http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 46, 2015

Pages 45–54

http://topology.nipissingu.ca/tp/

The Behavior of the Maximal Degree of the Khovanov Homology under Twisting

by

Keiji Tagami

Electronically published on April 4, 2014

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 46 (2015)
Pages 45-54

E-Published on April 4, 2014

THE BEHAVIOR OF THE MAXIMAL DEGREE OF THE KHOVANOV HOMOLOGY UNDER TWISTING

KEIJI TAGAMI

Abstract. In this paper, we study an asymptotic behavior of the maximal homological degree of the non-zero Khovanov homology groups under twisting.

1. Introduction

In [3], for each oriented link L, Mikhai Khovanov defined a graded chain complex whose graded Euler characteristic is equal to the Jones polynomial of L. Its homology groups are link invariants called the Khovanov homology groups. Throughout this paper, we only consider the rational Khovanov homology. The Khovanov homology has two gradings, homological degree i and q-grading j. By $\mathrm{KH}^i(L)$, we denote the homological degree i term of the Khovanov homology groups of a link L, and, by $\mathrm{KH}^{i,j}(L)$, we denote the homological degree i and q-grading j term of the Khovanov homology groups of L.

The maximal homological degree of the non-zero Khovanov homology groups of a link gives a lower bound of the minimal positive crossing number of the link (see Proposition 2.2). The minimal positive crossing number of a link is the minimal number of the positive crossings of diagrams of the link. From this fact, it seems that the Khovanov homology estimates the positivity of links.

Marko Stošić [4, Theorem 2] showed that the maximal homological degree of the non-zero Khovanov homology groups of the $(2k, 2kn)$-torus link is $2k^2n$. By using the same method as Stošić's, we [5, Corollary 1.2] proved that the maximal homological degree of the non-zero Khovanov

2010 *Mathematics Subject Classification.* Primary 57M27.
Key words and phrases. Khovanov homology, knot, twist.
The author was supported by JSPS KAKENHI Grant Number 25001362.

homology groups of the $(2k+1,(2k+1)n)$-torus link is $2k(k+1)n$. These results intimate that the maximal degree of the non-zero Khovanov homology groups grows as the number of full-twists grows.

Let L be an oriented link and C_p be a disk which intersects with L at p points transversely with the same orientations as in Figure 1. Then, for any positive integer n, we define a link $t_n(L;C_p)$ as the link obtained from L by adding n full-twists at C_p.

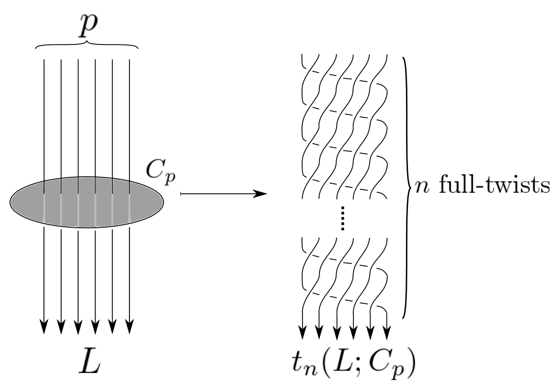


FIGURE 1. $t_n(L;C_p)$.

We consider the following question.

Question 1.1. Let L and C_p be as above and let $T_{p,pn}$ denote the positive (p,pn)-torus link. Does the following equality hold?

$$\lim_{n\to\infty}\frac{\max\{i\in\mathbf{Z}\mid \mathrm{KH}^i(t_n(L;C_p))\neq 0\}}{n}$$
$$=\lim_{n\to\infty}\frac{\max\{i\in\mathbf{Z}\mid \mathrm{KH}^i(T_{p,pn})\neq 0\}}{n}.$$

Note that Stošić [4] and the author [5] proved

(1.1)
$$\lim_{n\to\infty}\frac{\max\{i\in\mathbf{Z}\mid \mathrm{KH}^i(T_{p,pn}))\neq 0\}}{n}=\begin{cases}2k^2 & \text{if } p=2k,\\ 2k(k+1) & \text{if } p=2k+1.\end{cases}$$

We proved the following, providing evidence towards an affirmative answer to Question 1.1.

Theorem 1.2 ([5, Theorem 1.3 and Proposition 1.4])**.** *Let K be an oriented knot. Denote the (p, pn)-cabling of K by $K(p, pn)$ for positive integers p and n. Assume that each component of $K(p, pn)$ has an orientation induced by K, i.e., each component of $K(p, pn)$ is homologous to K in the tubular neighborhood of K. Put $c_+(K) := \min\{c_+(D) \mid D \text{ is a diagram of } K\}$, where $c_+(D)$ is the number of the positive crossings of D. If $n \geq 2c_+(K)$, then we have the following for any positive integer k.*

$$\max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(K(2k, 2kn)) \neq 0\} = 2k^2 n,$$
$$\begin{aligned} 2k(k+1)n &\leq \max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(K(2k+1, (2k+1)n)) \neq 0\} \\ &\leq 2k(k+1)n + c_+(K). \end{aligned}$$

In particular, we have

$$\lim_{n \to \infty} \frac{\max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(K(p, pn)) \neq 0\}}{n} = \begin{cases} 2k^2 & \text{if } p = 2k, \\ 2k(k+1) & \text{if } p = 2k+1. \end{cases}$$

In this paper, we consider Question 1.1 for $p = 2$. Precisely, we prove the following.

Theorem 1.3 (Main Theorem)**.** *Let L be an oriented link and let C be a disk which intersects L at two points with the same orientations as in Figure 2. Then we have*

$$\lim_{n \to \infty} \frac{\max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(t_{n/2}(L; C)) \neq 0\}}{n} = 1,$$

where $t_{n/2}(L; C)$ is the link obtained from L by adding n half twists at C.

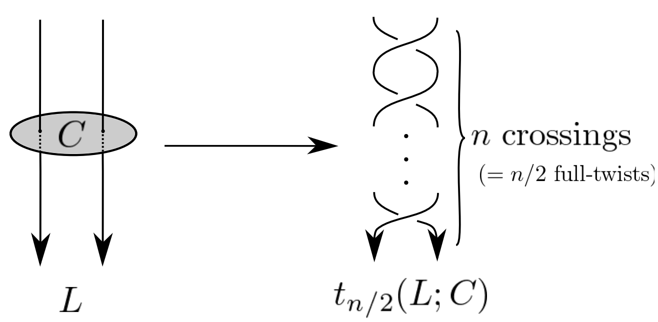


FIGURE 2. Definition of the link $t_{n/2}(L; C)$. The integer n is the number of added half twists.

There is Liam Watson's work [6] on an asymptotic behavior of the reduced Khovanov homology under twisting. Ignoring grading, he gave a nice relation between the reduced Khovanov homologies of $t_{N/2}(L;C)$ and $t_{(N+1)/2}(L;C)$ for sufficiently large N [6, Lemma 4.14].

This paper is organized as follows: In §2, we recall the definition of the Khovanov homology. In §3, we prove our main theorem (Theorem 1.3).

2. Khovanov Homology

2.1. The definition of the Khovanov homology.

In this subsection, we recall the definition of the (rational) Khovanov homology. Let L be an oriented link. Take a diagram D of L and an ordering of the crossings of D. For each crossing of D, we define 0-smoothing and 1-smoothing as in Figure 3. A smoothing of D is a diagram where each crossing of D is changed by either 0-smoothing or 1-smoothing. Let n be the number of the crossings of D. Then D has 2^n smoothings.

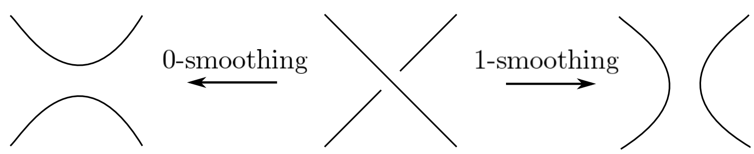


Figure 3. 0-smoothing and 1-smoothing.

By using the given ordering of the crossings of D, we have a natural bijection between the set of smoothings of D and the set $\{0,1\}^n$, where, to any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n$, we associate the smoothing D_ε where the i-th crossing of D is ε_i-smoothed. Each smoothing D_ε is a collection of disjoint circles.

Let V be a graded free $\mathbf{Q}$-module generated by 1 and X with $\deg(1) = 1$ and $\deg(X) = -1$. Let k_ε be the number of the circles of the smoothing D_ε. Put $M_\varepsilon = V^{\otimes k_\varepsilon}$. The module M_ε has a graded module structure, that is, for $v = v_1 \otimes \cdots \otimes v_{k_\varepsilon} \in M_\varepsilon$, $\deg(v) := \deg(v_1) + \cdots + \deg(v_{k_\varepsilon})$. Then define

$$C^i(D) := \bigoplus_{|\varepsilon|=i} M_\varepsilon\{i\},$$

where $|\varepsilon| = \sum_{i=1}^{m} \varepsilon_i$. Here, $M_\varepsilon\{i\}$ denotes M_ε with its gradings shift by i (for a graded module $M = \bigoplus_{j\in\mathbf{Z}} M^j$ and an integer i, we define the graded module $M\{i\} = \bigoplus_{j\in\mathbf{Z}} M\{i\}^j$ by $M\{i\}^j = M^{j-i}$).

The differential map $d^i \colon C^i(D) \to C^{i+1}(D)$ is defined as follows. Fix an ordering of the circles for each smoothing D_ε and associate the i-th tensor factor of M_ε to the i-th circle of D_ε. Take elements $\varepsilon, \varepsilon' \in \{0,1\}^n$

such that $\varepsilon_j = 0$ and $\varepsilon'_j = 1$ for some j and that $\varepsilon_i = \varepsilon'_i$ for any $i \neq j$. For such a pair $(\varepsilon, \varepsilon')$, we will define a map $d_{\varepsilon \to \varepsilon'} \colon M_\varepsilon \to M_{\varepsilon'}$ as follows.

In the case where two circles of D_ε merge into one circle of $D_{\varepsilon'}$, the map $d_{\varepsilon \to \varepsilon'}$ is the identity on all factors except the tensor factors corresponding to the merged circles where it is a multiplication map $m \colon V \otimes V \to V$ given by

$$m(1 \otimes 1) = 1,\ m(1 \otimes X) = m(X \otimes 1) = X,\ m(X \otimes X) = 0.$$

In the case where one circle of D_ε splits into two circles of $D_{\varepsilon'}$, the map $d_{\varepsilon \to \varepsilon'}$ is the identity on all factors except the tensor factor corresponding to the split circle where it is a comultiplication map $\Delta \colon V \to V \otimes V$ given by

$$\Delta(1) = 1 \otimes X + X \otimes 1,\ \Delta(X) = X \otimes X.$$

If there exist distinct integers i and j such that $\varepsilon_i \neq \varepsilon'_i$ and that $\varepsilon_j \neq \varepsilon'_j$, then define $d_{\varepsilon \to \varepsilon'} = 0$.

In this setting, we define a map $d^i \colon C^i(D) \to C^{i+1}(D)$ by $\sum_{|\varepsilon|=i} d^i_\varepsilon$, where $d^i_\varepsilon \colon M_\varepsilon \to C^{i+1}(D)$ is defined by

$$d^i(v) := \sum_{|\varepsilon'|=i+1} (-1)^{l(\varepsilon,\varepsilon')} d_{\varepsilon \to \varepsilon'}(v).$$

Here $v \in M_\varepsilon \subset C^i(D)$ and $l(\varepsilon, \varepsilon')$ is the number of 1's in front of (in our order) the factor of ε which is different from ε'.

We can check that $(C^i(D), d^i)$ is a cochain complex and we denote its i-th homology group by $H^i(D)$. We call these the unnormalized homology groups of D. Since the map d^i preserves the grading of $C^i(D)$, the group $H^i(D)$ has a graded structure $H^i(D) = \bigoplus_{j \in \mathbf{Z}} H^{i,j}(D)$ induced by that of $C^i(D)$. For any link diagram D, we define its Khovanov homology $\mathrm{KH}^{i,j}(D)$ by

$$\mathrm{KH}^{i,j}(D) = H^{i+n_-, j-n_++2n_-}(D),$$

where n_+ and n_- are the number of the positive and negative crossings of D, respectively. The grading i is called the homological degree and j is called the q-grading.

Theorem 2.1 ([1], [3])**.** *For any oriented link L and a diagram D of L, the homology group* $\mathrm{KH}(D)$ *is preserved under the Reidemeister moves. In this sense, we can denote* $\mathrm{KH}(L) = \mathrm{KH}(D)$*. Moreover, the graded Euler characteristic of the homology* $\mathrm{KH}(L)$ *equals the Jones polynomial of L, that is,*

$$V_L(t) = (q + q^{-1})^{-1} \sum_{i,j \in \mathbf{Z}} (-1)^i q^j \operatorname{rank} \mathrm{KH}^{i,j}(L) \Big|_{q=-t^{\frac{1}{2}}},$$

where $V_L(t)$ is the Jones polynomial of L.

The following is an immediate consequence of the definition.

Proposition 2.2. *For any oriented link L, we have*

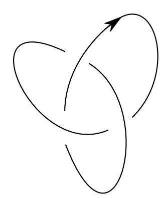
$$\max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(L) \neq 0\} \leq c_+(L),$$

where $c_+(L)$ is defined in Theorem 1.2.

Proof. For any diagram D of L, by the definition, we have $H^i(D) = 0$ if $i > c(D)$, where $c(D)$ is the number of the crossings of D. Hence, $\mathrm{KH}^i(L) = \mathrm{KH}^i(D) = 0$ for $i > c_+(D)$. Since $\mathrm{KH}^i(L)$ does not depend on the choice of D, we have $\mathrm{KH}^i(L) = 0$ for $i > c_+(K)$. $\square$

Example 2.3. For example, the Khovanov homology of the left-handed trefoil knot K (depicted in Figure 4) is given as follows.

$$\mathrm{KH}^{i,j}(K) = \begin{cases} \mathbf{Q} & \text{if } (i,j) = (0,-1),(0,-3),(-2,-5),(-3,-9), \\ 0 & \text{otherwise.} \end{cases}$$



j \ i	−3	−2	−1	0
−1				1
−3				1
−5		1		
−7				
−9	1			

FIGURE 4. The left-handed trefoil and the table of the Khovanov homology of the knot whose (i,j) element is $\dim_{\mathbf{Q}} \mathrm{KH}^{i,j}(K)$.

2.2. Main tool.

Our main tool is the following theorem proved by S. Wehrli [7] and Abhijit Champanerkar and Ilya Kofman [2]. The ***0**-smoothing* of a diagram D is the disjoint circles obtained from D by 0-smoothing all crossings (see Figure 3). We define the ***1**-smoothing* of a diagram D analogously. Then we have the following.

Theorem 2.4 ([7], [2])**.** *Let D be a link diagram. If $H^{i,j}(D) \neq 0$, we have*

$$s_1(D) - 2 - c(D) \leq j - 2i \leq 2 - s_0(D),$$

*where $c(D)$ is the number of the crossings of D and $s_0(D)$ and $s_1(D)$ are the numbers of the circles appearing in the **0**-smoothing and the **1**-smoothing of D, respectively.*

3. The Main Theorem and Its Proof

In this section, we prove our main theorem (Theorem 1.3). To prove this theorem, we first compute the maximal degree of the Jones polynomial and prove that it is proportional to the number n of twists if n is sufficiently large (Lemma 3.1). Since the Jones polynomial is the graded Euler characteristic of the Khovanov homology, we obtain corresponding bounds on the gradings in which it is supported.

Let L_0 be an oriented link and D_0 be a diagram of L_0. Let C be a disk as in Figure 5 and D_n be the diagram obtained from D_0 by adding n half twists at C as in Figure 5. The diagram D_n is a diagram of $t_{n/2}(L_0; C)$. Put $L_n := t_{n/2}(L_0; C)$.

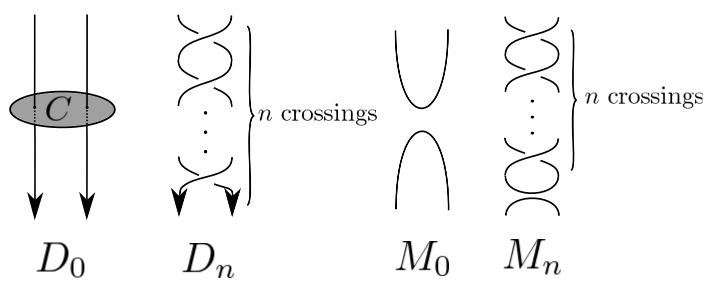


Figure 5. The diagrams D_0, D_n, M_0, and M_n.

To prove Theorem 1.3, we use the following lemma.

Lemma 3.1. *There is a positive integer N such that, for any $n \geq N$,*

$$\operatorname{maxdeg} V_{L_{n+1}}(t) - \operatorname{maxdeg} V_{L_n}(t) = \tfrac{3}{2},$$

where $\operatorname{maxdeg} V_L(t) := \max\{i \in \frac{1}{2}\mathbf{Z} \mid$ *the coefficient of t^i in $V_L(t)$ is not* $0\}$.

Proof. The Kauffman bracket of an (unoriented) link diagram is given as follows:

- $\langle \asymp\!\!\!\!\times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$,
- $\langle \bigcirc \rangle = 1$,
- $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^{2})\langle D \rangle$.

For any link L, the Jones polynomial is given by

$$V_L(t) = (-A)^{-3w(D)} \langle D \rangle \Big|_{A = t^{-\frac{1}{4}}},$$

where D is a diagram of L and $w(D)$ is the writhe of D. Hence, we have

(3.1)

$$\begin{aligned}
V_{L_n}(t) &= (-A)^{-3w(D_n)}(A\langle D_{n-1}\rangle + A^{-1}\langle M_{n-1}\rangle)\Big|_{A=t^{-\frac{1}{4}}} \\
&= (-A)^{-3w(D_{n-1})}(-A)^{-3}A\langle D_{n-1}\rangle\Big|_{A=t^{-\frac{1}{4}}} \\
&\quad + (-A)^{-3w(D_n)}A^{-1}(-A)^{-3(n-1)}\langle M_0\rangle\Big|_{A=t^{-\frac{1}{4}}} \\
&= -t^{\frac{1}{2}}V_{L_{n-1}}(t) + (-1)^{-3(w(D_n)+n-1)}A^{-3(w(D_0)+n)-3n+2}\langle M_0\rangle\Big|_{A=t^{-\frac{1}{4}}} \\
&= -t^{\frac{1}{2}}V_{L_{n-1}}(t) + (-1)^{-3(w(D_n)+n-1)}t^{\frac{3}{2}n}(A^{-3w(D_0)+2}\langle M_0\rangle)\Big|_{A=t^{-\frac{1}{4}}},
\end{aligned}$$

where M_n is the diagram depicted in Figure 5.

By way of contradiction, assume that for any positive integer n, $\operatorname{maxdeg} V_{L_n}(t) - \operatorname{maxdeg} V_{L_{n-1}}(t) \leq \frac{1}{2}$. Then we have $\operatorname{maxdeg} V_{L_n}(t) \leq \frac{1}{2}n + \operatorname{maxdeg} V_{L_0}(t)$. Hence, from (3.1), there is a positive integer N' such that

$$\operatorname{maxdeg} V_{L_n}(t) = \operatorname{maxdeg}(t^{\frac{3}{2}n}((A^{-3(w(D_0)+2)}\langle M_0\rangle)\Big|_{A=t^{-\frac{1}{4}}})(t))$$

for any $n \geq N'$. In particular, $\operatorname{maxdeg} V_{L_{N'+1}}(t) - \operatorname{maxdeg} V_{L_{N'}}(t) = \frac{3}{2}$. This is a contradiction. Hence, there is a positive integer N such that

$$\operatorname{maxdeg} V_{L_N}(t) - \operatorname{maxdeg} V_{L_{N-1}}(t) > \frac{1}{2}. \tag{3.2}$$

Then, we have

$$\operatorname{maxdeg} V_{L_{N+1}}(t) - \operatorname{maxdeg} V_{L_N}(t) = \frac{3}{2} > \frac{1}{2}.$$

In fact, by the skein relation $t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t)$ for a usual skein triple (L_+, L_-, L_0), we obtain

$$V_{L_{N+1}}(t) = t^2 V_{L_{N-1}}(t) + (t^{\frac{3}{2}} - t^{\frac{1}{2}})V_{L_N}(t).$$

From (3.2), we have

$$\operatorname{maxdeg} V_{L_{N+1}}(t) = \operatorname{maxdeg} t^{\frac{3}{2}} V_{L_N}(t) = \frac{3}{2} + \operatorname{maxdeg} V_{L_N}(t).$$

Inductively, we obtain $\operatorname{maxdeg} V_{L_{n+1}}(t) = \frac{3}{2} + \operatorname{maxdeg} V_{L_n}(t)$ for any $n \geq N$. □

Proof of Theorem 1.3. Let L_0, L_n, D_0, and D_n be as above. Note that $s_0(D_n) = s_0(D_0)$, where $s_0(D_0)$ and $s_0(D_n)$ are introduced in Theorem 2.4. By Theorem 2.4, if $H^{i,j}(D_n) \neq 0$, we have

$$j - 2i \leq 2 - s_0(D_0). \tag{3.3}$$

Since the graded Euler characteristic of the Khovanov homology is the Jones polynomial (Theorem 2.1), we obtain

$$\max\{j \in \mathbf{Z} \mid \mathrm{KH}^{*,j}(D) \neq 0\} \geq 2 \operatorname{maxdeg} V_D(t) + 1$$

for any link diagram D. Moreover,

(3.4)

$$\max\{j \in \mathbf{Z} \mid H^{*,j}(D) \neq 0\} \geq 2 \operatorname{maxdeg} V_D(t) + 1 - c_+(D) + 2c_-(D),$$

where $c_+(D)$ and $c_-(D)$ are the numbers of the positive and negative crossings of D, respectively. Put $f(D_n) := 2 \operatorname{maxdeg} V_{L_n}(t) + 1 - c_+(D_n) + 2c_-(D_n)$ and $k(D_n) := \frac{1}{2}(f(D_n) - 2 + s_0(D_0))$. From (3.3), (3.4), the definition of H^i (the unnormalized Khovanov homology), and Figure 6, we have

$$k(D_n) \leq \max\{i \in \mathbf{Z} \mid H^i(D_n) \neq 0\} \leq c(D_n) = c(D_0) + n. \tag{3.5}$$

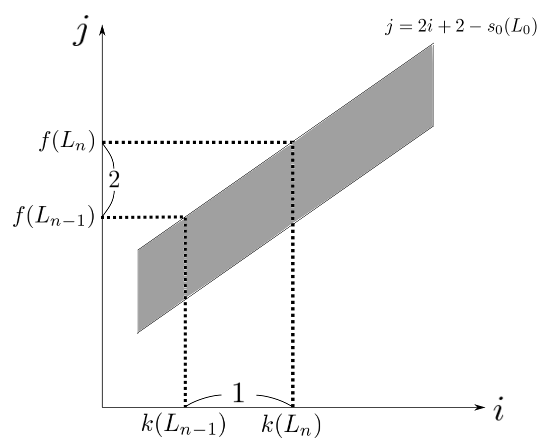


FIGURE 6. The gray parallelogram contains the support of $H^{i,j}$.

By Lemma 3.1, there is a positive integer N such that $k(D_{n+1}) - k(D_n) = 1$ for any $n \geq N$. In particular, $k(D_{n+1}) = n - N + k(D_{N+1})$. From (3.5), we have

$$n - N + k(D_{N+1}) \leq \max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(L_n) \neq 0\} + c_-(D_0) \leq c(D_0) + n.$$

This implies Theorem 1.3. $\square$

Acknowledgments. The author would like to thank Hitoshi Murakami for his encouragement and helpful comments. He also would like to thank the referee.

References

[1] Dror Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. **2** (2002), 337–370.

[2] Abhijit Champanerkar and Ilya Kofman, *Spanning trees and Khovanov homology*, Proc. Amer. Math. Soc. **137** (2009), no. 6, 2157–2167.

[3] Mikhai Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.

[4] Marko Stošić, *Khovanov homology of torus links*, Topology Appl. **156** (2009), no. 3, 533–541.

[5] Keiji Tagami, *The maximal degree of the Khovanov homology of a cable link*, Algebr. Geom. Topol. **13** (2013), no. 5, 2845–2896.

[6] Liam Watson, *Surgery obstructions from Khovanov homology*, Selecta Math. (N.S.) **18** (2012), no. 2, 417–472.

[7] S. Wehrli, *A spanning tree model for Khovanov homology*, J. Knot Theory Ramifications **17** (2008), no. 12, 1561–1574

Department of Mathematics; Tokyo Institute of Technology; Oh-okayama, Meguro, Tokyo 152-8551, Japan

E-mail address: `tagami.k.aa@m.titech.ac.jp`