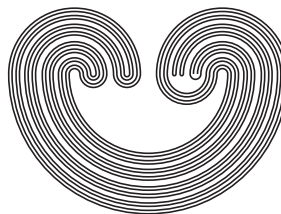


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TOPOLOGY PROCEEDINGS



Volume 46, 2015

Pages 135–143

<http://topology.nipissingu.ca/tp/>

PRODUCTS AND COUNTABLE DENSE HOMOGENEITY

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Electronically published on May 23, 2014

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

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ISSN: 0146-4124

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PRODUCTS AND COUNTABLE DENSE HOMOGENEITY

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ABSTRACT. Building on work of Stewart Baldwin and Robert E. Beaudoin, assuming Martin's Axiom, we construct a zero-dimensional separable metrizable space X such that X is countable dense homogeneous while X^2 is not. It follows from results of Michael Hrušák and Beatriz Zamora Avilés that such a space X cannot be Borel. Furthermore, X can be made homogeneous and completely Baire as well.

1. INTRODUCTION

As it is common in the literature about countable dense homogeneity, by *space* we will always mean “separable metrizable topological space.” By *countable* we will always mean “at most countable.” For all undefined topological notions, we refer to [15]. Our reference for descriptive set theory is [8]. For all other set-theoretic notions, we refer to [9]. Given a space X , we will denote by $\mathcal{H}(X)$ the group of homeomorphisms of X . Recall that a space X is *countable dense homogeneous* (briefly, CDH) if, for every pair (A, B) of countable dense subsets of X , there exists $h \in \mathcal{H}(X)$ such that $h[A] = B$.

The fundamental positive result in the theory of CDH spaces is the following (see [1, Theorem 5.2]). In particular, it shows that the Cantor set 2^ω , the Baire space ω^ω , the Euclidean spaces \mathbb{R}^n , the spheres S^n , and the Hilbert cube $[0, 1]^\omega$ are all examples of CDH spaces. See [2, §14, §15, and §16] for much more on this topic. Recall that a space is *strongly locally homogeneous* (briefly, SLH) if there exists a base \mathcal{B} for X such that

2010 *Mathematics Subject Classification.* 54G20, 03E50, 54H05.

Key words and phrases. countable dense homogeneous, Martin's Axiom, square. The author acknowledges the support of the FWF grant I 1209-N25.

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for every $U \in \mathcal{B}$ and $x, y \in U$, there exists $h \in \mathcal{H}(X)$ such that $h(x) = y$ and $h \upharpoonright (X \setminus U) = \text{id}_{X \setminus U}$.

Theorem 1.1 (Anderson, Curtis, and van Mill). *Every Polish SLH space is CDH.*

Using the fact that homeomorphisms permute the connected components, it is easy to see that the product $2^\omega \times S^1$ is not CDH. Therefore, countable dense homogeneity is not productive, even in the class of compact topological groups. However, the situation improves if we restrict our attention to zero-dimensional spaces. Let

$$\mathcal{C} = \{X : X \approx \kappa \oplus (\lambda \times 2^\omega) \oplus (\mu \times \omega^\omega), \text{ where } 0 \leq \kappa, \lambda, \mu \leq \omega\}$$

be the class of spaces that are homeomorphic to a countable disjoint sum of copies of 2^ω , ω^ω , and 1. Recall that a space is an *absolute Borel set* (or simply *Borel*) if it is a Borel subset of some Polish space. See [7, Lemma 2.2, Corollary 2.4, and Corollary 2.5] for a proof of the following theorem.

Theorem 1.2 (Hrušák and Zamora Avilés). *If X is a zero-dimensional Borel CDH space, then $X \in \mathcal{C}$.*

Proposition 1.3. *The class \mathcal{C} is closed under countable products.*

Proof. It is easy to verify directly that \mathcal{C} is closed under finite products. So let X be the product of a countably infinite subcollection of \mathcal{C} . Without loss of generality, we can assume that all factors of X are non-empty and infinitely many of them have size bigger than 1. It follows that X is a non-empty zero-dimensional Polish space with no isolated points. Therefore, X is homeomorphic to 2^ω (if X is compact) or to ω^ω (if X is not compact, hence nowhere compact). \square

Corollary 1.4. *Assume that I is countable and X_i is a zero-dimensional Borel CDH space for every $i \in I$. Then $\prod_{i \in I} X_i$ is CDH.*

Using a method of Stewart Baldwin and Robert E. Beaudoin (see §2), we will show that the “Borel” assumption in Corollary 1.4 cannot be dropped.

The following is our main result. The construction of the example is the content of §3, and the verification of its properties is the content of §4. Recall that a space X is *completely Baire* if every closed subspace of X is Baire. By a classical result of Hurewicz, a space is completely Baire if and only if it does not contain any closed subspace that is homeomorphic to \mathbb{Q} (see [15, Corollary 1.9.13]).

Theorem 1.5. *Assume MA(σ -centered). Then there exists a zero-dimensional CDH space X such that X^2 is not CDH.¹ Furthermore, X can be made homogeneous and completely Baire as well.*

We will not prove the second part of the theorem, since it can be obtained by exactly the same methods used in the proof of [3, Theorem 3.5], and it would make our construction unnecessarily cumbersome. In fact, those methods show that the space X can be made a homogeneous Bernstein set. Recall that a subset B of 2^ω is a *Bernstein set* if $B \cap K \neq \emptyset$ and $(2^\omega \setminus B) \cap K \neq \emptyset$ for every perfect subset K of 2^ω . Using the characterization mentioned above, it is easy to see that every Bernstein set is completely Baire.

We conclude this introduction with several open questions.

Question 1.6. Can the assumption of MA(σ -centered) in Theorem 1.5 be dropped?

Question 1.7. For which κ such that $2 \leq \kappa \leq \omega$ is there a zero-dimensional space X such that X^n is CDH for every $n < \kappa$ while X^κ is not?² Can X be homogeneous and completely Baire?

Notice that the space $X = 2^\omega \oplus S^1$ is CDH while X^2 is not. However, in the case $3 \leq \kappa \leq \omega$, we would not know the answer to Question 1.7 even if the zero-dimensionality requirement were dropped.

The *type* of a countable dense subset D of a space X is $\{h[D] : h \in \mathcal{H}(X)\}$. Clearly, a space is CDH if and only if it has exactly one type of countable dense subsets. Also notice that \mathfrak{c} is the maximum possible number of types of countable dense subsets of a space. In [6], Michael Hrušák and Jan van Mill started an investigation of this natural notion. In particular, [6, Theorem 4.5] gives a condition under which a space must have \mathfrak{c} types of countable dense subsets (see also [10, Theorem 14 and Theorem 16] for more specific statements). However, we were unable to answer the following question, even in the case $\kappa = \mathfrak{c}$.

¹It follows from recent results of Rodrigo Hernández-Gutiérrez, Michael Hrušák, and Jan van Mill [5] that such a space also exists in models obtained by adding at least ω_2 Cohen reals to a model of CH. Just consider $X = Y \oplus 2^\omega$, where Y is a meager CDH subspace of 2^ω (such a space exists in ZFC by [5, Theorem 4.1]). Notice that $Z = X^2 \setminus (2^\omega \times 2^\omega)$ is meager and has size \mathfrak{c} , so it is not CDH by the proof of [5, Theorem 4.4]. On the other hand, Z is preserved by every homeomorphism of X^2 , because it is the union of all meager open subsets of X^2 . Therefore, X^2 is not CDH. However, it is clear that X is neither homogeneous nor Baire.

²The case $\kappa = \omega$ was recently settled by Hernández-Gutiérrez, Hrušák, and van Mill [5, Theorem 4.8], who proved the existence of such a space in ZFC.

Question 1.8. For which cardinals κ such that $2 \leq \kappa \leq \mathfrak{c}$ is there a zero-dimensional CDH space X such that X^2 has exactly κ types of countable dense subsets?

By [7, Corollary 2.7], Theorem 1.2 extends to all projective spaces if one assumes the axiom of Projective Determinacy. Hence, the same is true for Corollary 1.4. Therefore, the following question seems natural.

Question 1.9. Is it consistent that there exists a zero-dimensional analytic CDH space X such that X^2 is not CDH? Co-analytic?

By considering $(\omega + 1) \times 2^\omega \approx 2^\omega$, one sees that a factor of a CDH product need not be CDH. Actually, van Mill [13] constructed a rigid space X such that $X^2 \approx [0, 1]^\omega$. Recall that a space is *rigid* if its only homeomorphism is the identity. In particular, X is a continuum that is not CDH, while X^2 is CDH.

Question 1.10. For which κ such that $2 \leq \kappa \leq \omega$ is there a space X such that X^n is not CDH for every $n < \kappa$ while X^κ is CDH?³ Can X be a continuum?

L. Brian Lawrence [11] constructed a non-trivial rigid zero-dimensional space X such that X^2 is homogeneous. But we do not know whether X^2 can be made CDH.

Question 1.11. Is there a zero-dimensional space X that is not CDH while X^2 is CDH? Can X be rigid?

2. RESULTS OF BALDWIN AND BEAUDOIN

Given an infinite cardinal λ , a subset D of a space X is λ -dense in X if $|D \cap U| = \lambda$ for every non-empty open subset U of X . The following results are [3, Lemma 3.1 and Lemma 3.2]. We present a simpler version of the proof of the first result. Similar posets were recently used in the proofs of [12, Lemma 22 and Lemma 25].

Theorem 2.1 (Baldwin and Beaudoin). *Assume MA(σ -centered). Let $\kappa < \mathfrak{c}$ be a cardinal. Suppose that A_α and B_α are countable dense subsets of 2^ω for each $\alpha < \kappa$. Also assume that $A_\alpha \cap A_\beta = \emptyset$ and $B_\alpha \cap B_\beta = \emptyset$ whenever $\alpha < \beta < \kappa$. Then there exists $f \in \mathcal{H}(2^\omega)$ such that $f[A_\alpha] = B_\alpha$ for every $\alpha < \kappa$.*

Proof. Consider the poset \mathbb{P} consisting of all pairs of the form $p = (g, \pi) = (g_p, \pi_p)$ such that, for some $n = n_p \in \omega$, the following requirements are satisfied. Let $A = \bigcup_{\alpha \in \kappa} A_\alpha$ and $B = \bigcup_{\alpha \in \kappa} B_\alpha$.

³Notice that $X = \omega + 1$ and $X = [0, 1]^d$ for every d such that $1 \leq d < \omega$ answer in the affirmative the case $\kappa = \omega$.

- $g = g_{\alpha_1} \cup \dots \cup g_{\alpha_m}$, where $\alpha_1 < \dots < \alpha_m < \kappa$ and each g_{α_i} is a finite bijection from A_{α_i} to B_{α_i} .
- π is a permutation of n^2 .
- $\pi(a \upharpoonright n) = g(a) \upharpoonright n$ for every $a \in \text{dom}(g)$.

Order \mathbb{P} by declaring $q \leq p$ if the following conditions are satisfied.

- $g_q \supseteq g_p$.
- $\pi_q(t) \upharpoonright n_p = \pi_p(t \upharpoonright n_p)$ for all $t \in n^2$.

For each $\ell \in \omega$, define

$$D_\ell = \{p \in \mathbb{P} : n_p \geq \ell\}.$$

Let $p = (g, \pi) \in \mathbb{P}$ with $n_p = n$, and let $\ell \in \omega$. Choose $n' \geq \ell, n$ big enough so that all $a \upharpoonright n'$ are distinct for $a \in \text{dom}(g)$ and all $b \upharpoonright n'$ are distinct for $b \in \text{ran}(g)$. Now it is easy to obtain a permutation π' of n'^2 such that $q = (g, \pi') \in \mathbb{P}$ and $q \leq p$. So each D_ℓ is dense in \mathbb{P} .

For each $a \in A$, define

$$D_a^{\text{dom}} = \{p \in \mathbb{P} : a \in \text{dom}(g_p)\}.$$

Given $p \in \mathbb{P}$ and $a \in A_\alpha \setminus \text{dom}(g_p)$, one can simply choose $b \in B_\alpha \setminus \text{ran}(g_p)$ such that $b \upharpoonright n_p = \pi_p(a \upharpoonright n_p)$. This choice will make sure that $q = (g_p \cup \{(a, b)\}, \pi_p) \in \mathbb{P}$. Furthermore, it is clear that $q \leq p$. So each D_a^{dom} is dense in \mathbb{P} .

For each $b \in B$, define

$$D_b^{\text{ran}} = \{p \in \mathbb{P} : b \in \text{ran}(g_p)\}.$$

As above, one can easily show that each D_b^{ran} is dense in \mathbb{P} .

It remains to show that \mathbb{P} is σ -centered. We will proceed as in [9, Exercise III.2.13]. It will be enough to construct $x_e : A \rightarrow B$ for $e \in \omega$ such that $g \subseteq x_e$ for some e whenever $g = g_p$ for some $p \in \mathbb{P}$. Let $\{f_\alpha : \alpha < \kappa\}$ be an independent family of functions (see [9, Exercise III.2.12]). In particular, each $f_\alpha : \omega \rightarrow \omega$, and, given any $j_1, \dots, j_m \in \omega$ and $\alpha_1 < \dots < \alpha_m < \kappa$, there exists $e \in \omega$ such that $f_{\alpha_1}(e) = j_1, \dots, f_{\alpha_m}(e) = j_m$. Enumerate as $\{d_j^\alpha : j \in \omega\}$ all finite bijections from A_α to B_α . It is easy to check that defining

$$x_e = \bigcup_{\alpha \in \kappa} d_{f_\alpha(e)}^\alpha$$

for every $e \in \omega$ yields the desired functions. (Notice that \mathbb{P} would be σ -centered even if $\kappa = \mathfrak{c}$. However, in that case, we would have too many dense sets.)

Since $|A| < \mathfrak{c}$ and $|B| < \mathfrak{c}$, the collection of dense sets

$$\mathcal{D} = \{D_\ell : \ell \in \omega\} \cup \{D_a^{\text{dom}} : a \in A\} \cup \{D_b^{\text{ran}} : b \in B\}$$

also has size less than \mathfrak{c} . Therefore, by $\text{MA}(\sigma\text{-centered})$, there exists a \mathcal{D} -generic filter $G \subseteq \mathbb{P}$. To define $f(x)(i)$, for a given $x \in 2^\omega$ and $i \in \omega$, choose any $p \in G$ such that $i \in n_p$ and set $f(x)(i) = \pi_p(x \upharpoonright n_p)(i)$. \square

Corollary 2.2 (Baldwin and Beaudoin). *Assume $\text{MA}(\sigma\text{-centered})$. Let $\kappa < \mathfrak{c}$ be a cardinal. Suppose that A_α and B_α are λ_α -dense subsets of 2^ω for each $\alpha < \kappa$, where each $\lambda_\alpha < \mathfrak{c}$ is an infinite cardinal. Also assume that $A_\alpha \cap A_\beta = \emptyset$ and $B_\alpha \cap B_\beta = \emptyset$ whenever $\alpha < \beta < \kappa$. Then there exists $f \in \mathcal{H}(2^\omega)$ such that $f[A_\alpha] = B_\alpha$ for every $\alpha < \kappa$.*

Proof. Notice that each A_α , being a λ_α -dense subset of 2^ω , can be partitioned into λ_α countable dense subsets of 2^ω . The same holds for each B_α . Since $\text{MA}(\sigma\text{-centered})$ implies that \mathfrak{c} is regular (see, for example, [9, Theorem III.3.61 and Lemma III.1.26]), we can apply Theorem 2.1. \square

The results in this section were originally employed by Baldwin and Beaudoin [3, Theorem 3.5] to construct a homogeneous CDH Bernstein set under $\text{MA}(\sigma\text{-centered})$. We remark that their proof contains a small inaccuracy. Using their notation, given λ -dense $A, B \subseteq X_\alpha$, it is not possible to use Corollary 2.2 to find $g \in \mathcal{H}(2^\omega)$ such that $g[A] = B$, $g[B] = A$, $g[X_\gamma \setminus (A \cup B)] = X_\gamma \setminus (A \cup B)$, and $g[Y_\gamma] = Y_\gamma$, since A and B might not be disjoint. However, this is easily fixed by requiring instead that $g[A] = B$, $g[X_\gamma \setminus A] = X_\gamma \setminus B$, and $g[Y_\gamma] = Y_\gamma$, as we do in the next section.

3. THE CONSTRUCTION

Let Q and R be any two disjoint countable dense subsets of 2^ω . Given $i \in 2$, denote by $\pi_i : 2^\omega \times 2^\omega \rightarrow 2^\omega$ the natural projection on the i -th coordinate. Given $S \subseteq 2^\omega$ and a subgroup \mathcal{H} of $\mathcal{H}(2^\omega)$, let

$$\mathcal{H}[S] = \{h(z) : z \in S, h \in \mathcal{H}\}$$

denote the closure of S under the action of \mathcal{H} .

Enumerate as $\{(A_\alpha, B_\alpha) : \alpha < \mathfrak{c}\}$ all pairs of countable dense subsets of 2^ω , making sure that each pair is listed cofinally often. Enumerate as $\{g_\alpha : \alpha < \mathfrak{c}\}$ all homeomorphisms satisfying the following conditions.

- $g_\alpha : T_\alpha \rightarrow T_\alpha$, where T_α is a \mathbf{G}_δ subset of $2^\omega \times 2^\omega$.
- $Q^2 \subseteq T_\alpha$.
- $\pi_0 \upharpoonright (g_\alpha[Q^2])$ is injective.

Notice that each T_α is dense in $2^\omega \times 2^\omega$. In particular, if M is meager in $2^\omega \times 2^\omega$, then $M \cap T_\alpha$ is meager in T_α . Also notice that each T_α is a Polish space.

By transfinite recursion, we will construct two increasing sequences $\langle X_\alpha : \alpha < \mathfrak{c} \rangle$ and $\langle Y_\alpha : \alpha < \mathfrak{c} \rangle$ of subsets of 2^ω and an increasing sequence $\langle \mathcal{H}_\alpha : \alpha < \mathfrak{c} \rangle$ of subgroups of $\mathcal{H}(2^\omega)$.

By induction, we will make sure that the following requirements are satisfied for every $\alpha < \mathfrak{c}$.

- (1) $|X_\alpha|, |Y_\alpha|, |\mathcal{H}_\alpha| \leq \max\{|\alpha|, \omega\}$.
- (2) $X_\alpha \cap Y_\alpha = \emptyset$.
- (3) $h[X_\alpha] = X_\alpha$ and $h[Y_\alpha] = Y_\alpha$ for all $h \in \mathcal{H}_\alpha$.
- (4) $X_{\alpha+1} \setminus X_\alpha$ and $Y_{\alpha+1} \setminus Y_\alpha$ are $\max\{|\alpha|, \omega\}$ -dense in 2^ω .
- (5) If $A_\alpha \cup B_\alpha \subseteq X_\alpha$ and $X_\alpha \setminus (A_\alpha \cup B_\alpha)$ is $\max\{|\alpha|, \omega\}$ -dense in 2^ω , then there exists $f \in \mathcal{H}_{\alpha+1}$ such that $f[A_\alpha] = B_\alpha$.
- (6) There exists $(x, y) \in X_{\alpha+1}^2 \cap T_\alpha$ such that $\pi_0(g_\alpha(x, y)) \in Y_{\alpha+1}$.

Start the construction by letting $X_0 = Q$, $Y_0 = R$, and $\mathcal{H}_0 = \{\text{id}_{2^\omega}\}$.

Take unions at limit stages. At a successor stage $\alpha + 1$, assume that X_β , Y_β , and \mathcal{H}_β are given for every $\beta \leq \alpha$. We will start by defining $\mathcal{H}_{\alpha+1}$, making sure that condition (5) is satisfied. Let $\lambda = \max\{|\alpha|, \omega\}$. If $(A_\alpha \cup B_\alpha) \not\subseteq X_\alpha$ or $X_\alpha \setminus (A_\alpha \cup B_\alpha)$ is not λ -dense in 2^ω , simply let $\mathcal{H}_{\alpha+1} = \mathcal{H}_\alpha$. Now assume that $(A_\alpha \cup B_\alpha) \subseteq X_\alpha$ and $X_\alpha \setminus (A_\alpha \cup B_\alpha)$ is λ -dense in 2^ω . By applying Corollary 2.2 with $\kappa = 3$, $\lambda_0 = \omega$, and $\lambda_1 = \lambda_2 = \lambda$, one obtains $f \in \mathcal{H}(2^\omega)$ such that $f[A_\alpha] = B_\alpha$, $f[X_\alpha \setminus A_\alpha] = X_\alpha \setminus B_\alpha$, and $f[Y_\alpha] = Y_\alpha$. Let $\mathcal{H}_{\alpha+1} = \langle \mathcal{H}_\alpha \cup \{f\} \rangle$. For the rest of the proof, let $\mathcal{H} = \mathcal{H}_{\alpha+1}$.

Next, we will make sure that condition (6) is satisfied. Define

$$C_h^i = \{(x, y) \in T_\alpha : h(\pi_i(x, y)) = \pi_0(g_\alpha(x, y))\}$$

for $h \in \mathcal{H}$ and $i \in 2$. Clearly, each C_h^i is closed. We claim that they are also nowhere dense. We will prove this only for C_h^0 , since a similar argument works for C_h^1 . In order to get a contradiction, assume that $U, V \subseteq 2^\omega$ are non-empty open sets such that $h(x) = \pi_0(g_\alpha(x, y))$ whenever $(x, y) \in (U \times V) \cap T_\alpha$. Fix $q, r, r' \in Q$ such that $r \neq r'$ and $(q, r), (q, r') \in U \times V$. Then $\pi_0(g_\alpha(q, r)) = h(q) = \pi_0(g_\alpha(q, r'))$, contradicting the assumption that $\pi_0 \upharpoonright (g_\alpha[Q^2])$ is injective.

Since $\text{MA}(\sigma\text{-centered})$ obviously implies $\text{MA}(\text{countable})$, which is equivalent to $\text{cov}(\text{meager}) = \mathfrak{c}$ (see [4, Theorem 7.13]), and $|\mathcal{H}|, |X_\alpha|, |Y_\alpha| < \mathfrak{c}$ by condition (1), there exists $(x, y) \in T_\alpha$ such that

$$(x, y) \notin g_\alpha^{-1}[\pi_0^{-1}[X_\alpha] \cap T_\alpha] \cup \bigcup_{i \in 2} \pi_i^{-1}[Y_\alpha] \cup \bigcup_{h \in \mathcal{H}, i \in 2} C_h^i.$$

Notice that $\mathcal{H}[X_\alpha] \cap \mathcal{H}[Y_\alpha] = X_\alpha \cap Y_\alpha = \emptyset$ by conditions (2) and (3) and by our choice of f . The set $g_\alpha^{-1}[\pi_0^{-1}[X_\alpha] \cap T_\alpha]$ guarantees that $\mathcal{H}[X_\alpha] \cap \mathcal{H}[\{\pi_0(g_\alpha(x, y))\}] = \emptyset$. The sets $\pi_i^{-1}[Y_\alpha]$ guarantee that $\mathcal{H}[\{x, y\}] \cap \mathcal{H}[Y_\alpha] = \emptyset$. The sets C_h^i guarantee that $\mathcal{H}[\{x, y\}] \cap \mathcal{H}[\{\pi_0(g_\alpha(x, y))\}] = \emptyset$.

Combining the above observations, one sees that

$$\mathcal{H}[X_\alpha \cup \{x, y\}] \cap \mathcal{H}[Y_\alpha \cup \{\pi_0(g_\alpha(x, y))\}] = \emptyset.$$

It follows that it is possible to construct $X_{\alpha+1} \supseteq \mathcal{H}[X_\alpha \cup \{x, y\}]$ and $Y_{\alpha+1} \supseteq \mathcal{H}[Y_\alpha \cup \{\pi_0(g_\alpha(x, y))\}]$ that satisfy the requirement (4), while still maintaining (1), (2), and (3). This can be done in λ stages, adding one point to each from every non-empty clopen subset of 2^ω and closing under the action of \mathcal{H} at each stage.

In the end, set $X = \bigcup_{\alpha < \mathfrak{c}} X_\alpha$.

4. THE VERIFICATION

We will start by showing that X is CDH. So fix a pair (A, B) of countable dense subsets of X . Since $\text{cf}(\mathfrak{c}) > \omega$, there exists $\alpha < \mathfrak{c}$ such that $A \cup B \subseteq X_\alpha$. Now fix $\beta \geq \alpha + 1$ such that $(A, B) = (A_\beta, B_\beta)$. Notice that $X_\beta \setminus (A_\beta \cup B_\beta)$ is $\max\{|\beta|, \omega\}$ -dense in 2^ω by condition (4). Therefore, by condition (5), there exists $f \in \mathcal{H}_{\beta+1}$ such that $f[A_\beta] = B_\beta$. Condition (3) guarantees that $f[X] = X$, so $f \upharpoonright X$ is the desired homeomorphism.

In order to show that X^2 is not CDH, we will employ the following classical result, which is a well-known tool for “killing” homeomorphisms (see [14] for several interesting applications). For a proof of Theorem 4.1, see [8, Theorem 3.9 and Exercise 3.10].

Theorem 4.1 (Lavrentiev). *Let Z be a Polish space and $S \subseteq Z$. Every homeomorphism $f : S \rightarrow S$ extends to a homeomorphism $g : T \rightarrow T$, where $T \supseteq S$ is a \mathbf{G}_δ subset of Z .*

Let D be a countable dense subset of X^2 such that $\pi_0 \upharpoonright D$ is injective. Such a subset is easy to construct using the fact that X has no isolated points. Assume, in order to get a contradiction, that $f : X^2 \rightarrow X^2$ is a homeomorphism such that $f[Q^2] = D$. By Theorem 4.1, there exists a homeomorphism $g : T \rightarrow T$ that extends f , where $T \supseteq X^2 \supseteq Q^2$ is a \mathbf{G}_δ subset of $2^\omega \times 2^\omega$. Since we enumerated all such homeomorphisms, we must have $g = g_\alpha$ and $T = T_\alpha$ for some $\alpha < \mathfrak{c}$. By conditions (6) and (2), there exists $(x, y) \in X^2 \cap T_\alpha$ such that $\pi_0(g_\alpha(x, y)) \notin X$, contradicting the fact that $g_\alpha(x, y) = g(x, y) = f(x, y) \in X^2$.

Acknowledgments. The author is grateful to the anonymous referee for spotting a serious problem in an early version of this article. He also thanks Lyubomyr Zdomsky for useful discussions, and, in particular, for pointing out the footnote to Theorem 1.5.

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