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TOPOLOGY PROCEEDINGS

Volume 46, 2015

Pages 29–32

http://topology.nipissingu.ca/tp/

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Electronically published on April 1, 2014

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 46 (2015)
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UNICOHERENCE OF THE n^{th}-FOLD SYMMETRIC PRODUCT SUSPENSION OF A CONTINUUM

ENRIQUE CASTAÑEDA-ALVARADO AND ALEJANDRO ILLANES

Abstract. For a metric continuum X, let $F_n(X)$ be the hyperspace of all nonempty subsets of X with at most n elements and let $SF_1^n(X)$ be the continuum $F_n(X)/F_1(X)$, that is, $SF_1^n(X)$ is the quotient space obtained by identifying the set $F_1(X)$ to a point in $F_n(X)$. In this paper we show that for every continuum X, $SF_1^2(X)$ is unicoherent. This answers a question by Enrique Castañeda-Alvarado and Javier Sánchez-Martínez.

1. Introduction

A *continuum* is a compact connected metric space, with more than one point. Given a metric continuum X, we consider the n^{th}*-symmetric product* $F_n(X)$ defined as

$$F_n(X) = \{A \subset X : A \text{ is nonempty and } A \text{ contains at most } n \text{ points}\}.$$

This space is considered with the Hausdorff metric.

Given $0 < m < n$, we consider the quotient space

$$SF_m^n(X) = F_n(X)/F_m(X),$$

which is obtained by shrinking to a point the subset $F_m(X)$ of $F_n(X)$.

Spaces of the form $F_1^n(X)$ are called n^{th}-fold symmetric product suspension of a continuum.

2010 *Mathematics Subject Classification.* Primary 54B20, Secondary, 54F55, 54F15.

Key words and phrases. continuum, hyperspace, symmetric product, unicoherence.

This paper was partially supported by the project "Hiperespacios topológicos (0128584)" of Consejo Nacional de Ciencia y Tecnología (CONACYT), 2009, the project "Teoría de Continuos, Hiperespacios y Sistemas Dinámicos" (IN104613) of PAPIIT, DGAPA, UNAM, and Consejo Mexiquense de Ciencia y Tecnología (COMECYT) by agreement to implement the "Séptimo Taller de Investigación en Continuos e Hiperespacios."

A connected space X is *unicoherent* provided that $A \cap B$ is connected whenever A and B are closed connected subsets of X such that $X = A \cup B$.

Unicoherence of symmetric products has been studied since 1954 [7] and it is known that

(a) for each continuum X and each $n \geq 3$, $F_n(X)$ is unicoherent [9, Theorem 8],

(b) if X is a locally connected continuum, then $F_2(X)$ is unicoherent if and only X is unicoherent (see [7] and [8, Theorem 1.6]),

(c) there exists a unicoherent continuum X such that $F_2(X)$ is not unicoherent [2].

In a recent paper [3], unicoherence of continua $SF_m^n(X)$ was considered. If $0 < m < n$ and $3 \leq n$, from the facts that $F_n(X)$ is unicoherent [9, Theorem 8] and monotone mappings between continua preserve unicoherence [12, Corollary 13.35], we have that $SF_m^n(X)$ is unicoherent [3, Theorem 5.1]. So, the interesting case to study is the unicoherence of $SF_1^2(X)$. In [1], Franco Barragán claims that, for the continuum X_0 given by Enrique Castañeda-Alvarado in [2], $SF_1^2(X_0)$ is not unicoherent. In [3], it is shown that

(a) Barragán's claim is wrong since $SF_1^2(X_0)$ is unicoherent [3, Theorem 5.15],

(b) for some classes of continua X, $SF_1^2(X)$ is unicoherent [3, §3 and §5],

(c) if X is the continuum with the form of the Greek letter theta, then $SF_1^2(X)$ is not unicoherent [3, Example 3.16]. (This claim is wrong as we will see in Theorem 2.2.)

In [3, Question 5.16], it was also asked,

> Does there exist a unicoherent continuum X for such that $SF_1^2(X)$ is not unicoherent?

The main result of our paper answers this question by showing that for each continuum X, $SF_1^2(X)$ is unicoherent.

2. Main Result

A *mapping* is a continuous function. We denote by S^1 the unit circle in the plane, by $\mathbb{R}$ the set of real numbers, and by $e : \mathbb{R} \to S^1$ the exponential mapping given by $e(t) = (\cos(2\pi t), \sin(2\pi t))$.

Theorem 2.1. *If X is a locally connected continuum, then $SF_1^2(X)$ is unicoherent.*

Proof. According to [5, Theorem 3], we only have to show that if $f : SF_1^2(X) \to S^1$ is a mapping, then there exists a mapping $\psi : SF_1^2(X) \to \mathbb{R}$ such that $e \circ \psi = f$. Let $f : SF_1^2(X) \to S^1$ be a mapping. Consider the quotient map $q : F_2(X) \to SF_1^2(X)$ and let $\varphi = f \circ q : F_2(X) \to S^1$. Then φ is a mapping such that $\varphi|_{F_1(X)}$ is constant. By [8, Lemma 1.5], φ is homotopic to a constant mapping. Thus, there exists a map $h : F_2(X) \to \mathbb{R}$ such that $\varphi = e \circ h$ [10]. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} F_2(X) & \overset{q}{\to} & SF_1^2(X) \\ h\downarrow & & f\downarrow \\ \mathbb{R} & \overset{e}{\to} & S^1 \end{array}$$

Fix an element $p_0 \in X$. Given $t \in X$, $e(h(\{p_0\})) = f(q(\{p_0\})) = f(q(\{t\})) = e(h(\{t\}))$. Thus, $e(h(\{p_0\})) = e(h(\{t\}))$. This implies that $h(F_1(X)) \subset e^{-1}(e(h(\{p_0\})))$. Since $e^{-1}(e(h(\{p_0\})))$ is discrete and $F_1(X)$ is connected, we have that $h(F_1(X))$ is a one-point set. Hence, h preserves the fibers of the quotient map q. By the transgression theorem [4, Chapter 3.2, Theorem 3.2], there exists a mapping $\psi : SF_1^2(X) \to \mathbb{R}$ such that $\psi \circ q = h$. It is easy to show that $e \circ \psi = f$. Therefore, $SF_1^2(X)$ is unicoherent. □

Theorem 2.2. *If X is a continuum, then $SF_1^2(X)$ is unicoherent.*

Proof. By [6, p. 186], X is an inverse limit of locally connected continua. That is, there exist a sequence $\{X_r\}_{r=1}^{\infty}$ of locally connected continua and a sequence $\{f_r\}_{r=1}^{\infty}$ of mappings such that for each r, $f_r : X_{r+1} \to X_r$ and X is homeomorphic to

$$\varprojlim\{X_r, f_r\} = \{\{x_r\}_{r=1}^{\infty} \in \prod_{r=1}^{\infty} X_r : \text{for each } r, x_r = f_r(x_{r+1})\}.$$

By [3, Theorem 4.4], $SF_1^2(X)$ is homeomorphic to

$$\varprojlim\{SF_1^2(X_r), SF_1^2(f_r)\},$$

where $SF_1^2(f_r) : SF_1^2(X_{r+1}) \to SF_1^2(X_r)$ is the induced mapping (see [3, p. 313]). By Theorem 2.1, each continuum $SF_1^2(X_r)$ is locally connected and unicoherent. By the remark following Corollary 1 in [11], we conclude that $SF_1^2(X)$ is unicoherent. □

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