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T. Nasri, B. Mashayekhy, and H. Mirebrahimi

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Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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ON QUASITOPOLOGICAL HOMOTOPY GROUPS OF INVERSE LIMIT SPACES

T. NASRI, B. MASHAYEKHY, AND H. MIREBRAHIMI

Abstract. The paper is devoted to study the behavior of quasitopological homotopy groups on inverse limit spaces. More precisely, we present some conditions under which the quasitopological homotopy group of an inverse limit space and especially a product space is a topological group. Finally, we give some conditions for countability of homotopy groups.

1. Introduction and Motivation

Endowed with the quotient topology induced by the natural surjective map $q : \Omega^n(X,x) \to \pi_n(X,x)$, where $\Omega^n(X,x)$ is the nth loop space of (X,x) with the compact-open topology, the familiar homotopy group $\pi_n(X,x)$ becomes a quasitopological group which is called the quasitopological nth homotopy group of the pointed space (X,x), denoted by $\pi_n^{qtop}(X,x)$ (see [2], [3], [4], and [17]).

It was claimed by Daniel K. Biss [2] that $\pi_1^{qtop}(X,x)$ is a topological group. However, Jack S. Calcut and John D. McCarthy [5] and Paul Fabel [12] showed that there is a gap in the proof of [2, Proposition 3.1]. The misstep in the proof is repeated by H. Ghane et al. [17] to prove that $\pi_n^{qtop}(X,x)$ is a topological group [17, Theorem 2.1] (see also [5]).

Calcut and McCarthy [5] showed that $\pi_1^{qtop}(X,x)$ is a homogeneous space and more precisely, Jeremy Brazas [3] mentioned that $\pi_1^{qtop}(X,x)$ is a quasitopological group in the sense of [1].

Calcut and McCarthy [5] proved that for a path connected and locally path connected space X, $\pi_1^{qtop}(X)$ is a discrete topological group if and

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only if X is semilocally 1-connected (see also [3]). Ali Pakdaman et al. [24] show that for a locally $(n-1)$-connected space X, $\pi_n^{qtop}(X,x)$ is discrete if and only if X is semilocally n-connected at x (see also [17]). Also, they prove that the quasitopological fundamental group of every small loop space is an indiscrete topological group.

Fabel [12] and [13] and Brazas [3] presented some spaces for which their quasitopological homotopy groups are not topological groups. Moreover, despite Fabel's result [12] that says the quasitopological fundamental group of the Hawaiian earring is not a topological group, Ghane et al. [18] proved that the quasitopological nth homotopy group of an n-Hawaiian like space is a prodiscrete metrizable topological group for all $n \geq 2$. By *n-Hawaiian like space* X, we mean the natural inverse limit, $\varprojlim(Y_i^n, y_i^*)$, where $(Y_i^n, y_i^*) = \bigvee_{j\leq i}(X_j^n, x_j^*)$ is the wedge of X_j^n's in which X_j^n's are $(n-1)$-connected, locally $(n-1)$-connected, semilocally n-connected, and compact CW spaces. Hence, it seems interesting to find out when $\pi_n^{qtop}(X,x)$ is or is not a topological group.

The recent result inspires the authors to study the behavior of quasitopological homotopy groups on inverse limit spaces. Let I be a partially ordered set, and let $\{(X_i,x_i),\varphi_{ij}\}_I$ be an inverse system of pointed topological spaces, where $\varphi_{ij} : (X_j,x_j) \to (X_i,x_i)$ is a pointed continuous map for all $i \leq j$. Suppose $(X,x) = \varprojlim(X_i,x_i)$ is the inverse limit space of the above inverse system. Since $\pi_n^{qtop}(-)$ is a functor from the category of pointed topological spaces Top_* to the category of quasitopological groups $qTopGrp$, there exists a natural continuous homomorphism $\beta_n^{qtop} : \pi_n^{qtop}(X,x) \to \varprojlim \pi_n^{qtop}(X_i,x_i)$. The main goal of the paper is to present some inverse limit spaces and especially some product spaces whose quasitopological nth homotopy groups are topological groups for $n \geq 1$.

The paper is organized as follows. The crucial misstep in proving that $\pi_n^{qtop}(X,x)$ is a topological group is that, for the natural quotient map $q : \Omega^n(X,x) \to \pi_n(X,x)$, the map $q \times q$ can fail to be a quotient map [12, Theorem 1]. In §2, we intend to obtain some conditions in which the product map $q \times q$ is a quotient map for some spaces. Using this fact, we can present a class of product spaces whose quasitopological nth homotopy groups are topological groups. One of the main results of §2 is as follows:

Corollary 2.7. *Let $(X,x) = \prod_{i\in I}(X_i,x_i)$, where X_i's are second countable spaces whose nth homotopy groups are countable and Hausdorff. Then the isomorphism $\pi_n^{qtop}(X,x) \cong \prod_{i\in I}\pi_n^{qtop}(X_i,x_i)$ holds in topological groups.*

Note that Biss [2, Proposition 5.2] claimed the above result without any conditions, but his proof has the mentioned misstep. Another main result of §2 is as follows:

Corollary 2.12. *Let* $(X,x) = \varprojlim(X_i,x_i)$ *be the inverse limit space of an inverse system* $\{(X_i,x_i),\varphi_{ij}\}_I$*, where* I *is countable. Suppose that* X_i *is second countable,* $\pi_n^{qtop}(X_i,x_i)$ *is Hausdorff for all* $i \in I$*,* $\pi_n^{qtop}(X,x)$ *is second countable and the map* $\beta_n : \pi_n(X,x) \to \varprojlim \pi_n(X_i,x_i)$ *is a group monomorphism. Then* $\pi_n^{qtop}(X,x)$ *is a topological group.*

One of the main conditions of the above result is assuming that $\beta_n : \pi_n(X,x) \to \varprojlim \pi_n(X_i,x_i)$ is a group monomorphism. Using [10], [14], and [15], we point out some classes of spaces for which the map β_n is a group monomorphism. Moreover, we recall conditions under which homotopy group functors commute with inverse limits.

In §3, we intend to generalize some results of G. R. Conner and J. W. Lamoreaux [8] on the countability of $\pi_1(X,x)$ to obtain some conditions on a space to have a countable homotopy group. For this, we show that some properties of a space can be transferred to its loop space.

2. Quasitopological Homotopy Groups of Inverse Limits

First, we intend to obtain some conditions on a topological space X under which $\pi_n^{qtop}(X,x)$ is a topological group. The crucial misstep in proving that $\pi_n^{qtop}(X,x)$ is a topological group is that for the natural quotient map $q : \Omega^n(X,x) \to \pi_n(X,x)$, the map $q \times q$ can fail to be a quotient map (see [12, Theorem 1]). We are going to find some conditions under which the product of quotient maps $q \times q$ is also a quotient map. Ernest Michael [22] introduced a new class of quotient maps, called bi-quotient maps. A map $f : X \to Y$ is called a *bi-quotient map* if, whenever $y \in Y$ and $\mathcal{U}$ is a covering of $f^{-1}(y)$ by open subsets of X, finitely many $f(U)$, where $U \in \mathcal{U}$, cover some neighborhood of y in Y [22, Definition 1.1]. He showed [22, Theorem 1.2] that any product (finite or infinite) of bi-quotient maps is also a bi-quotient map. Thus, it is sufficient to see under which conditions the quotient map q is a bi-quotient map. We need the following interesting results of Michael [22].

Lemma 2.1 ([22, Corollary 3.5])**.** *Let* $f : X \to Y$ *be a quotient map, where* Y *is Hausdorff and* X *is second countable. Then* f *is a bi-quotient map if and only if* Y *is second countable.*

A topological space X is called a *k-space* if a set $A \subseteq X$ is closed whenever $A \cap K$ is closed in K for every compact $K \subseteq X$. Note that locally compact spaces and first countable spaces are k-spaces [22].

Lemma 2.2 ([22, Theorem 1.5]). *If $f_i : X_i \to Y_i$, $i = 1, 2$, are quotient maps and X_1 and $Y_1 \times Y_2$ are both Hausdorff k-spaces, then $f_1 \times f_2$ is a quotient map.*

We shall also need the following well-known result.

Theorem 2.3 ([9, Theorem 12.5.2]). *If X is second countable and Y is locally compact and second countable, then the function space X^Y is second countable. In particular, if X is second countable, then $\Omega^n(X, x)$ is also second countable for all $x \in X$ and $n \in \mathbb{N}$.*

The following theorems are two of the main results of this section.

Theorem 2.4. *Suppose that X is second countable and $\pi_n^{qtop}(X, x)$ is Hausdorff and second countable. Then $\pi_n^{qtop}(X, x)$ is a topological group.*

Proof. Since X is second countable, by Theorem 2.3, $\Omega^n(X, x)$ is second countable. Now, by Lemma 2.1, the quotient map q is a bi-quotient map. Hence, by [22, Theorem 1.2], the product map $q \times q$ is a bi-quotient map and so it is a quotient map which yields the result. □

Theorem 2.5. *Let X be a metric space. If one of the following conditions holds, then $\pi_n^{qtop}(X, x)$ is a topological group.*

(i) *$\pi_n^{qtop}(X, x)$ is Hausdorff and first countable.*
(ii) *$\pi_n^{qtop}(X, x)$ is Hausdorff and locally compact.*

Proof. (i) Since X is metric, the loop space $\Omega^n(X, x)$ is metric by [20, Theorem 1 of Chapter IV §44.V] and so it is Hausdorff and first countable. Now, by Lemma 2.2, the product map $q \times q$ is a quotient map which implies the result.

(ii) Similar to (i). □

Now, we want to present a class of product spaces whose quasitopological nth homotopy groups are topological groups. Let $(X, x) = \prod_{i \in I}(X_i, x_i)$ and consider the following commutative diagram:

$$\begin{array}{ccc} \Omega^n(X,x) & \xrightarrow{\phi} & \prod_{i\in I}\Omega^n(X_i,x_i) \\ \Big\downarrow q & & \prod_{i\in I} q_i \Big\downarrow \\ \pi_n^{qtop}(X,x) & \xrightarrow{\beta_n^{qtop}} & \prod_{i\in I}\pi_n^{qtop}(X_i,x_i), \end{array} \tag{2.1}$$

where ϕ is the natural homeomorphism using the fact that loop spaces preserve products and $q_i : \Omega^n(X_i, x_i) \to \pi_n^{qtop}(X_i, x_i)$ is the natural quotient map for all $i \in I$. Since the map q is a quotient map and the map $\beta_n : \pi_n(X, x) \to \prod_{i \in I} \pi_n(X_i, x_i)$ is a group isomorphism, if we show that $\prod_{i \in I} q_i$ is also a quotient map, then $\beta_n^{qtop} : \pi_n^{qtop}(X, x) \to$

$\prod_{i\in I}\pi_n^{qtop}(X_i,x_i)$ will be an isomorphism of quasitopological groups. We are going to study the case in which the quotient map q_i is a bi-quotient map for all $i\in I$. The following theorem is another main result of this section.

Theorem 2.6. *Let $\{(X_i,x_i)\}_{i\in I}$ be a family of second countable spaces such that $\pi_n^{qtop}(X_i,x_i)$ is Hausdorff and second countable for all $i\in I$. Then the map $\prod q_i:\prod_{i\in I}\Omega^n(X_i,x_i)\to\prod_{i\in I}\pi_n^{qtop}(X_i,x_i)$ is a quotient map.*

Proof. Since X_i is second countable, by Theorem 2.3, $\Omega^n(X_i,x_i)$ is also second countable for all $i\in I$. Since $\pi_n^{qtop}(X_i,x_i)$ is Hausdorff and second countable, by Lemma 2.1, the quotient map $q_i:\Omega^n(X_i,x_i)\to\pi_n^{qtop}(X_i,x_i)$ is a bi-quotient map for all $i\in I$. Hence, by [22, Theorem 1.2], the product map $\prod q_i$ is a bi-quotient map and so it is a quotient map. □

It is well known that if $(X,x)=\prod_{i\in I}(X_i,x_i)$, then $\pi_n(X,x)\cong\prod_{i\in I}\pi_n(X_i,x_i)$. However, it is known that the functor π_n^{qtop} does not preserve products in general. For example, consider the Hawaiian earring (HE). Paul Fabel [12] proved that the product $q\times q:\Omega(HE)\times\Omega(HE)\longrightarrow\pi_1^{qtop}(HE)\times\pi_1^{qtop}(HE)$ is not a quotient map, and hence the topology of $\pi_1^{qtop}(HE\times HE)$ is strictly finer than $\pi_1^{qtop}(HE)\times\pi_1^{qtop}(HE)$. Moreover, Fabel [13] showed that for each $n\geq 1$, there exists a compact, path connected, metric space X such that multiplication is discontinuous in $\pi_n^{qtop}(X,x)$. Hence, π_n^{qtop} does not preserve the product $X\times X$.

Corollary 2.7. *Let $(X,x)=\prod_{i\in I}(X_i,x_i)$, where X_i's are second countable spaces whose quasitopological nth homotopy groups are second countable and Hausdorff. Then the isomorphism $\pi_n^{qtop}(X,x)\cong\prod_{i\in I}\pi_n^{qtop}(X_i,x_i)$ holds in topological groups.*

Proof. The result holds from Theorem 2.4 and Theorem 2.6 and using diagram (2.1). □

Remark 2.8. Let $\{(X_i,x_i)\}_{i\in I}$ be a family of metric spaces such that $\pi_n^{qtop}(X_i,x_i)$ is Hausdorff and first countable or Hausdorff and locally compact for all $i\in I$. Then by Theorem 2.5 and a similar proof of Theorem 2.6 and Corollary 2.7, we have the following results, respectively.

(i) The map $\prod q_i:\prod_{i\in I}\Omega^n(X_i,x_i)\to\prod_{i\in I}\pi_n^{qtop}(X_i,x_i)$ is a quotient map.

(ii) The isomorphism $\pi_n^{qtop}(X,x)\cong\prod_{i\in I}\pi_n^{qtop}(X_i,x_i)$ holds in topological groups.

Proposition 2.9. *Let $(X,x) = \prod_{i\in I}(X_i,x_i)$, where X_i's are locally $(n-1)$-connected and semilocally n-connected at x_i. Then the isomorphism $\pi_n^{qtop}(X,x) \cong \prod_{i\in I}\pi_n^{qtop}(X_i,x_i)$ holds in topological groups.*

Proof. Since X_i is locally $(n-1)$-connected and semilocally n-connected at x_i, by [24, Theorem 6.7], $\pi_n^{qtop}(X_i,x_i)$ is discrete. Hence, $\prod_{i\in I} q_i : \prod_{i\in I}\Omega^n(X_i,x_i) \to \prod_{i\in I}\pi_n(X_i,x_i)$ is open and the result follows from diagram (2.1). □

Note that the above proposition holds if we replace the conditions locally $(n-1)$-connected and semilocally n-connected with locally n-connected (see [24, Theorem 6.6]). The examples mentioned after Theorem 2.6 show that the assumptions of second countability of $\pi_n^{qtop}(X_i,x_i)$'s in Corollary 2.7 and semilocally n-connectedness in Proposition 2.9 are essential.

Theorem 2.10. *If $\pi_n^{qtop}(X,x)$ is discrete, then X is semilocally n-connected at x.*

Proof. The result holds by a similar argument of [2, Theorem 5.1], [5, Lemma 3.1], and [17, Theorem 3.2]. □

The following result seems interesting.

Corollary 2.11. *Let $X = \prod_{i\in I} X_i$, where the X_i's are locally n-connected. Then the following statements hold.*

(i) *If X is locally n-connected, then all but finitely many of the X_i's are n-connected.*
(ii) *The space X is semilocally n-connected if all but finitely many of the X_i's are n-connected.*

Proof. (i) Since X is locally n-connected, $\pi_n^{qtop}(X)$ is discrete [24, Theorem 6.6]. By Proposition 2.9, $\prod_{i\in I}\pi_n^{qtop}(X_i)$ is discrete. Since the X_i's are locally n-connected, $\pi_n^{qtop}(X_i)$ is discrete. But a product of infinitely many discrete spaces having more than one point is not discrete. Thus, we conclude that all but finitely many of the X_i's are n-connected.

(ii) Applying Proposition 2.9, we conclude that $\prod_{i\in I}\pi_n^{qtop}(X_i) \cong \pi_n^{qtop}(X)$ is discrete. Hence, the result holds by Theorem 2.10. □

As an immediate consequence of the above corollary, $\prod S^n$ is semilocally k-connected for all $k < n$, and it is not locally n-connected.

Corollary 2.12. *Let $(X,x) = \varprojlim(X_i,x_i)$ be the inverse limit space of an inverse system $\{(X_i,x_i),\varphi_{ij}\}_I$, where I is countable. Suppose that X_i is second countable, $\pi_n^{qtop}(X_i,x_i)$ is Hausdorff for all $i\in I$, $\pi_n^{qtop}(X,x)$ is* *second countable, and the map* $\beta_n : \pi_n(X,x) \to \lim\limits_{\leftarrow} \pi_n(X_i,x_i)$ *is a group monomorphism. Then* $\pi_n^{qtop}(X,x)$ *is a topological group.*

Proof. Since X_i's are second countable and I is countable, the space X is second countable by [9, Theorem 8.6.2]. Since β_n^{qtop} is an injective continuous map and $\pi_n^{qtop}(X_i,x_i)$'s are Hausdorff, $\pi_n^{qtop}(X,x)$ is Hausdorff. Hence, Theorem 2.4 implies the result. □

Fabel [12] proved that the quasitopological fundamental group of the Hawaiian earring is not a topological group. Hence, the hypothesis of $\pi_n^{qtop}(X,x)$ being second countable is essential in Corollary 2.12.

One of the main conditions of the above result is assuming that $\beta_n : \pi_n(X,x) \to \lim\limits_{\leftarrow} \pi_n(X_i,x_i)$ is a group monomorphism. In the following we point out some classes of spaces for which the map β_1 is a group monomorphism.

Remark 2.13. Let $(X,x) = \lim\limits_{\leftarrow}(X_i,x_i)$ be compact and let the X_i's be compact polyhedra. Using [15, Remark 1], the natural homomorphism $\beta_1 : \pi_1(X,x) \to \lim\limits_{\leftarrow} \pi_1(X_i,x_i)$ is a monomorphism if one of the following conditions holds:

(1) X is either a 1-dimensional compact Hausdorff space or X is a 1-dimensional separable metric space (see [10, Corollary 1.2]).
(2) X is any subset of a closed surface M^2 (see [15, Theorem 5]).
(3) X is a tree of n-manifolds (well balanced if $n = 2$) (see [14, Theorem 3.1]).

Remark 2.14. Let $(X,x) = \lim\limits_{\leftarrow}(X_i,x_i)$ be the inverse limit space of an inverse system $\{(X_i,x_i),\varphi_{ij}\}_I$ and consider the following commutative diagram:

$$\begin{array}{ccc} \Omega^n(X,x) & \xrightarrow{\phi} & \lim\limits_{\leftarrow} \Omega^n(X_i,x_i) \\ \downarrow q & & Q \downarrow \\ \pi_n^{qtop}(X,x) & \xrightarrow{\beta_n^{qtop}} & \lim\limits_{\leftarrow} \pi_n^{qtop}(X_i,x_i), \end{array} \tag{2.2}$$

where ϕ is the natural homeomorphism and Q is the restriction of the product of the quotient maps q_i. Since the map q is a quotient map, if we assume that the map $\beta_n : \pi_n(X,x) \to \lim\limits_{\leftarrow} \pi_n(X_i,x_i)$ is a group isomorphism and show that Q is also a quotient map, then $\beta_n^{qtop} : \pi_n^{qtop}(X,x) \to \lim\limits_{\leftarrow} \pi_n^{qtop}(X_i,x_i)$ will be an isomorphism of quasitopological groups. Now, if the map $\prod q_i$ is a quotient map (see Theorem 2.6 and Remark 2.8(i)), the X_i's are Hausdorff and $\lim\limits_{\leftarrow} \Omega^n(X_i,x_i)$ is a saturated subspace of

$\prod_{i\in I}(X_i,x_i)$ with respect to $\prod q_i$, then Q is a quotient map [23, Theorem 22.1].

One of the main conditions of Remark 2.14 is assuming that $\beta_n : \pi_n(X,x) \to \varprojlim \pi_n(X_i,x_i)$ is a group isomorphism. In the following, we recall some conditions on the X_i's in which the map β_n is a group isomorphism.

The following interesting result of Joel M. Cohen [7] is essential.

Theorem 2.15. *Let $(X,x) = \varprojlim(X_i,x_i)$ be the inverse limit space of an inverse system $\{(X_i,x_i),\varphi_{ij}\}_I$, where the X_i's are Hausdorff and the consecutive maps φ_{ij} have the covering homotopy property for compactly generated spaces. Then the following short exact sequence exists for all $n \geq 1$.*

$$0 \to \varprojlim{}^1 \pi_{n+1}(X_i,x_i) \to \pi_n(X,x) \xrightarrow{\beta_n} \varprojlim \pi_n(X_i,x_i) \to 0, \tag{2.3}$$

where $\varprojlim^1$ is the first derived functor of the inverse limit functor.

Remark 2.16. To prove that β_n is a group isomorphism, it is sufficient to see under which conditions $\varprojlim^1 \pi_{n+1}(X_i,x_i) = 0$. With the assumptions of Theorem 2.15, the natural homomorphism $\beta_n : \pi_n(X,x) \to \varprojlim \pi_n(X_i,x_i)$ is a group isomorphism, for all $n \geq 1$, if $\{(X_i,x_i)\}$ is a movable inverse system. Indeed, movability of X implies that $\varprojlim \pi_n(X_i,x_i)$ is movable by [21, Remark 6.1.1], and therefore it has the Mittag-Leffler property [21, Corollary 6.2.4]. Now, by [21, Theorem 6.2.10], we have $\varprojlim^1 \pi_n(X_i,x_i) = 0$.

3. Countability of Homotopy Groups

In this section, we intend to generalize some results of Conner and Lamoreaux [8] on the countability of $\pi_1(X,x)$. For this, we show that some properties of topological spaces can be transferred from X to the loop space $\Omega^n(X,x)$ for some $x \in X$.

Lemma 3.1. *Let X be locally $(n-1)$-connected space. If $\Omega^{(n-1)}(X,x)$ is semilocally simply connected at the constant $(n-1)$-loop e_x, then X is semilocally n-connected at x for all $n \geq 2$.*

Proof. Let $n = 2$. Since $\Omega(X,x)$ is semilocally simply connected, there exists a basic element V of the constant loop e_x such that the homomorphism $\phi : \pi_1(V,e_x) \to \pi_1(\Omega(X,x),e_x)$, induced by inclusion, is trivial where $V = \bigcap_{j=1}^{m} < K_j, U_j >$. Put $U = \bigcap_{j=1}^{m} U_j$ and consider the following commutative diagram:

$$\begin{array}{ccc} \pi_1(V, e_x) & \xrightarrow{\phi} & \pi_1(\Omega(X,x), e_x) \\ \uparrow{\scriptstyle \theta^*} & & \uparrow{\scriptstyle \theta} \\ \pi_2(U, x) & \xrightarrow{\psi} & \pi_2(X,x), \end{array}$$

where $\theta : \pi_2(X,x) \to \pi_1(\Omega(X,x), e_x)$ is given by $\theta([g]) = [\bar{g}]$, where $\bar{g}(t)(s) = g(t,s)$ and θ^* is its restriction on U. Since $\theta : \pi_2(X,x) \to \pi_1(\Omega(X,x), e_x)$ is an isomorphism and the homomorphism ϕ is trivial, the homomorphism $\psi : \pi_2(U,x) \to \pi_2(X,x)$ is trivial. Hence, X is semilocally 2-connected at x. The result holds by induction on $n \geq 2$ similarly. □

Note that the converse of this fact has been shown by Hidekazu Wada [25, Remark]. The first result on countability of homotopy groups is as follows.

Theorem 3.2. *Let X be an $(n-1)$-connected, locally $(n-1)$-connected, semilocally n-connected, and separable metric space. Then $\pi_n(X,x)$ is countable.*

Proof. It is easy to see that if X is $(n-1)$-connected, then $\Omega^{(n-1)}(X,x)$ is path connected. Wada [25, Corollary] proved that if X is locally $(n-1)$-connected, then $\Omega^{(n-1)}(X,x)$ is locally path connected and so it is locally connected. Also, he showed [25, Remark] that the $(n-1)$-loop space of a semilocally n-connected space is semilocally simply connected. Since X is a separable metric space, so is $\Omega^{(n-1)}(X,x)$ [19, Ch. II, §22.III, Theorem]. Therefore, $\Omega^{(n-1)}(X,x)$ satisfies the hypotheses of [8, Lemma 2.2] which implies that $\pi_1(\Omega^{(n-1)}(X,x), e_x) \cong \pi_n(X,x)$ is countable. □

A space X is called *n-homotopically Hausdorff* at $x \in X$ if, for any essential n-loop α based at x, there is an open neighborhood U of x for which α is not homotopic (rel $\dot{I}^n$) to any n-loop lying entirely in U. X is said to be n-homotopically Hausdorff if it is n-homotopically Hausdorff at any $x \in X$ (see [16]).

Consider $\overline{\Omega^n(X,x)}$ as the space of homotopy classes rel $\dot{I}^n$ of n-loops at x in X. If p is an n-loop at x and U is an open neighborhood of x, then we define $O^n(p,U)$ to be the collection of homotopy classes of n-loops rel $\dot{I}^n$ containing n-loops of the form $p * \alpha$, where α is an n-loop in U at x. It is routine to check that the collection $O^n(p,U)$ is a basis for $\overline{\Omega^n(X,x)}$. In the following, we show that $\overline{\Omega^n(X,x)}$ is Hausdorff if and only if X is n-homotopically Hausdorff at x. (See also [6] for the case $n = 1$.)

Lemma 3.3. *$\overline{\Omega^n(X,x)}$ is Hausdorff if and only if X is n-homotopically Hausdorff at x.*

Proof. Let f be an essential n-loop based at x. Since $\overline{\Omega^n(X,x)}$ is Hausdorff, there are neighborhoods $W = \bigcup_{i\in I} O^n(p_i, U_i)$ of f and $W' = \bigcup_{i\in I} O^n(q_i, V_i)$ of e_x such that $W \cap W' = \emptyset$. Put $V = \bigcup_{i\in I} V_i$, then $x \in V$. If h is an n-loop in V and $h \simeq f$ rel $\dot{I^n}$, then $[h] \in W \cap W'$, which is a contradiction.

Conversely, let $[f], [g] \in \overline{\Omega^n(X,x)}$ such that $[f] \neq [g]$, then $g^{-1} * f \not\simeq e_x$. Since X is n-homotopically Hausdorff, there is an open neighborhood U of x for which $g^{-1}f$ is not homotopic (rel $\dot{I^n}$) to any n-loop lying entirely in U. Put $W = O^n(f, U)$ and $W' = O^n(g, U)$. We show that $W \cap W' = \emptyset$. Let $[h] \in W \cap W'$, then $[g * \alpha'] = [h] = [f * \alpha]$, where α and α' are n-loops in U. Thus, $[g^{-1} * f] = [\alpha' * \alpha^{-1}]$, which is a contradiction. □

Lemma 3.4. *$\overline{\Omega^n(X,x)}$ with the above topology is homeomorphic to $\overline{\Omega(\Omega^{(n-1)}(X,x), e_x)}$, where $\Omega^{(n-1)}(X,x)$ is equipped with the compact-open topology for all $n \geq 2$.*

Proof. Consider the homeomorphism $\phi : \overline{\Omega^n(X,x)} \to \overline{\Omega(\Omega^{(n-1)}(X,x), e_x)}$ defined by $\phi([f]) = [\overline{f}]$, where $\overline{f}(t)(s_1, s_2, \cdots, s_{n-1}) = f(t, s_1, s_2, \cdots, s_{n-1})$. □

Lemma 3.5. *Let $n \geq 2$. Then a space X is n-homotopically Hausdorff at x if and only if $\Omega^{(n-1)}(X,x)$ is homotopically Hausdorff at e_x for any $x \in X$.*

Proof. By Lemma 3.3, X is n-homotopically Hausdorff at x if and only if $\overline{\Omega^n(X,x)}$ is Hausdorff. By Lemma 3.4, $\overline{\Omega^n(X,x)}$ is homeomorphic to $\overline{\Omega(\Omega^{(n-1)}(X,x), e_x)}$. Also, $\overline{\Omega(\Omega^{(n-1)}(X,x), e_x)}$ is Hausdorff if and only if $\Omega^{(n-1)}(X,x)$ is homotopically Hausdorff at e_x for any $x \in X$. □

Now, the second result on countability of homotopy groups is as follows.

Proposition 3.6. *Suppose that X is a second countable, locally $(n-1)$-connected, and n-homotopically Hausdorff space at x which is not semilocally n-connected at this point. Then $\pi_n(X,x)$ is uncountable.*

Proof. This follows from [8, Lemma 2.3], Theorem 2.3, Lemma 3.5, and Lemma 3.1. □

The following corollary is a consequence of Theorem 3.2 and Proposition 3.6.

Corollary 3.7. *If X is an $(n-1)$-connected, locally $(n-1)$-connected, separable metric space, then the following statements are equivalent.*

(i) *X is semilocally n-connected.*
(ii) *X is n-homotopically Hausdorff and $\pi_n(X)$ is countable.*

Definition 3.8. Let $i : X \to Y$ be an embedding of one path-connected space into another. Then we say that X is a *π_n-retract of Y* if there exists a homomorphism $r : \pi_n(Y) \to \pi_n(X)$ such that the composition $ri_* : \pi_n(X) \to \pi_n(X)$ is an isomorphism. In this case, the homomorphism r is called a *π_n-retraction for X in Y*. Also, X is called a *π_n-neighborhood retract in Y* if X is a π_n-retract of one of its open neighborhoods in Y.

Definition 3.9. A separable metric space X is called a *π_n-absolute neighborhood retract (π_n-ANR)* if whenever X is a subspace of a separable metric space Y, then X is a π_n-neighborhood retract in Y.

Lemma 3.10. *Let Y be locally $(n-1)$-connected and semilocally n-connected and let X be a π_n-retract of Y. Then X is semilocally n-connected.*

Proof. Let $x \in X$. Since Y is semilocally n-connected, there exists a neighborhood V of x in Y such that $j_* : \pi_n(V, x) \to \pi_n(Y, x)$ is trivial. Since X is a π_n-retract of Y, there exists a homomorphism $r : \pi_n(Y) \to \pi_n(X)$ such that the composition $ri_* : \pi_n(X) \to \pi_n(X)$ is an isomorphism. We show that the induced mapping of inclusion $j'_* : \pi_n(V \cap X, x) \to \pi_n(X, x)$ is trivial. For this, consider the following commutative diagram:

$$\begin{array}{ccc} \pi_n(V \cap X, x) & \xrightarrow{j'_*} & \pi_n(X, x) \\ \Big\downarrow{\scriptstyle i'_*} & & {\scriptstyle i_*}\Big\downarrow \\ \pi_n(V, x) & \xrightarrow{j_*} & \pi_n(Y, x), \end{array}$$

where the map i'_* is induced by the inclusion. Now, the homomorphism j_* is trivial and i_* is injective, so j'_* is trivial. Therefore, X is semilocally n-connected. □

Corollary 3.11. *Let X be a separable metric space. If X is π_n-ANR, then it is semilocally n-connected.*

Proof. Since X is a separable metric space, it follows from the proof of the Urysohn metrization theorem [23, Theorem 4.1] that X can be embedded as a subspace of the Hilbert cube $Q = \prod_{i=1}^{\infty}[0, 1]$. Now, since X is a π_n-ANR, we can choose an open set U in the Hilbert cube such that X is a π_n-retract in U. Since U is semilocally n-connected, by Lemma 3.10, X is also semilocally n-connected. □

Proposition 3.12. *Let X be an $(n-1)$-connected, locally $(n-1)$-connected, separable metric space in which $\pi_n(X, x)$ is free. Then X is semilocally n-connected at x.*

Proof. This follows from [8, Theorem 2.6] and Lemma 3.1. □

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(Nasri, Mashayekhy, Mirebrahimi) Department of Pure Mathematics; Center of Excellence in Analysis on Algebraic Structures; Ferdowsi University of Mashhad; P.O.Box 1159-91775; Mashhad, Iran

E-mail address, Nasri: ta_na8@stu-mail.um.ac.ir

E-mail address, Mashayekhy: bmashf@um.ac.ir

E-mail address, Mirebrahimi: h_mirebrahimi@um.ac.ir