepsilon) \in \mathcal{B}_\Gamma$. For each n, find an open neighborhood V_n of x_n so that $\overline{V_n} \subseteq U_n \cap \varepsilon^{-1}((\frac{2}{3}\varepsilon(x_n),\infty))$ and define the continuous function $h_n : \overline{V_n} \to [0, \frac{\varepsilon(x_n)}{n+1}]$ such that $h_n(x_n) = \frac{\varepsilon(x_n)}{n+1}$ and $h_n(\overline{V_n} \setminus V_n) = \{0\}$. It is easy to see that $h \in L \cap B(f_0,\varepsilon)$, where $h : X \to \mathbb{R}$ is defined via

$$h = \begin{cases} h_n, & \text{on } \overline{V_n}, \text{ whenever } n \geq 1, \\ 0, & \text{on } X \setminus \bigcup_{n\geq 1} \overline{V_n}. \end{cases}$$

Since $C_\Gamma(X)$ is countably tight, there is a countable subset $L' = \{f_n : n \geq 1\}$ of L so that $f_0 \in cl_\Gamma(L')$. For each $n \geq 1$, choose $k_n \geq 1$ such that $f_n(x_n) < k_n/2$ and define the function $\eta : X \to (0,\infty)$ via

$$\eta(x) = \begin{cases} \frac{f_n(x)}{k_n}, & \text{if } x = x_n \ (n \geq 1), \\ \frac{1}{2}, & \text{if } x \notin D. \end{cases}$$

It follows that $\eta \in LSC^+(X)$ and $L' \cap B(f_0,\eta) = \emptyset$, a contradiction. □

Now we turn to characterizing 2nd countability and related properties of the graph topology.

Theorem 5.2. *Let X be a Tychonoff space. The following are equivalent:*

(1) $C_\Gamma(X)$ *is 2nd countable,*

(2) $C_\Gamma(X)$ *has a countable network,*
(3) $C_\Gamma(X)$ *is separable,*
(4) $C_\Gamma(X)$ *has ccc,*
(5) X *is compact and metrizable.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4) are trivial; as for (4)⇒(5), note that $\tau_u \subseteq \tau_\Gamma$ implies $(C(X), \tau_u)$ has ccc, so it is (metrizable) separable, which is equivalent to (5) by [20, Theorem 4.2.4]. Finally, (5)⇒(1), since $\tau_\Gamma = \tau_u$ if X is compact, and then $(C(X), \tau_u)$ is separable and metrizable. □

References

[1] G. Beer and S. Naimpally, *Graphical convergence of continuous functions*, Acta Math. Hungar. **140** (2013), no. 4, 305–315.

[2] A. Bouziad, *Filters, consonance and hereditary Baireness*, Topology Appl. **104** (2000), no. 1-3, 27–38.

[3] Jiling Cao and Zbigniew Piotrowski, *Two variations of the Choquet game*, Kyungpook Math. J. **44** (2004), no. 4, 495–504.

[4] Gustave Choquet, *Lectures on Analysis. Vol. I: Integration and Topological Vector Spaces.* Ed. Jerrold E. Marsden, Tim Lance, and Stephen S. Gelbart. New York-Amsterdam: W. A. Benjamin, Inc., 1969.

[5] G. Debs, *Espaces héréditairement de Baire*, Fund. Math. **129** (1988), no. 3, 199–206.

[6] G. Di Maio, L. Holá, D. Holý, and R. A. McCoy, *Topologies on the space of continuous functions*, Topology Appl. **86** (1998), no. 2, 105–122.

[7] Eric K. van Douwen, *Nonnormality or hereditary paracompactness of some spaces of real functions*, Topology Appl. **39** (1991), no. 1, 3–32.

[8] Ryszard Engelking, *General Topology.* Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[9] Gary Gruenhage, *Generalized metric spaces* in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 423–501.

[10] S. P. Gul'ko and G. A. Sokolov, *P-points in N^* and the spaces of continuous functions*, Topology Appl. **85** (1998), no. 1-3, 137–142.

[11] J. D. Hansard, *Function space topologies*, Pacific J. Math. **35** (1970), 381–388.

[12] R. C. Haworth and R. A. McCoy, *Baire Spaces.* Dissertationes Math. (Rozprawy Mat.) 141 (1977).

[13] Ľubica Holá and Dušan Holý, *Spaces of lower semicontinuous set-valued maps*, Math. Slovaca **63** (2013), no. 4, 863–870.

[14] Ľ. Holá and R. A. McCoy, *Compactness in the fine and related topologies*, Topology Appl. **109** (2001), no. 2, 183–190.

[15] Alexander S. Kechris, *Classical Descriptive Set Theory.* Graduate Texts in Mathematics, 156. New York: Springer-Verlag, 1995.

[16] Norman Levine, *On the graph topology for function spaces*, Kyungpook Math. J. **24** (1984), no. 2, 101–113.

[17] John Mack, *On a class of countably paracompact spaces*, Proc. Amer. Math. Soc. **16** (1965), 467–472.

[18] J. E. Mack and D. G. Johnson, *The Dedekind completion of $C(\mathcal{X})$*, Pacific J. Math. **20** (1967), 231–243.

[19] R. A. McCoy, *Fine topology on function spaces*, Internat. J. Math. Math. Sci. **9** (1986), no. 3, 417–424.

[20] Robert A. McCoy and Ibula Ntantu, *Topological Properties of Spaces of Continuous Functions.* Lecture Notes in Mathematics, 1315. Berlin: Springer-Verlag, 1988.

[21] Ernest Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[22] Somashekhar Amrith Naimpally, *Graph topology for function spaces*, Trans. Amer. Math. Soc. **123** (1966), 267–272.

[23] N. Noble, *Products with closed projections*, Trans. Amer. Math. Soc. **140** (1969), 381–391.

[24] John C. Oxtoby, *Cartesian products of Baire spaces*, Fund. Math. **49** (1960/1961), 157–166.

[25] H. E. White, Jr., *Topological spaces that are α-favorable for a player with perfect information*, Proc. Amer. Math. Soc. **50** (1975), 477–482.

[26] H. H. Wicke and J. M. Worrell, Jr., *On the open continuous images of paracompact Čech complete spaces*, Pacific J. Math. **37** (1971), no. 1, 265–275.

[27] László Zsilinszky, *On β-favorability of the strong Choquet game*, Colloq. Math. **125** (2011), no. 2, 233–243.

(Holá) Academy of Sciences; Institute of Mathematics; Štefánikova 49, 81473 Bratislava, Slovakia
E-mail address: `hola@mat.savba.sk`

(Zsilinszky) Department of Mathematics and Computer Science; The University of North Carolina at Pembroke; Pembroke, NC 28372 USA
E-mail address: `laszlo@uncp.edu`