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SHORE AND CENTER POINTS OF A CONTINUUM

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ABSTRACT. We consider shore points, strong center points, and center points not only in λ-dendroids but in any non-degenerate continuum, and we present some relations between them. We give some conditions to show when a continuum is an arvum continuum, and also we give some conditions to show when, for a closed subset A of a continuum X, $T(A)$ is a shore set of X.

1. PRELIMINARIES

A *continuum* is a nonempty, compact, connected metric space. A *subcontinuum* of a continuum X is a continuum contained in X.

For a continuum X, $C(X) = \{A \subset X : A$ is a subcontinuum of $X\}$. It is known that $C(X)$ with the topology induced by the Hausdorff metric is a continuum [6, p. 52, Theorem 4.2].

A nonempty subset A of a continuum X is a *shore set* if, for every $\varepsilon > 0$, there is a subcontinuum C of X such that $A \cap C = \emptyset$ and $H(X, C) < \varepsilon$, with H the Hausdorff metric.

In particular, when $A = \{p\} \subset X$, and A is a shore set, we say that the point p is a *shore point* of X.

It is easy to see that in an interval, end points are shore points, every point of a Knaster continuum is a shore point, and every subcontinua of a Knaster continuum is a shore set.

A point p of a continuum X is a *cut point* if $X \setminus \{p\}$ is not connected.

It is clear that if p is a shore point of a continuum X, then p is not a cut point of X. The converse is not always true. Take two harmonic fans A_1 and A_2 such that $A_1 \cap A_2 = \{v_2\}$ and v_2 is the vertex in A_2 but in

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A_1, the point v_2 is the end point of the limit arc and v_2v is the limit arc in A_2. If we take a point $p \in v_2v$, then p is a shore point, but p is not a cut point.

A point p of a continuum X is a *center point* if there are two points a and b in X, such that for every $\varepsilon > 0$ there is a subcontinuum C of X containing p and $diam\,(C) < \varepsilon$ and there are open sets U and V containing a and b, respectively, such that every arc in X from U to V intersects C.

If p is a center point with respect to the points a and b, we say that p is an *ab-center*.

A point p of a continuum X is a *strong center point* if there are open sets U and V such that every arc in X from U to V contains p.

The definition of center is due to Piotr Minc who proved that every dendroid has at least one center [4, Theorem 3.6]. It is clear from the definition that every point that is a strong center point is a center point. The converse is not always true [7, Example 1].

In some cases we only use shore (center, strong center, cut) instead of shore point (center, strong center, cut) of X.

We define an *arvum continuum* as a continuum for which every point is a shore point.

The Knaster continuum is an arvum continuum.

2. Shore, Cut, Center, and Strong Center Points

In this section, we present some relationships between center, strong center, cut, and shore points. We prove that in every continuum a cut point is always a center point and strong center point, and it is not a shore point.

A center point is not necessarily a cut point. So we show that if a continuum X is locally connected, then every center point is a cut point and strong center point, and, in this case, the set of the center points and the set of the shore points are disjoint.

Proposition 2.1. *Let X be a continuum and let p be a cut point of X. Then*

(1) *p is a strong center and, by a consequence, p is a center;*
(2) *p is not a shore point.*

Proof. If p is a cut point, then $X \setminus \{p\} = M \cup K$, where M and K are open, not empty, and disjoint subsets of X.

(1) Let $a \in M$ and let $b \in K$. If there is an arc α from M to K, then $p \in \alpha$; in the other way, $\alpha \subset X \setminus \{p\}$ and $M \cap K \neq \emptyset$, a contradiction. Therefore, p is a strong center and, as a consequence, p is a center.

(2) It is clear that $M \cup \{p\}$ and $K \cup \{p\}$ are subcontinua of X. Let $\varepsilon_1 = H(M \cup \{p\}, K \cup \{p\})$ and $\varepsilon = \frac{\varepsilon_1}{4}$. If there is a subcontinuum B of X such that $H(X, B) < \varepsilon$, then $p \in B$; otherwise, by using that B is connected, we have that either $B \subset M$ or $B \subset K$. So, $H(X, B) > \varepsilon$, a contradiction. Therefore, p is not a shore point. □

From the previous proposition, we obtain the following corollary.

Corollary 2.2. *Let X be a continuum. If X has a cut point, then X has at least one strong center point.*

It is not always true that every strong center point is a cut point; the Warsaw circle has points that are strong center points but no cut points, but if we add the hypothesis of local connectedness, then it is true.

Theorem 2.3. *Let X be a locally connected continuum. Then the following conditions are equivalent:*

(1) *p is not a shore point*;
(2) *p is a cut point*;
(3) *p is a strong center point*;
(4) *p is a center point.*

Proof. (1) implies (2). We suppose that p is not a cut point and we show that p is a shore point.

Let $\varepsilon > 0$ and $p \in X$. Since p is not a cut point of X, $X \setminus \{p\}$ is connected. Then there is a subcontinuum C of X such that $p \notin C$ and $H(C, X) < \varepsilon$ [6, p. 137, Exercise 8.45]. Therefore, p is a shore point.

By using Proposition 2.1, (2) implies (1), and (2) implies (3), and (3) implies (4) are clear.

(4) implies (3). Let p be an ab-center and $\varepsilon > 0$. Then there is C in $C(X)$ and open subsets U and V of X such that $p \in C$, $diam(C) < \varepsilon$, $a \in U$, $b \in V, C \cap U = \emptyset = C \cap V$, and every arc from U to V intersects C.

Let U' and V' be open connected subsets of X such that $a \in U' \subset U$ and $b \in V' \subset V$.

Every arc from a to b contains p. To see this, we suppose that there is an arc L from a to b such that $p \notin L$.

Let $r = d(L, \{p\})$ and $\varepsilon_1 = \frac{r}{4}$. If there is $B \in C(X)$ such that $p \in B$ and $diam(B) < \varepsilon_1$, since $C \cap U = \emptyset = C \cap V$, then $B \cap L = \emptyset$, which is a contradiction.

Now, we see that every arc from U' to V' contains p. Let uv be an arc from U' to V'. By using [6, p. 132, Theorem 8.26], U' is arcwise connected, then there is an arc from a to u contained in U'; we denote this arc by au. In the same way, there is an arc vb in V' from v to b.

We consider L' as the union of the arcs au, uv, and vb; then L' is an arc from a to b. This implies that $p \in L'$. Then $p \in C \cap U$ or $p \in C \cap V$, a contradiction. Therefore, $p \in uv$.

(3) implies (2). We suppose that p is not a cut point and we show that p is not a strong center point.

Since p is not a cut point, by [6, p. 132, Theorem 8.26], $X \setminus \{p\}$ is arcwise connected, so p is not a strong center point. □

2.1. Shore and center points in an arcwise connected continuum.

Jo Heath and Van C. Nall in [1, §4] define for a dendroid D and $x, y \in D$ the set $Q_x(y) = \{z \in D : y \in zx\}$.

Let X be an arcwise connected continuum; we define for every two points $z, y \in X$, $Q_z(y) = \{x \in X : y \in zx \text{ for some arc } zx\}$.

It is not difficult to see that the set $Q_z(y)$ is not empty, $y \in Q_z(y)$, and $Q_z(y)$ is arcwise connected.

The following lemma was proved by Nall for dendroids [7, Lemma 4]. Our proof is similar to Nall's proof.

Lemma 2.4. *Let X be a continuum. If A is a proper subset of X such that $X \setminus A$ is arcwise connected, then A is a shore set of X if and only if $Int(A) = \emptyset$.*

Proof. If A is a shore set, it follows by the definition that $Int(A) = \emptyset$. Now we assume that $X \setminus A$ is arcwise connected and $Int(A) = \emptyset$. Let $\varepsilon > 0$ and let $D_1, D_2, D_3, ..., D_n$ be open and nonempty subsets of X such that $X \subset \bigcup_{i=1}^{n} D_i$ and $diam(D_i) < \frac{\varepsilon}{4}$.

Since $Int(A) = \emptyset$, let $y \in X \setminus A$ and $d_i \in D_i \setminus A$ such that $yd_i = \alpha_i$ is an arc from y to d_i for each $i \in \{1, 2, 3, ..., n\}$. We consider $D = \bigcup_{i=1}^{n} \alpha_i$. D is a subcontinuum of X such that $D \cap A = \emptyset$ and $H(D, X) < \varepsilon$. Therefore, A is a shore set. □

Nall [7, Theorem 1] showed that, for every dendroid D, if p is a strong center point of X, then p is not a shore point; we see that it is always true for every arcwise connected continuum.

Theorem 2.5. *Let X be an arcwise connected continuum. Then every point which is not a center point is a shore point.*

Proof. Let $p \in X$ such that p is not a center point, then p is not a strong center point. We consider two cases: first, when $X \setminus \{p\}$ is arcwise connected and second, when $X \setminus \{p\}$ is not arcwise connected.

Case 1: $X \setminus \{p\}$ is an arcwise connected set.

Since $Int(\{p\}) = \emptyset$ and $X \setminus \{p\}$ is arcwise connected, then by Lemma 2.4, p is a shore point.

Case 2: $X \setminus \{p\}$ is not an arcwise connected set.

For every $y \in X \setminus \{p\}$, we denote by A_y the arcwise component in $X \setminus \{p\}$ that contains y.

Now, if there is $y \in X \setminus \{p\}$ such that A_y is a dense subset of X, then p is a shore point.

We suppose that $X \setminus \overline{A_y} \neq \emptyset$ for each $y \in X \setminus \{p\}$ and there is $y \in X \setminus \{p\}$ such that $Int(A_y) \neq \emptyset$. Let U be an open subset in X such that $U \subset A_y$ and let V be an open subset of X such that $V \subset X \setminus \overline{A_y}$, then $V \cap A_y = \emptyset$.

We affirm that every arc from V to U contains p.

Let $\alpha = vu$ be an arc from V to U and $p \in \alpha$. Since $u \in A_y$, let β be an arc from u to y contained in A_y. Then $\alpha \cup \beta$ is an arc from v to y such that $p \notin \alpha \cup \beta$; this implies that $v \in A_y$. So, p is a strong center, a contradiction. Then for every $y \in X \setminus \{p\}$, we have that $Int(A_y) = \emptyset$. Then p is a shore point. □

We observe that in the previous theorem it is necessary that X be an arcwise connected continuum. As an example, we take a simple closed curve and two rays that converge to the simple closed curve, one in the interior and the other outside.

2.2. Shore sets.

It is always true that if $T(A)$ is a shore set of a continuum X, then A is a shore set of X. The converse is not always true; for example, in the Knaster continuum for every point p, $\{p\}$ is a shore set, but $T(\{p\}) = Knaster$.

We want to know when, for a given closed subset A of a dendroid D, $T(A)$ is a shore set of D. First, we give some definitions about the function T.

Given a compactum X, the *power set of* X, denoted by $\mathcal{P}(X)$, is $\mathcal{P}(X) = \{A : A \subset X\}$.

Let X be a compactum. Defined $T : \mathcal{P}(X) \to \mathcal{P}(X)$ by $T(A) = \{x \in X :$ for each subcontinuum W of X such that $x \in Int(W)$, we have that $W \cap A \neq \emptyset\}$, for each $A \in \mathcal{P}(X)$. The function T is called Jones's set function T.

It is known that for each compactum X and $A \in \mathcal{P}(X)$, $T(A)$ is a closed subset of X, and if X is a continuum and A is a subcontinuum of X, then $T(A)$ is a subcontinuum of X [3, p. 149, Theorem 3.1.21].

A continuum X is *T-additive* if, for each pair A and B of closed subsets of X, $T(A \cup B) = T(A) \cup T(B)$.

If D is a dendroid, we denote by $C_e(D) = \{z \in D : z \text{ is a center of } D\}$. Minc proved that for every dendroid D, $C_e(D) \neq \emptyset$ [4, Theorem 3.6].

For every two points $x, y \in D$, we denote by xy the arc from x to y in D.

Proposition 2.6. *Let X be an aposyndetic continuum. If X is T-additive and A is a closed shore set of X, then $T(A)$ is a shore set of X.*

Proof. Since X is aposyndetic and T-additive, X is locally connected [3, p. 161, Theorem 3.1.47]. This implies that $T(A) = A$ [3, p. 153, Theorem 3.1.31]. Therefore, $T(A)$ is a shore set of X. □

Corollary 2.7. *Let D be a dendroid and let A be a closed shore set of D. If D is aposyndetic, then $T(A)$ is a shore set of D.*

Proof. Every dendroid D is T-additive [3, p. 160, Theorem 3.1.45]. □

Proposition 2.8. *Let D be a dendroid, let p be a center of D, and let A be a closed subset of D. If $T(A) \subset Q_p(y)$ for some $y \in T(A)$ and $T(A) \cap C_e(D) = \emptyset$, then $T(A)$ is a shore set of D.*

Proof. Let p be a center of D, let A be a closed subset of D, and let $y \in T(A)$ such that $T(A) \subset Q_p(y)$. Since $y \in T(A)$, $T(A) \cap C_e(D) = \emptyset$ and, by using [7, Lemma 1], we have that $Int(Q_p(y)) = \emptyset$. It is easy to see that $D \backslash Q_p(y)$ is arcwise connected. Then, by Lemma 2.4, $Q_p(y)$ is a shore set of D. Hence, $T(A) \subset Q_p(y)$ and $T(A)$ is a shore set of D. □

Theorem 2.9. *Let D be a dendroid and let A be a closed subset of D such that $A \cap C_e(D) = \emptyset$ and A has at most a finite number of components. Then A is a shore set of D.*

Proof. Let $p \in D$ be a center point and $K_1, K_2, \ldots, K_n$ be the components of A.

Since D is hereditarily arcwise connected, then, for each $i \in \{1, 2, ..., n\}$, there is a unique $y_i \in K_i$ such that $K_i \subset Q_p(y_i)$.

Since $A \cap C_e(D) = \emptyset$ and, by [7, Theorem 1], for every $a \in A$, a is not a strong center point, then, by [7, Lemma 1], we have that $Int(Q_p(y_i)) = \emptyset$, so $Int(K_i) = \emptyset$. Then, by Lemma 2.4, $Q_p(y)$ is a shore set of D. Then K_i is a shore set of D. By using [2, Theorem 3], A is a shore set of D. □

In the previous theorem, it is necessary that $A \cap C_e(D) = \emptyset$ and A has at most a finite number of components. We present two examples to see it.

Example 2.10. Let F be the harmonic fan with vertex in $u = (0, 0)$ and the end points $\left(1, \frac{1}{n}\right)$ $n \in \mathbb{N}$ with limit arc $uv, v = (1, 0)$. We consider $A = uv$; then $A \cap C_e(D) \neq \emptyset$ and A is not a shore set of D.

Example 2.11. We take D to be the Cantor fan and let $A = Cantor\ set \times \left\{\frac{1}{2}\right\}$, so $A \cap C_e(D) = \emptyset$, A has uncountable many components, and A is not a shore set of D.

By Theorem 2.9, it is easy to prove the following propositions.

Proposition 2.12. *Let D be a dendroid and let A be a closed subset of D such that $T(A) \cap C_e(D) = \emptyset$ and $T(A)$ has at most a finite number of components. Then $T(A)$ is a shore set of D.*

Proposition 2.13. *Let D be a dendroid and let A be a closed subset of D. If $T(A)$ is connected and $T(A) \cap C_e(D) = \emptyset$, then $T(A)$ is a shore set of D.*

Corollary 2.14. *Let D be a dendroid and let A be a shore continua of D. If $T(A) \cap C_e(D) = \emptyset$, then $T(A)$ is a shore set of D.*

3. Arvum Continua

In this section, we show that every indecomposable continuum is an arvum continuum and give some conditions for a continuum to be an arvum continuum.

We remember that an arvum continuum is as a continuum for which every point is a shore point. The Knaster continuum and the Warsaw circle are examples of arvum continua.

Proposition 3.1. *If X is an indecomposable continuum, then X is an arvum continuum.*

Proof. Let $x \in X$. We denote by $k(x)$ the composant from x in X and let $y \in X \backslash k(x)$. Then X is irreducible between x and y. Since $k(y)$ is dense in X, we have that x is a shore point. □

It is not difficult to see that the following proposition gives a sufficient condition for a continuum to be an arvum continuum.

Proposition 3.2. *Let X be a continuum. If, for every $\varepsilon > 0$, there are two disjoint subsets A and B of X such that $H(A, X) < \varepsilon$ and $H(B, X) < \varepsilon$, then X is an arvum continuum.*

With the same argument that we use in Lemma 2.4, we could obtain the following lemma.

Lemma 3.3. *Let X be an arcwise connected continuum and $z \in X$. If $X \backslash \overline{Q_z(y)}$ is empty for some $y \in X \backslash \{z\}$, then z is a shore point.*

Proposition 3.4. *Let X be an arcwise connected continuum. If $X \backslash \overline{Q_z(y)}$ is empty for every pair z and y in X, then X is an arvum continuum.*

Proof. The proposition follows immediately from Lemma 3.3. □

Corollary 3.5. *If X is a dendroid and $X \setminus \overline{Q_z(y)}$ is empty for every pair z and y in X, then X is an arvum dendroid.*

Proposition 3.6. *If X is an arcwise connected continuum without center points, then X is an arvum continuum.*

Proof. The proposition follows immediately from Theorem 2.5. □

4. Questions

In this small section, we present some questions related to this paper.

Question 4.1. Is it true that if A is a shore set of a dendroid D, then $Int\,(T(A)) = \emptyset$.

Question 4.2. For a closed subset A of a continuum X, when is $T(A)$ a shore set of X?

Question 4.3. For a shore set A of a continuum X, when is $T(A)$ a shore set of X?

Question 4.4. What continua without cut points have center points?

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