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STRONGLY PSEUDORADIAL SPACES

JEREMY BRAZAS AND PAUL FABEL

Abstract. The "weakly Hausdorff" property for pseudoradial spaces fails to be naturally characterized by unique convergence of transfinite sequences. In response, we develop the category **SPsRad** of strongly pseudoradial spaces, compactly generated spaces whose closed sets are determined by globally continuous maps from well-ordered spaces. Categorically **SPsRad** is the coreflective hull of the class of well-ordered spaces, and **SPsRad** is Cartesian closed. The strongly pseudoradial weakly Hausdorff spaces admit a natural characterization involving unique extensions of injective maps of well-ordered spaces. We also obtain analogs in **SPsRad** of the fact that for sequential spaces, sequential compactness is equivalent to countable compactness.

1. Introduction

This paper introduces a category **SPsRad** of strongly pseudoradial spaces, a natural generalization of sequential spaces. The space X is *strongly pseudoradial* if, for each non-closed set $A \subset X$, there exists (with the order topology), a noncompact well-ordered space α and a map $f : \alpha \cup \{\infty_\alpha\} \to X$ so that $f(\alpha) \subset A$ and $f(\infty_\alpha) \notin A$. Here, $\alpha \cup \{\infty_\alpha\}$ denotes the familiar one point compactification of α.

Our motivation is the observation that the following fact about sequential spaces does not (as shown in Example 5.3) generalize naturally to pseudoradial spaces. If X is a sequential space, then each convergent sequence in X has a unique limit if and only if each compact subspace of

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X is closed. On the other hand, this fact generalizes naturally to strongly pseudoradial spaces (Corollary 4.2).

Recall that the space X is *sequential* ([12], [13]) if, for each non-closed $A \subset X$, there exists a convergent sequence $a_n \to x$ so that $\{a_n\} \subset A$ and $x \notin A$; that X is *compactly generated* (*CG*) ([21], [22]) if, for each non-closed $A \subset X$, there exists a compact Hausdorff space K and a map $f : K \to X$ such that $f(K) \cap A$ is not closed in X; and that X is *pseudoradial* (*PsRad*) ([1], [14], [24]) if, for each non-closed $A \subset X$, there exists an unbounded well-ordered set α, a function $f : \alpha \cup \{\infty_\alpha\} \to A$ so that f is continuous at ∞_α, $f(A) \subset X$, and $f(\infty_\alpha) \notin A$. Thus, each sequential space is strongly pseudoradial, and every strongly pseudoradial space is both pseudoradial and compactly generated. Example 5.4 shows that a compactly generated pseudoradial space need not be strongly pseudoradial.

If the sequential space X is T_1, then non-closed sets are detected by convergent sequences of distinct points. The natural generalization (Theorem 3.5) in **SPsRad** employs injective maps of regular cardinals with the order topology.

Recall the space X is *US* if convergent sequences in X have unique limits; X is *weakly Hausdorff* (*WH*) if maps of compact T_2 spaces into X have closed image ([21], [22]); and X is a *KC space* if compact subsets of X are closed. For general spaces $T_2 \Rightarrow KC \Rightarrow WH \Rightarrow US \Rightarrow T_1$, and the implications are strict [25]. If X is CG, then $WH \iff KC$, and if X is sequential, then $US \iff WH \iff KC$. For strongly pseudoradial spaces, the natural generalization of the US property is *unique strong pseudoradial convergence*: For each map $f : \alpha \to X$ of a non-compact well-ordered space α, there is at most one continuous extension to $\alpha \cup \{\infty_\alpha\}$. As noted, we show that a strongly pseudoradial space X is KC if and only if it has unique strong pseudoradial convergence. This follows from Theorem 4.1 which asserts that maps of compact well-ordered spaces into such spaces X have closed image. Example 5.3 shows this result does not translate naturally to the traditional pseudoradial category.

In contrast with the fact that the category **PsRad** of pseudoradial spaces fails to be Cartesian closed [4], the convenient categorical properties (in the sense of [21]) of **SPsRad** are established in section 6. The examples in section 5 illustrate various ways in which the main results are best possible and also establish strict relationships among various categories under consideration.

Theorem 7.10 is an analog (or generalization modulo Lemma 7.16) of the fact that for sequential spaces, countable compactness is equivalent to sequential compactness. However, in **SPsRad**, the relationship between compactness and related notions quickly leads to deep waters in axiomatic set theory, even for first countable countably compact spaces ([10], [8], [9], [17]), (and much more generally for pseudoradial spaces ([7], [5], [6], [17], [18], [19], [24])). For example, in **SPsRad**, a natural analog of "sequentially compact" is *strongly pseudoradially compact* (*SPC*). The space X has the latter property if, for each noncompact well-ordered space α and each map $f : \alpha \to X$, there exists $\beta \subset \alpha$, closed and cofinal, so that $f|\beta$ is continuously extendable to the one point compactification $\beta \cup \{\infty_\beta\}$. By inspection, this notion is equivalent to compactness for well-ordered spaces, but in general, the notions are inequivalent and plausibly unrelated. Assuming Jensen's combinatorial diamond principle $\diamondsuit$, Ostaszewski spaces ([17], [16]) exist and are noncompact strongly pseudoradial spaces satisfying SPC. On the other hand, in **SPsRad**, every compact weakly Hausdorff space is SPC (Corollary 7.11). The authors were unable to settle whether all compact, T_1 **SPsRad** spaces are SPC (Problem 7.17).

2. Definitions and Preliminaries

By a *well-ordered space* X, we mean a well-ordered set with the order topology generated by half open intervals $(a, b] = \{x \in X | a < x \leq b\}$ (and by closed intervals $[m, b]$ if m is minimal in X).

If α is a noncompact well-ordered space, $\alpha \cup \{\infty_\alpha\}$ denotes the (unique) well-ordered one point compactification of α, (i.e., the compact space obtained by attaching a maximal point ∞_α to α whose basic neighborhoods are of the form $(a, \infty_\alpha])$.

Remark 2.1. If α is a noncompact well-ordered space and $\beta \subset \alpha$, then the subspace topology of β coincides with the order topology of β if and only if β is a compact subspace of $\alpha \cup \{\infty_\alpha\}$ or (if β is not compact and taking closure in $\alpha \cup \{\infty_\alpha\}$), $\sup\beta = \overline{\beta} \backslash \beta$.

If α is a well-ordered set, a subset $K \subset \alpha$ is *cofinal* if, for each $x \in \alpha$, there exists $k \in K$ so that $x \leq k$. K is an *initial segment* if $K = \alpha$ or $K = [0, x) = \{y \in \alpha | 0 \leq y < x\}$ for some $x \in \alpha$.

If A and B are sets, then $|A| < |B|$ means there exists an injection from A into B and no injection is surjective, and $|A| = |B|$ means there exists a bijection from A onto B. An *ordinal* is a well-ordered set. A *cardinal* α is a well-ordered set so that $|\beta| < |\alpha|$ for every proper initial segment β. A *regular cardinal* α is a cardinal such that if $\beta \subset \alpha$ and β is cofinal, then there exists an order preserving bijection from β onto α. (We are formally ignoring the standard notion "cofinality.")

If A and B are subsets of the linearly ordered set $(S, <)$, the notation $A < B$ means $a < b$ for all $a \in A$ and all $b \in B$.

By the *directed topology* on the well-ordered set $\alpha\cup\{\infty_\alpha\}$, we mean the (generally finer) space with topology generated by sets $\{a\}$ and $(a,\infty_\alpha]$, with $a\in\alpha$. The space X is *strongly pseudoradial* if, for each non-closed $A\subset X$, there exists a noncompact well-ordered space α and a map $f:\alpha\cup\{\infty_\alpha\}\to X$ so that $f(\alpha)\subset A$ and $f(\infty_\alpha)\notin A$. If the same conclusion holds with respect to the directed topology on $\alpha\cup\{\infty_\alpha\}$, we obtain the generally weaker property that X is *pseudoradial.* The space X has *unique strong pseudoradial convergence* if $f(\infty_\alpha)=g(\infty_\alpha)$ for all noncompact well-ordered spaces α and all pairs of maps $f,g:\alpha\cup\{\infty_\alpha\}\to X$ such that $f|\alpha=g|\alpha$. If the same conclusion holds with respect to the directed topology on $\alpha\cup\{\infty_\alpha\}$, we obtain the generally stronger property that X has *unique pseudoradial convergence.*

A space X is *sequentially compact* if each sequence in X has a convergent subsequence and X is *countably compact* if each countable open cover of X has a finite subcover.

3. Characterizing Strongly Pseudoradial Spaces

The main result of this section (Corollary 3.6) generalizes to strongly pseudoradial spaces the following fact about sequential spaces: The space X is sequential if and only if, for each non-closed $A\subset X$, there exists $a\in A$ such that $\overline{\{a\}}\backslash A\neq\emptyset$ or there exists a convergent sequence of *distinct* points $a_n\to b$ so that $\{a_1,a_2,\ldots\}\subset A$ and $b\notin A$ (i.e., there exists a continuously extendable injective map $f:\alpha\to A$ of the regular cardinal $\omega=\{1,2,3\ldots\}$, so that $f(\infty_\omega)\notin A$).

The following lemmas are well known or straightforward. While likely apparent to the reader proficient in elementary set theory [15], care is needed to ensure that extra topological or structural conditions are met. For example, in Lemma 3.1, merely invoking the axiom of choice does not guarantee the useful extra properties that β is closed or cofinal.

Lemma 3.1. *Suppose α is a noncompact well-ordered space and $\{A_i\}$ is a partition of α into bounded sets (indexed by a set I). Then there exists a closed cofinal subspace $\beta\subset\alpha$ so that $|A_i\cap\beta|\leq 1$ for all $i\in I$.*

Proof. If $x\in\alpha$, let A_{i_x} denote the (unique) element of $\{A_i\}$ so that $x\in A_{i_x}$. Let 0 denote the minimal element of α. Let $\beta_0=\{0\}$.

Suppose $x\in\alpha\cup\{\infty_\alpha\}$ and $\beta_a\subset\alpha$ has been defined for all $a<x$. Let $\gamma_x=\cup_{a<x}\beta_a$. Let $m_x=\sup\gamma_x$ in the space $\alpha\cup\{\infty_\alpha\}$. Suppose the following three conditions hold: (i) $x\leq m_x$ (ii) if $\{a,b\}\subset\gamma_x$ and $a<b$, then $A_{i_a}<\{b\}$, (iii) $\gamma_x\cup\{m_x\}$ is compact.

Observe that conditions (i), (ii), and (iii) are true in case $x=0$. If $x>0$, proceed as follows.

Case 1: If $m_x = \infty_\alpha$, let $\beta = \gamma_x$ and the theorem is proved by conditions (ii) and (iii) and the fact that γ_x is unbounded in α.

Case 2: Suppose that $m_x < \infty_\alpha$. Obtain $r_x \in \alpha$ minimal so that $A_{i_{m_x}} < r_x$. Define $\beta_x = \gamma_x \cup \{m_x\} \cup \{r_x\}$. Observe that β_x is compact since $\gamma_x \cup \{m_x\}$ is compact (by the induction hypothesis). Note $m_x \in A_{i_{m_x}}$, and hence $m_x < r_x$. Note that $\gamma_{x+1} = \beta_x$ and $m_{x+1} = r_x$.

We now check that conditions (i) to (iii) are preserved at the index $x+1$. To check (iii) recall that $\gamma_{x+1} \cup \{m_{x+1}\} = \beta_x$, which is compact, as shown above. To check (i), recall that $m_{x+1} = r_x$ and $m_x < r_x$ as shown above. By hypothesis, $x \leq m_x$, and thus $x + 1 \leq m_x + 1 \leq r_x = m_{x+1}$. To check (ii), suppose that $\{a, b\} \subset \gamma_{x+1}$ and $a < b$. Recall that $\gamma_x \leq m_x < r_x$. If γ_x is compact, then $m_x \in \gamma_x$. If $b \in \gamma_x$, then $a \in \gamma_x$, and $A_{i_a} < b$ by the induction hypothesis. If $b = m_x$ and γ_x is not compact, obtain c so that $a < c < b$, and thus, by the induction hypothesis, $A_{i_a} < c < b$. Suppose that $b = r_x$. If $a < m_x$, then $A_{i_a} < m_x < b$, and if $a = m_x$, then $A_{i_{m_x}} < r_x$ by definition of r_x. In case $x = \infty_\alpha$, we achieve Case 1. □

Lemma 3.2. *Suppose that α and J are unbounded well-ordered sets and $h : \alpha \to J$ is a bijection. There exists a subset $K \subset \alpha$ so that $h|K$ is order preserving and $h(K)$ is cofinal in J.*

Proof. Manufacture K as follows. Let k_1 denote the minimal element of α. Let $K_1 = \{k_1\}$. Observe that $h|K_1$ is order preserving and $K_1 < h^{-1}(y)$ if $h(K_1) < \{y\}$. Observe that $[k_1, \infty) \cap K_1 \neq \emptyset$. Suppose that $i \in \alpha \cup \infty$ and K_j is defined for each $j < i$. Suppose that $h|(\cup_{j<i} K_j)$ is order preserving and $\cup_{j<i} K_j < h^{-1}(y)$ if $h(\cup_{j<i} K_j) < \{y\}$. Suppose that $[j, \infty) \cap K_j \neq \emptyset$ for each $j < i$.

Case 1: If $h(\cup_{j<i} K_j)$ is cofinal, let $K = \cup_{j<i} K_j$, and observe that the lemma is proved.

Case 2: If $h(\cup_{j<i} K_j)$ is not cofinal, by hypothesis there exists $k_i \in \alpha$ minimal so that $\cup_{j<i} K_j < \{k_i\}$ and $h(\cup_{j<i} K_j) < \{h(k_i)\}$. Let $K_i = \cup_{j<i} K_j \cup \{k_i\}$. Observe that $h|K_i$ is order preserving. Minimality of k_i ensures $K_i < h^{-1}(y)$ if $h(K_i) < \{y\}$. By the induction hypothesis, $j < k_i$ for each $j < i$. Thus, $i \leq k_i$. Case 1 is reached by the time $\alpha = \infty$. □

Lemma 3.3. *If α is an unbounded well-ordered set and β has minimal order type among ordinals $\beta \subset \alpha$ such that β is cofinal in α, then β is a regular cardinal.*

Proof. Given well-ordered sets γ and β, we say $\gamma \prec \beta$ if γ can be embedded as a proper initial segment of β and $\gamma \preceq \beta$ if $\gamma \prec \beta$ or γ is isomorphic to β. To obtain a contradiction obtain α of minimal order type so that the result fails.

If $\beta \prec \alpha$, then obtain a regular cardinal γ cofinal in β. Observe that $\gamma \preceq \beta$ and γ is cofinal in α. By minimality of β, we conclude γ is isomorphic to β, and thus β is a regular cardinal, a contradiction.

By definition, if β is isomorphic to α and α is a cardinal, then α is a regular cardinal. If α is not a cardinal, obtain $x \in \alpha$ minimal so that $|[0,x)| = |\alpha|$. Obtain a bijection $h : [0,x) \to \alpha$ and apply Lemma 3.2 to obtain $K \subset [0,x)$ so that $h|K$ is order preserving and cofinal in α. Since $K \prec \alpha$ and α is a minimal counterexample, obtain a regular cardinal γ cofinal in K and note $h|\gamma$ is cofinal in α, and we have a contradiction since $\gamma \prec \alpha$. □

We can easily ensure that cofinal sets are closed in the order topology.

Lemma 3.4. *Suppose α is a well-ordered space. There exists a closed cofinal subspace $\beta \subset \alpha$ so that β is a regular cardinal.*

Proof. Obtain by Lemma 3.3 a (possibly non closed) cofinal $\gamma \subset \alpha$ so that γ is a regular cardinal. Let $\partial\gamma = \overline{\gamma}\backslash\gamma$. For each $x \in \partial\gamma$, obtain $k_x \in \gamma$ minimal so that $x < k_x$. Let K denote the union of such points k_x. Let $\beta = \overline{\gamma}\backslash K$. Note K is open in the space α, and hence β is closed. The order preserving bijection $\beta \to \gamma$ fixing $\gamma\backslash K$ pointwise and sending $x \in \partial\gamma$ to $k_x \in K$ shows β has the same order type as γ, and thus β is a regular cardinal. Since γ is cofinal in α, $\gamma\backslash K$ is cofinal in α (since for each $k_x \in K$ the next element of γ is not in K). Thus, β is cofinal in α. □

In a similar manner to T_1 sequential spaces, we can detect non-closed sets in a T_1 strongly pseudoradial space with extendable injective maps of regular cardinals.

Theorem 3.5. *A space X is strongly pseudoradial if and only if, for each non-closed $A \subset X$, there exists $a \in A$ such that $\overline{\{a\}}\backslash A \neq \emptyset$ or there exists (with the order topology) a regular cardinal γ and a continuous injection $\kappa : \gamma \cup \{\infty_\gamma\} \to X$ such that $\kappa(\gamma) \subset A$ and $\kappa(\infty_\gamma) \notin A$.*

Proof. Suppose X is strongly pseudoradial and $A \subset X$ is not closed. Suppose there does not exist $a \in A$ and $b \notin A$ so that $b \in \overline{\{a\}}$. Obtain a noncompact ordinal β and a map $f : \beta \cup \{\infty_\beta\} \to X$ such that $f(\beta) \subset A$ and $f(\infty_\beta) \notin A$. For $a \in A$, let $S_a = f^{-1}(a) \subset \beta$. Note S_a is bounded since, otherwise, continuity of f at ∞_β shows $f(\infty_\beta)$ is a limit point of the singleton $\{a\}$. Thus, the sets $\{S_a\}$ form a partition of α into bounded sets. By Lemma 3.1, there exists a closed cofinal set $\alpha \subset \beta$ such that $f|\alpha$ is one to one. By Lemma 3.4, there exists a closed and cofinal regular ordinal $\gamma \subset \alpha$. Let $\kappa = f|\gamma$.

For the converse, suppose $A \subset X$ is not closed. If there exists $a \in A$ and $b \notin A$ such that $b \in \overline{\{a\}}$, employ the map of $\omega \cup \{\infty_\omega\}$ sending

$n \to a$ and $\infty_\omega \to b$. If no such a exists, employ the advertised map κ and conclude that X is strongly pseudoradial. □

As one might expect, $|X|$ is the lower bound on the size of cardinals needed to detect non-closed sets in the T_1 strongly pseudoradial space X.

Corollary 3.6. *Suppose X is a space and α is the cardinal such that $|X| = |\alpha|$. Then X is strongly pseudoradial if and only if, for each non-closed $A \subset X$, there exists $a \in A$ such that $\overline{\{a\}}\backslash A \neq \emptyset$ or there exists (with the order topology) a regular cardinal β (with $|\beta| \leq |\alpha|$) and a continuous injection $h : \beta \cup \{\infty_\beta\} \to X$ such that $h(\beta) \subset A$ and $h(\infty_\beta) \notin A$.*

4. Unique Strong Pseudoradial Convergence

A sequential space X is a *US* space if and only if X is a *KC* space. Corollary 4.2 provides a strong analogue in the **SPsRad** category. Example 5.3 shows the natural analogue fails in the **PsRad** category.

Theorem 4.1. *Suppose X is strongly pseudoradial and has unique strong pseudoradial convergence. Suppose α is a noncompact well-ordered space and $h : \alpha \cup \{\infty_\alpha\} \to X$ is a continuous injection. Then $h(\alpha \cup \{\infty_\alpha\})$ is closed in X.*

Proof. To obtain a contradiction, suppose the claim is false. Obtain an ordinal of minimal order type of the form $\alpha \cup \{\infty_\alpha\}$ such that $Y = h(\alpha \cup \{\infty_\alpha\})$ is not closed in X, but so that $h([0, a])$ is closed in X for all $a \in \alpha$. Since X is T_1 and strongly pseudoradial and since Y is not closed in X, obtain a noncompact ordinal β and a continuous injection $g : \beta \cup \{\infty_\beta\} \to X$ so that $g(\beta) \subset Y$ and $g(\infty_\beta) \notin Y$. If $h(\infty_\alpha) \notin g(\beta)$, let $\gamma = \beta$. If $h(\infty_\alpha) \in g(\beta)$, let $\gamma = \beta\backslash[0_\beta, g^{-1}h(\infty_\alpha)]$. Define $\kappa : \gamma \cup \{\infty_\beta\} \to Y$ so that $\kappa|\gamma = g|\beta$ and define $\kappa(\infty_\beta) = h(\infty_\alpha)$. Note that the injection $g|(\gamma \cup \{\infty_\beta\})$ is continuous. Thus, since X has strong unique transfinite convergence, κ is not continuous. Since $\kappa|\gamma$ is continuous, κ is not continuous at the point ∞_β. Thus, there exists an open set $U \subset X$ so that $h(\infty_\alpha) \in U$ and $(b, \infty_\beta]\backslash\kappa^{-1}(U) \neq \emptyset$ for all $b \in \gamma$. Since $\kappa(\infty_\beta) \in U$, $(b, \infty_\beta]\backslash\kappa^{-1}(U) = (b, \infty_\beta)\backslash\kappa^{-1}(U)$. Hence, since $\kappa|\gamma$ is continuous, $(b, \infty_\beta)\backslash\kappa^{-1}(U)$ is a nonempty open subspace of γ for all $b \in \gamma$. Let $K = \gamma\backslash\kappa^{-1}(U)$. Observe that K is a noncompact closed subspace of $\gamma, \kappa(K) \cap U = \emptyset$, and $K \cup \{\infty_\beta\}$ is compact. Since h is continuous at ∞_α, there exists $a \in \alpha$ so that $h(i) \in U$ if $a < i$. Hence, $\kappa(K) \subset h([0_\alpha, a])$. By hypothesis, $h([0_\alpha, a])$ is closed in X. The injective map $g|(K \cup \{\infty_\beta\})$ shows $g(\infty_\beta)$ is a limit point of $g(K)$, and thus $g(\infty_\beta)$ is in the closed set $h([0, a])$. This contradicts the fact that $g(\infty_\beta) \notin Y = h(\alpha \cup \{\infty_\alpha\})$. □

Corollary 4.2. *Suppose X is strongly pseudoradial. The following are equivalent:*

(1) *X has unique strong pseudoradial convergence.*
(2) *If C is a compact well-ordered space and $h : C \to X$ is a continuous injection, then h is a closed embedding.*
(3) *X is weakly Hausdorff.*
(4) *X is a KC space.*

Proof. By definition, (4) $\Rightarrow$ (3) $\Rightarrow$ (2) for all spaces X. To show (2) $\Rightarrow$ (4), suppose $A \subset X$ is not closed. By Theorem 3.5 and Theorem 4.1, obtain a noncompact ordinal α and a closed embedding $\kappa : \alpha \cup \{\infty_\alpha\} \to X$ so that $\kappa(\alpha) \subset A$ and $\kappa(\infty_\alpha) \notin A$. Since κ is a closed map, $\kappa(\alpha) = im(\kappa) \cap A$ is a closed subspace of the space A. Since the closed subspace $\kappa(\alpha)$ of A is not compact, A is not a compact space.

(1) $\Rightarrow$ (2) follows directly from Theorem 4.1. If (1) is false, obtain a noncompact ordinal α and an extendable continuous injection $f : \alpha \to X$ with non-unique extensions $\infty_\alpha \to x$ and $\infty_\alpha \to y$ with $x \neq y$. Then y is a limit point of $f(\alpha) \cup \{x\}$, and hence (2) fails. Thus, (2) $\Rightarrow$ (1). □

5. Examples and Counterexamples

The examples in this section illustrate various ways in which the main results of this paper are best possible and also establish **SPsRad** is a proper subcategory of **PsRad** $\cap$ **CG**.

The (1) $\Rightarrow$ (2) implication of Corollary 4.2 yields closed embeddings for pseudoradial spaces. Given (1), if we drop the pseudoradial hypothesis, we could hope in principle to obtain embeddings that are not necessarily closed. The following example, obtained by attaching an extra point to the Arens–Fort space, shows this is generally hopeless.

Example 5.1. *There exists a US space Y so that with the order topology on $\omega \cup \{\infty_\omega\}$, there exists a continuous bijection $h : \omega \cup \{\infty_\omega\} \to Y$ so that h is **not** a homeomorphism.*

Proof. The well-known Arens-Fort space [20, Example 26] is a countable space with a single non-isolated point, such that no sequence of isolated points converges to the non-isolated point. In particular, we may take the Arens-Fort space to be the natural numbers $X = \{1, 2, 3 \ldots\}$ so that each single-point set $\{2\}, \{3\}, \ldots$ is open, $\{1\}$ is a limit point of $\{2, 3, 4 \ldots\}$, and such that no sequence in $\{2, 3, 4, \ldots\}$ converges to 1. Let $Y = X \cup \{\infty\}$ be the space obtained by attaching an extra point to X with basic open sets $\{\infty, n, n+1, \ldots\}$ at the added point. Note that Y is a non-Hausdorff, US space. The natural identity function $id : \omega \cup \{\infty_\omega\} \to Y$ is a continuous bijection but not a homeomorphism. □

Corollary 3.6 requires that the cardinal α satisfies $|\alpha| = |X|$, and this cannot be relaxed, even if α is a non-regular cardinal, as shown in the following example.

Example 5.2. *Let α be a non-regular cardinal with the order topology (for example, let $\alpha = \cup\alpha_n$, the union of nested regular cardinals $\alpha_1 < \alpha_2 < \ldots$). Let $X = \alpha \cup \{\infty_\alpha\}$. Observe that there is no limit to the length of proper initial segments of α needed to detect non-closed sets $A \subset X$. Thus, α is the minimal cardinal which makes Corollary 3.6 true, despite the fact that no non-closed set $A \subset X$ requires $\alpha \cup \{\infty_\alpha\}$ to detect the failure of A to be closed.*

The following example illustrates the failure of the pseudoradial analogue of Corollary 4.2 in the **PsRad** category.

Example 5.3. *Let α be an uncountable, unbounded well-ordered set with the discrete topology and attach two unrelated maximal points. In particular, let $X = \alpha \cup \{x, y\}$ with $x \neq y$, declare $\alpha < x$ and $\alpha < y$, and let $(a, x]$ and $(a, y]$ be basic open sets for $a \in \alpha$. Then X is a pseudoradial KC space, but X does not have unique pseudoradial convergence.*

The following example shows **SPsRad** is a proper subcategory of **PsRad** $\cap$ **CG**

Example 5.4. *Let X be an uncountable, well-ordered set with the discrete topology and let $Y = X \cup \{\infty\}$ denote the Alexandroff compactification of X. Then Y is a compact Hausdorff pseudoradial space, but Y is not strongly pseudoradial.*

Proof. Since Y is a compact Hausdorff space, A is not closed in Y if and only if A is not compact. Thus, $id : Y \to Y$ shows Y is compactly generated.

To see that Y is pseudoradial, suppose $A \subset Y$ and A is not closed. Then ∞ is a limit point of A. Notice $A \cup \infty$ is a well-ordered set. Let $A' \cup \infty' = A \cup \infty$ with the directed topology and consider the inclusion function $j : A' \cup \infty' \to Y \cup \infty$. If $U \subset Y$ is an open set such that $\infty \in U$, then $Y \backslash U$ is finite. Select $y \in Y$ such that $Y \backslash U < \{y\}$. Hence, if $y < a$ and $a \in A$, then $j(a) \in U$. Thus, Y is transfinite sequential.

To see that Y is not strongly pseudoradial, suppose α is a noncompact well-ordered space and suppose $f : \alpha \cup \{\infty_\alpha\} \to Y$ is a map such that $f(\alpha) \subset A$. Note if $i \in \alpha$, then $f([0, i])$ is a compact subset of the discrete space A, and hence $f([0, i])$ is finite.

To obtain a contradiction, suppose for each integer $N \geq 1$ there exists $i_N \in \alpha$ such that $|f([0, i_N])| = N$. Then $i_1 < i_2 \ldots$.

Since X is uncountable, there exists a limit $i \in \alpha$. Then $|f([0, i])| > N$ for each N. However, $|f([0, i])|$ is finite and we have a contradiction. Since

f is continuous at ∞_α, we conclude that $f(\infty_\alpha) \in f(\alpha)$, and hence Y is not strongly pseudoradial. □

Every strongly pseudoradial space is compactly generated; however, pseudoradial spaces need not be compactly generated, as illustrated in the following example.

Example 5.5. (**PsRad**$\backslash$**CG** $\neq \emptyset$) *Suppose α is a minimal, uncountable well-ordered set and $X = \alpha \cup \{\infty_\alpha\}$ with the directed topology generated by sets $\{\beta\}$ with $\beta \in \alpha$ and $(\beta, \infty_\alpha]$. Then X is pseudoradial but is not compactly generated.*

Proof. Suppose $A \subset X$ and A is not closed in X. Then $\overline{A}\backslash A \neq \emptyset$, and hence $\{\infty_\alpha\} = \overline{A}\backslash A$ since ∞_α is the only limit point of X.

To see that $X \in$ **PsRad**, let $Y = A \cup \{\infty_\alpha\}$ with the subspace topology. Note that Y is a well-ordered set, A is unbounded in Y, and Y enjoys the directed topology. Thus, inclusion $id : Y \to X$ is continuous, $id(A) \subset A$, and $id(\infty_\alpha) \notin A$.

To see that $X \notin$ **CG**, note that α is not closed in X. Suppose K is a compact T_2 space and $f : K \to X$ is a map. Let $B = f^{-1}(\alpha)$. It suffices to prove B is closed in K. Note $f(K)$ is compact. To obtain a contradiction, suppose that $f(K)$ is infinite. Obtain j_1 minimal in $f(K)$ and note that $j_1 \in \alpha$. Suppose that $n > 1$ and $j_1 < j_2 \ldots < j_{n-1}$ have been selected such that $j_i \in \alpha \cap f(K)$. Obtain $j_n \in f(K)$ minimal such that $j_{n-1} < j_n$. Note that $j_n \in \alpha$. Let $C = \{j_1, j_2, \ldots\}$. Since α is uncountable, there exists $l \in \alpha$ such that $C < \{l\}$. Note that $f(K)$ is Hausdorff since X is Hausdorff and note that C is a closed subspace of X. Hence, C is a closed subspace of $f(K)$, and thus C is compact. However, C is an infinite space with the discrete topology, and thus C is not compact, and we have a contradiction. Thus, $f(K)$ is a finite set. Let $f(K)\backslash\{\infty_\alpha\} = \{k_1, k_2, \ldots, k_n\}$. Since X is Hausdorff, $\{k_i\}$ is closed for each i. Thus, since f is continuous, B is the union of finitely many closed sets $f^{-1}\{k_i\}$. Hence, $f^{-1}(\alpha)$ is closed. Thus, $X \notin$ **CG**. □

Lemma 5.6. *Suppose X is a Hausdorff space and there exists a countably infinite set $A = \{a_1, a_2, \ldots\} \subset X$ such that A is not closed and such that each convergent sequence in A is eventually constant. Then X is not pseudoradial.*

Proof. Let α be an unbounded, well-ordered set and $\alpha \cup \{\infty_\alpha\}$ has the directed topology. To obtain a contradiction, suppose $f : \alpha \cup \{\infty_\alpha\} \to X$ is a map such that $f(\alpha) \subset A$ and $f(\infty_\alpha) \notin A$. Let $S_n = f^{-1}(a_n)$ and let $k_n = \sup S_n$. Note that the singleton $f(S_n) \subset A$. Hence, since X is a T_2 space, $f(\overline{S_n}) = f(S_n)$. Thus, $k_n < \infty_\alpha$ for all n (since otherwise $f(\infty_\alpha) \in A$). Let $s_n = \max\{k_1, \ldots, k_n\}$. Note $s_1 \leq s_2 \ldots$ and $s_n <$

∞_α. Let $s = \sup\{s_n\}$ and note $s = \infty_\alpha$ (since otherwise we obtain the contradiction $f((s, \infty_\alpha)) \cap A = \emptyset$). By construction, the sequence $s_n \to \infty_\alpha$. By continuity of f, $f(s_n) \to f(\infty_v)$. Thus, $\{f(s_n)\}$ is a convergent sequence in A, and hence $\{f(s_n)\}$ is eventually constant. Thus, $f(\infty_\alpha) \in A$, and we have a contradiction. □

The following example shows that **CG**$\setminus$**PsRad** $\neq \emptyset$

Example 5.7. Let ω denote the natural numbers with the discrete topology and let $X = \beta\omega$, the Stone-Čech compactification of ω [11]. Then X is a compactly generated space but is not pseudoradial.

Proof. Since $\beta\omega$ is compact Hausdorff, it is compactly generated. Let $A = \omega \subset X$. By construction, ω is not closed in $\beta\omega$. Each bounded real valued function $f : \omega \to \mathbb{R}$ is the restriction of a map $f : \beta\omega \to \mathbb{R}$. Thus, each convergent sequence in ω is eventually constant. Otherwise, there exists a convergent subsequence $n_1 < n_2 \ldots$ with $n_i \to x \in X$. Since X is Hausdorff, the bounded real valued map such that $f(n_{2i}) = 0$ and $f(m) = 1$ if $m \neq n_{2i}$ cannot be continuously extended to $\{x, \omega\}$, and we have a contradiction. Now apply Lemma 5.6 to see that $\beta\omega$ is not pseudoradial. □

6. On the Category of Strongly Pseudoradial Spaces

Let **SPsRad** denote the full subcategory of the usual category **Top** of topological spaces and continuous functions whose objects are strongly pseudoradial.

Recall that if $\mathscr{A}$ is any class of topological spaces, the *coreflective hull* of $\mathscr{A}$ is the subcategory $CH(\mathscr{A})$ of **Top** whose objects are the spaces homeomorphic to a quotient of a topological sum of the objects in $\mathscr{A}$. For instance, the sequential category **Seq** is the coreflective hull of the singleton $\{\omega + 1\}$ and the compactly generated category **CG** is the coreflective hull of the class of all compact Hausdorff spaces.

Proposition 6.1. *If $\mathscr{S}$ is the class of well-ordered spaces of the form $\alpha \cup \{\infty_\alpha\}$ where α is a non-compact well-ordered space, then $CH(\mathscr{S}) =$* **SPsRad**.

Proof. It suffices to show that a space X is strongly pseudoradial if and only if X is the quotient of a topological sum of elements of $\mathscr{S}$.

If X is strongly pseudoradial, then for each non-closed set $A \subset X$, there is a non-compact ordinal β_A and a map $f_A : \beta_A \cup \{\infty_{\beta_A}\} \to X$ such that $f(\beta_A) \subset A$ and $f(\infty_{\beta_A}) \notin A$. It is straightforward to see that X is the quotient of $\coprod_A (\beta_A \cup \{\infty_{\beta_A}\})$ using the maps f_A. The other direction follows from the fact that every quotient of a strongly pseudoradial space is strongly pseudoradial. The proof of this fact is the same as the well-known proof that the quotient of a sequential space is sequential. □

Corollary 6.2. *If $\mathscr{O}$ is the class of well-ordered spaces, then $CH(\mathscr{O}) =$* **SPsRad**.

Proof. Certainly $\mathscr{S} \subset \mathscr{O}$; thus, $CH(\mathscr{S}) \subseteq CH(\mathscr{O})$. Recall that every well-ordered space β is a retract of an element of $\mathscr{S}$ (see [3, Proposition 2.5] for a more general statement). Thus, $CH(\mathscr{O}) = CH(\mathscr{S})$. □

Since **SPsRad** is a coreflective hull of a class of compact Hausdorff spaces, we are motivated to use [2, Theorem 4.4] to show that **SPsRad** inherits the structure of a Cartesian closed category (which is, in fact, a convenient category of topological spaces, in the sense of [21], since it contains the sequential category [2, Proposition 7.3]). We follow the usual construction of an internal product and function space for coreflective hulls [2].

A subset $A \subset X$ is $\mathscr{S}$-closed if for every non-compact well-ordered space α and map $f : \alpha \cup \{\infty_\alpha\} \to X$ with $f(\alpha) \subset A$, then $f(\infty_\alpha) \in A$. Observe that the $\mathscr{S}$-closed sets determine a topology on the underlying set of X which is finer than the topology of X. Let $\mathbf{S}X$ denote the resulting strongly pseudoradial space. The functor $\mathbf{S} : \mathbf{Top} \to \mathbf{SPsRad}$ is a coreflection in the sense that it is right adjoint to the inclusion $\mathbf{SPsRad} \to \mathbf{Top}$.

This construction provides the internal categorical product $X \times_{\mathscr{S}} Y = \mathbf{S}(X \times Y)$ for **SPsRad**. The internal mapping spaces are constructed as follows. Given strongly pseudoradial spaces X and Y, the set of all continuous functions $X \to Y$ is denoted $M(X, Y)$. For any non-compact well-ordered space α, map $t : \alpha \cup \{\infty_\alpha\} \to X$, and open set $U \subset Y$, let $W(t, U) = \{f \in M(X, Y) | ft(\alpha \cup \{\infty_\alpha\}) \subseteq U\}$. The *$\mathscr{S}$-open topology* on $M(X, Y)$ is the topology generated by all of the subbasic sets of the form $W(t, U)$. Let $M_{\mathscr{S}}(X, Y)$ denote $M(X, Y)$ with the $\mathscr{S}$-open topology.

It is not necessarily true that $M_{\mathscr{S}}(X, Y)$ is strongly pseudoradial. Therefore, we take the function space in **SPsRad** to be the coreflection $\mathbf{S}M_{\mathscr{S}}(X, Y)$. We arrive at the main result of this section: **SPsRad** is a Cartesian closed coreflective subcategory of **Top**. This result is in contrast with the fact that the category **PsRad** of pseudoradial spaces is not Cartesian closed [4]. Moreover, since **SPsRad** contains the sequential category, **SPsRad** is a "convenient category" in the sense of [21].

Theorem 6.3. *The category* **SPsRad** *with $\mathscr{S}$-product $X \times_{\mathscr{S}} Y$ and function space $\mathbf{S}M_{\mathscr{S}}(X, Y)$ is Cartesian closed, i.e., for any strongly pseudoradial spaces X, Y, and Z, there is a natural homeomorphism*

$$\mathbf{S}M_{\mathscr{S}}(X, \mathbf{S}M_{\mathscr{S}}(Y, Z)) \cong \mathbf{S}(X \times_{\mathscr{S}} Y, Z).$$

Proof. The theorem follows directly from [2, Theorem 4.4] once the following two conditions are verified: (1) For each $\alpha \cup \{\infty_\alpha\}$ and $\beta \cup \{\infty_\beta\} \in \mathscr{S}$, the direct product $(\alpha \cup \{\infty_\alpha\}) \times (\beta \cup \{\infty_\beta\})$ is strongly pseudoradial, and (2) $\mathscr{S}$ is a regular class of spaces [2, Definition 2.2] ($\mathscr{S}$ is regular if, for each element $\gamma \in S$ with $S \in \mathscr{S}$, every neighborhood U of γ in S contains a closed neighborhood C for which there is a surjection $s : B \to C$ with $B \in \mathscr{S}$.

(1) Suppose $\alpha \cup \{\infty_\alpha\}$, $\beta \cup \{\infty_\beta\} \in \mathscr{A}$ and, without loss of generality, that $\beta < \alpha$. Since $(\alpha \cup \{\infty_\alpha\}) \times (\beta \cup \{\infty_\beta\})$ is compact, it is a closed subset of $(\alpha \cup \{\infty_\alpha\})^2$. Since $(\alpha \cup \{\infty_\alpha\})^2$ is strongly pseudoradial (See Lemma 6.5 below) and **SPsRad** is closed under taking closed subsets, $(\alpha \cup \{\infty_\alpha\}) \times (\beta \cup \{\infty_\beta\})$ is strongly pseudoradial.

(2) Suppose $\alpha \cup \{\infty_\alpha\} \in \mathscr{S}$. If $\gamma \in \alpha \cup \{\infty_\alpha\}$ is an isolated point, we set $B = \alpha \cup \{\infty_\alpha\}$ and the constant map $s : B \to C = \{\gamma\}$ suffices. If γ is a limit point of $\alpha \cup \{\infty_\alpha\}$, we may assume $U = (\gamma_0, \gamma]$ for $\gamma_0 < \gamma$. For any $\gamma_0 < \gamma' < \gamma$, we have $[\gamma', \gamma] = [\gamma', \gamma) \cup \{\gamma\} \in \mathscr{S}$. Thus, we set $B = C = [\gamma', \gamma]$ and let $s : B \to C$ be the identity map. □

The main difficulty in the proof of the above theorem is verifying that the product of two well-ordered spaces is strongly pseudoradial. The following technical lemma is required for the proof of Lemma 6.5.

Lemma 6.4. *Suppose K is a compact, well-ordered space with minimal element 0. Suppose $(M, m) \in K \times K$. Suppose $B \subset [0, M] \times [0, m]$ such that for all $k < M$, $([0, k] \times [0, m]) \cap B$ is closed in $[0, M] \times [0, m]$ and such that $\emptyset = (\{M\} \times [0, m]) \cap B = ([0, M] \times \{m\}) \cap B$. Suppose $(M, m) \in \overline{B} \setminus B$. Then there exists a limit point $l \in [0, M]$ and a map $f : [0, l] \to [0, M] \times [0, m]$ such that $f([0, l)) \subset B$ and $f(l) = (M, m)$.*

Proof. We will define $f : [0, l] \to [0, M] \times [0, m]$ so that if $f(k) = (x(k), y(k))$, then each of the maps x and y is strictly increasing.

To achieve this, at each stage of the definition of f we make as little strict progress in the direction of $[0, M]$ as possible, while guaranteeing positive progress in the direction of $[0, m]$. Thus, to implement the transfinite recursive definition of f, if k is not a limit point of $[0, M]$, (and working within B) $f(k)$ is defined so that starting at $f(k - 1)$ we move our current abscissa as little as possible strictly to the right subject to the demand that strict vertical progress is possible at the new abscissa. Then, having selected our new abscissa, we then claim as much vertical progress as possible. If k is a limit point of $[0, M]$, then the continuity of $f|_{[0,k)}$ (and compactness of $[0, M] \times [0, m]$) forces the definition of $f(k)$ to be the unique value such that $f|_{[0,k]}$ is continuous.

Before defining f, we build a few basic observations following directly from our hypotheses and previous observations.

Observation 0. For all $k < M$, $(\{k\} \times [0, m]) \cap B$ is closed in $[0, M] \times [0, m]$.

Observation 1. M is a limit point of $[0, M]$. (To obtain a contradiction, suppose otherwise. Then $\{M\}$ is open in $[0, M]$, and hence, since (M, m) is a limit point of B, the open set $\{M\} \times [0, m]$ contains a point $(M, y_m) \in B$ such that $(M, y_m) \neq (M, m)$, contradicting the hypothesis that $\emptyset = (\{M\} \times [0, m]) \cap B$.)

Observation 2. By a symmetric argument applied to Observation 1, m is a limit point of $[0, m]$.

Observation 3. By observations 1 and 2, basic open sets $U \times V$ of $[0, M] \times [0, m]$ containing (M, m) are of the form $(a, M] \times (b, m]$ with $a < M$ and $b < m$.

Observation 4. If W is an open set of $[0, M] \times [0, m]$ such that $(M, m) \in W$, then, since (M, m) is a limit point of B, Observation 3 ensures there exists $(x, y) \in W \cap B$ such that $x < M$ and $y < m$, and also, for each $(x, y) \in W \cap B$, there exists $(x^*, y^*) \in W \cap B$ such that $x < x^* < M$ and $y < y^* < m$.

For each $k \in [0, M]$, define $B_k = (\{k\} \times [0, m]) \cap B$. If $B_k \neq \emptyset$ (by Observation 0), let m_k be minimal such that $B_k \subset \{k\} \times [0, m_k]$. (Thus, (k, m_k) is the "maximal" element of B_k .)

Note $B \neq \emptyset$ since $\overline{B} \neq \emptyset$. Obtain $x_0 \in [0, M]$ minimal such that $B_{x_0} \neq \emptyset$. Define $f(0) = (x_0, m_0)$ and let $y_0 = m_0$. By hypothesis, $x_0 < M$ and $y_0 < m$.

Suppose $k \in [0, M]$ and $f(i) \in B$ for all $i < k$ so that all of the following hold:

(i) If $i < k$, then $f(i) = (x_i, y_i) \in B$.

(ii) If $i < k$, then $i \leq y_i$ and $i \leq x_i$.

(iii) If $i < j < k$, then $x_i < x_j < M$ and $y_i < y_j < m$.

(iv) $f|_{[0,k)}$ is continuous.

If $k - 1$ exists, obtain $x_k \in [0, M]$ minimal such that $x_{k-1} < x_k < M$, $B_k \neq \emptyset$, and $y_{k-1} < m_k < m$. Define $f(k) = (x_k, m_k)$ and let $y_k = m_k$.

To see that $f(k)$ is well defined, let $W = (x_{k-1}, M] \times (y_{k-1}, m]$. Observation 4 ensures the existence of the desired $f(k)$. Conditions (i) and (iii) are preserved by definition. To check condition (ii), let $i = k - 1$. Thus, $k - 1 \leq x_{k-1}$ and $k - 1 \leq y_{k-1}$. Thus, $(k - 1) + 1 \leq x_k$ and $(k - 1) + 1 \leq y_k$. For condition (iv), notice that $[0, k + 1) = [0, k) \cup \{k\}$ and $[0, k) = [0, k - 1]$. Thus, $[0, k - 1)$ is the union of two disjoint closed sets, and hence continuity of $f|_{[0,k+1)}$ follows from the familiar pasting from general topology.

If $k-1$ does not exist, then k is a limit point of K, and define $f(k) = (\sup_{i<k}\{x_i\}, \sup_{i<k}\{y_i\}) = (x_k, y_k)$. Condition (iii) and the l.u.b. property of K ensure $f(k)$ is well defined.

To check continuity of $f|_{[0,k]}$, suppose W is a basic open set of $[0,M]\times[0,m]$. Let $U = f|^{-1}_{[0,k]}(W)$. To check (iv), if $(M,m)\notin W$, then continuity of $f|_{[0,k)}$ ensures U is open in $[0,k)$ and (since $[0,k)$ is open in $[0,k]$), U is open in $[0,k]$. Suppose $(M,m)\in W$. Then (since $f|_{[0,k)}$ is increasing), there exists $a\in[0,M)$ such that $f^{-1}_{[0,k)}(W) = (a,k)$. Thus, $f^{-1}_{[0,k]}(W) = (a,k)\cup\{k\} = (a,k]$. Since $(a,k]$ is open in $[0,k]$, we conclude that $f|_{[0,k]}$ is continuous. By definition, $f|_{[0,k+1)} = f|_{[0,k]}$, and thus condition (iv) is preserved since $f|_{[0,k+1)}$ is continuous.

Condition (ii) for $i<k$, combined with continuity of $f|_{[0,k]}$, ensures condition (ii) is preserved for $i\le k+1$.

By definition, $f(k) = (x_k, y_k)$. Preservation of the remaining conditions depend on whether $f(k)\in B$ or not.

Case 1. If $f(k)\in B$, then by hypothesis of the lemma, $x_i<M$ and $y_i<m$. Moreover, since $\{x_i\}$ and $\{y_i\}$ for $i\le k$ are strictly transfinite sequences, condition (iii) is preserved.

Case 2. Suppose $f(k)\notin B$. Then, by continuity of f, $f(k)$ is a limit point of B.

Since (M,m) is the only missing limit point of B, we have $f(k) = (M,m)$. Taking $l=k$ completes the proof of the lemma. □

Lemma 6.5. *If α is a non-compact well-ordered space, then the product $(\alpha\cup\{\infty_\alpha\})\times(\alpha\cup\{\infty_\alpha\})$ (with the standard product topology) is strongly pseudoradial.*

Proof. Suppose A is a non-closed subset of $(\alpha\cup\{\infty_\alpha\})\times(\alpha\cup\{\infty_\alpha\})$.

We seek a well-ordered subspace $\beta\subset\alpha$ such that $\overline{\beta}\backslash\beta = \{\infty_\beta\}$ contains only the minimum element of $\{\gamma\in\alpha|\gamma>\beta\}$ and a map $f:\beta\cup\{\infty_\beta\}\to(\alpha\cup\{\infty_\alpha\})\times(\alpha\cup\{\infty_\alpha\})$ such that $f(\beta)\subset A$ and $f(\infty_\beta)\notin A$.

First, we reduce as follows to the case that each "vertical or horizontal slice" of A is closed in $(\alpha\cup\{\infty_\alpha\})\times(\alpha\cup\{\infty_\alpha\})$. For each $x\in\alpha\cup\{\infty_\alpha\}$, define $B_x = A\cap(\{x\}\times(\alpha\cup\{\infty_\alpha\}))$. Note each subspace $\{x\}\times(\alpha\cup\{\infty_\alpha\})$ is closed in $(\alpha\cup\{\infty_\alpha\})\times(\alpha\cup\{\infty_\alpha\})$ and is also canonically homeomorphic to the strongly pseudoradial space $\alpha\cup\{\infty_\alpha\}$. Thus, if there exists $x\in\alpha\cup\{\infty_\alpha\}$ such that B_x is not closed in $(\alpha\cup\{\infty_\alpha\})\times(\alpha\cup\{\infty_\alpha\})$, then there exists a non-compact well-ordered subspace $\beta\subset\alpha\cup\{\infty_\alpha\}$ and a map $f:\beta\cup\{\infty_\beta\}\to\{x\}\times(\alpha\cup\{\infty_\alpha\})$ such that $f(\beta)\subset B_x$ and $f(\infty_\beta)\notin B_x$. Since $\{x\}\times(\alpha\cup\{\infty_\alpha\})$ is closed in $(\alpha\cup\{\infty_\alpha\})\times(\alpha\cup\{\infty_\alpha\})$, it follows that $f(\infty_\beta)\in\{x\}\times(\alpha\cup\{\infty_\alpha\})$. Hence, $f(\infty_\beta)\notin A$ and we have the desired map f.

After applying a symmetric argument to slices of the form $(\alpha\cup\{\infty_\alpha\})\times\{y\}$, we have reduced to the case that the subspaces $(\{x\}\times(\alpha\cup\{\infty_\alpha\}))\cap A$ and $(\alpha\cup\{\infty_\alpha\})\times\{y\})\cap A$ are closed in $(\alpha\cup\{\infty_\alpha\})\times(\alpha\cup\{\infty_\alpha\})$ for all $\{x,y\}\subset(\alpha\cup\{\infty_\alpha\})$.

Let K be a compact set. Note that $(\{0\}\times K)\cap A$ is closed and $(K\times K)\cap A$ is not closed. Hence, there exists $M\in K$ minimal such that $([0,M]\times K)\cap A$ is not closed. By our assumptions, $([0,M]\times\{0\})\cap A$ is closed. Thus, there exists $m\in K$ minimal such that $([0,M]\times[0,m])\cap A$ is not closed in $K\times K$.

Let $C=([0,M]\times[0,m])\cap A$. Since $[0,M]\times[0,m]$ is closed in $K\times K$, C is not closed in $[0,M]\times[0,m]$. Hence, $\overline{C}\backslash C\neq\emptyset$ and $\overline{C}\backslash C\subset[0,M]\times[0,m]$. Suppose $(x,y)\in\overline{C}\backslash C$. To obtain a contradiction, suppose $x<M$. Then (x,y) is a limit point of the closed set $([0,x]\times[0,m])\cap A$, and hence $(x,y)\in A$, a contradiction. Thus, $x=M$. To obtain a contradiction, suppose $y<m$. Then (M,y) is a limit point of the closed set $([0,M]\times[0,y])\cap A$, and hence $(M,y)\in A$, a contradiction. Hence, $\overline{C}\backslash C=\{(M,m)\}$.

Note that (M,m) is not in the closed set $(([0,M]\times\{m\})\cup(\{M\}\times[0,m]))\cap C$. Obtain $a\in[0,M)$ and $b\in[0,m)$ such that $([a+1,M]\times\{m\})\cap C=(\{M\}\times[b+1,m])\cap C=\emptyset$. Note that $[a+1,M]\times[b+1,m]$ is clopen in $[0,M]\times[0,m]$. Let $B=[a+1,M]\times[b+1,m]\cap C$. Note that (K,B,M,m) satisfies the hypothesis of Lemma 6.4 and obtain a limit point $l\in[0,M]$ and a map $f:[0,l]\to\overline{B}$ such that $f([0,l))\subset B\subset A$ and $f(l)\notin B$. Let $\beta=[0,l)$ and $\infty_\beta=l$. By definition, $\overline{B}\backslash B\subset\overline{C}\backslash C\subset\overline{A}\backslash A$, and hence $f(l)\in A$. □

7. Compactness and Related Properties in SPsRad

We seek to generalize Proposition 7.1 below, a basic fact in **SEQ.** Our main result is Theorem 7.10. Lemma 7.16 shows Theorem 7.10 is a generalization of Proposition 7.1, provided we restrict our attention to so-called *UW* spaces. Corollary 7.11 generalizes in **SPsRad** the fact that compact weakly Hausdorff sequential spaces are sequentially compact.

Proposition 7.1. *If X is a sequential space, then X is countably compact if and only if X is sequentially compact* [12].

The following strengthening of "sequentially compact" is strict as shown by the minimal uncountable well-ordered space.

Definition 7.2. The space X is *strongly pseudoradially compact* if, for each noncompact well-ordered space α and each map $f:\alpha\to X$, there exists β closed and cofinal in α so that $f|\beta$ is continuously extendable at ∞_α.

Corollary 7.11 shows that every compact weakly Hausdorff space $X \in$ **SPsRad** is strongly pseudoradially compact, but the proof exploits the fact that maps of compact ordinals into X have closed image. To absorb various difficulties created when X is not weakly Hausdorff, we adjust our definitions as follows.

Definition 7.3. Suppose X is a space and α is a noncompact well-ordered space. A map $f : \alpha \to X$ is *decent* if there exists a map $g_f : \alpha \to X$ so that $g(i) \in \overline{\{f(i)\}}$ for all $i \in \alpha$, and $g(C)$ is closed for all compact $C \subset \alpha$.

Definition 7.4. If X is a space, declare X *decently strongly pseudoradially compact* if, for each noncompact well-ordered space α and each decent map $f : \alpha \to X$, there exists β closed and cofinal in α so that $f|\beta$ is continuously extendable at ∞_α.

Equivalent to the standard finite open subcover formulation, recall a space X is compact if and only if $\emptyset \neq \cap_{i \in I} A_i$ for all collections of closed sets $\{A_i\}$ with the finite intersection property. Weaker than compact, the closed set formulation of "countably compact" is $\emptyset \neq \cap_{n=1}^{\infty} L_n$ for each nested sequence of closed sets $\ldots L_3 \subset L_2 \subset L_1$, and this motivates the following definition.

Definition 7.5. The space X is *decently compact* if for each decent map $f : \alpha \to X$, then $\emptyset \neq \cap_{i \in \alpha} L_i$ if $L_i = \overline{\cup_{j=i}^{\infty_\alpha} f(j)}$.

We observe basic facts about decent maps and decently compact spaces as follows.

Remark 7.6. Suppose the map $g_f : \alpha \to X$ shows the map $f : \alpha \to X$ is decent. Then $\{g(i)\}$ is closed in X since $\{i\}$ is compact. If X is T_1, then $g_f = f$ since $\{f(i)\} = \overline{\{f(i)\}}$. If X is weakly Hausdorff, then each map $f : \alpha \to X$ is decent. If X is compact, then X is decently compact. If X is a space and the map $g_f : \alpha \to X$ shows f is decent, then $\cap_{i \in \alpha}(\overline{\cup_{j=i}^{\infty_\alpha} g_f(j)}) \subset \cap_{i \in \alpha}(\overline{\cup_{j=i}^{\infty_\alpha} f(j)})$ (since $\cup_{j=i}^{\infty_\alpha} g_f(j) \subset \cup_{j=i}^{\infty_\alpha} \overline{\{f(j)\}} \subset \overline{\cup_{j=i}^{\infty_\alpha} f(j)}$). Since g_f is decent, the space X is decently compact if and only if $\emptyset \neq \cap_{i \in \alpha}(\overline{\cup_{j=i}^{\infty_\alpha} f(j)})$ for all decent maps $f : \alpha \to X$ such that $\overline{\{f(i)\}} = \{f(i)\}$ for all $i \in \alpha$.

The following elementary lemma is used in the proof of Lemma 7.8.

Lemma 7.7. *Suppose $X \in$ **SPsRad**, Y is a space, and $f : X \to Y$ is a function. Then f is continuous if and only if $f \circ g$ is continuous whenever α is a noncompact well-ordered space and $g : \alpha \cup \{\infty_\alpha\} \to X$ is a map.*

Proof. If f is continuous, then the composition of maps $f \circ g$ is continuous. If f is not continuous, obtain $A \subset Y$ closed so that $B = f^{-1}(A)$ is not closed. Obtain a well-ordered space α and map $g : \alpha \cup \{\infty_\alpha\}$ so that $g(\alpha) \subset B$ and $g(\infty_\alpha) \notin B$. Then $f \circ g$ is not continuous since, if $f \circ g$ were continuous, we obtain the contradiction $(f \circ g)^{-1}(A) = \alpha \cup \{\infty_\alpha\}$. $\square$

Lemma 7.8. *Suppose X is strongly pseudoradial and decently compact. Suppose $f : \alpha \to X$ is decent and $f(i) = \overline{\{f(i)\}}$ for all $i \in \alpha$. Then there exists β closed and cofinal in α, so that $f|\beta$ is continuously extendable to $f|\beta \cup \{\infty_\alpha\}$.*

Proof. Note if g_f shows f is decent, then $g_f = f$. In particular, $f(C)$ is closed for all compact $C \subset X$. If $f^{-1}(x)$ is unbounded for some $x \in X$, then $x = \overline{\{x\}}$, and letting $\beta = f^{-1}(x)$, we have β closed and cofinal in α, and $f|\beta$ is continuously extendable mapping $\infty_\beta \to x$. If $f^{-1}(x)$ is bounded for all x, then, by lemmas 3.1 and 3.4, there exists a regular cardinal α_1 closed and cofinal in α so that $f|\alpha_1$ is one to one. Since $f|\alpha_1$ is decent, we may assume, without loss of generality, that α is a regular cardinal and f is one to one. Observe that $\emptyset = \cap_{i\in\alpha} f([i, \infty_\alpha))$ since f is one to one. To obtain a contradiction, suppose there exists β closed and cofinal in α so that $f(\beta \cap [i, \infty_\alpha))$ is closed in X for all $i \in \alpha$. Then $\emptyset = \cap_{i\in\alpha} f(\beta \cap [i, \infty_\alpha))$, contradicting the fact that X is decently compact. Thus, there exists $i \in \alpha$ so that $f[i, \infty_\alpha)$ is not closed. Since $[i, \infty_\alpha)$ is closed and cofinal in α, the subspace $[i, \infty_\alpha)$ is a regular cardinal and thus, without loss of generality, we may assume $f(\alpha)$ is not closed in X.

Since $f(\alpha)$ is not closed in X, by Theorem 3.5, obtain a noncompact regular cardinal γ and an extendable injective map $g : \gamma \to f(\alpha)$ so that $g(\infty_\gamma) \notin f(\alpha)$. Define $h : \gamma \to \alpha$ via $h = f^{-1} \circ g$. Let $\beta = h(\gamma)$. The plan is to show h is a closed homeomorphism and β is cofinal in α.

Since h is one-to-one, $|\gamma| \leq |\alpha|$. Thus, since γ is a regular cardinal if $C \subset \gamma$ is bounded, $|C| < |\gamma| \leq |\alpha|$ and, in particular, $h(C)$ is bounded in α.

To show that h is continuous, suppose κ is a noncompact well-ordered space and $p : \kappa \cup \{\infty_\kappa\} \to \gamma$ is a map. Then $im(p)$ is compact and thus bounded. Thus, by the previous paragraph, $im(h(p))$ is bounded in α. Obtain a compact $K \subset \alpha$ so that $im(h(p)) \subset K$. By hypothesis, $f|K$ is a closed embedding. Thus, $f^{-1}|f(K)$ is continuous. Hence, $f^{-1}g\kappa$ is continuous. Thus, by Lemma 7.7 h is continuous.

Suppose $A \subset \gamma$ and $B = h(A)$. If A is compact, then (since h is continuous), B is compact, and hence B is closed. If A is unbounded, then $g(\infty_\gamma)$ is a limit point of $g(A)$, and hence $g(\infty_\gamma)$ is a limit point of $f(B) = f(f^{-1}(g(A)))$. Thus, B is unbounded since otherwise we obtain the contradiction $g(\infty_\gamma) \in f(\overline{B})$. If A is closed and unbounded, to see that B is closed, suppose otherwise and obtain minimal $l \in \overline{B} \backslash B$. Let $B_1 = [0, l]$ and $A_1 = h^{-1}(B_1)$. Then A_1 is unbounded since otherwise we obtain the contradiction $l \in h(A_1) \subset B_1$. Thus, applying the same argument as shown above, we deduce that $g(\infty_\gamma)$ is a limit point of $g(A_1)$ and obtain the contradiction that B_1 is unbounded.

Since $h : \gamma \to \beta$ is a homeomorphism, $f|\beta$ is continuously extendable, with $\infty_\alpha \to g(\infty_\gamma)$. □

Lemma 7.9. *Suppose α is an unbounded well-ordered space and $g : \alpha \cup \{\infty_\alpha\} \to X$ is a map such that $\{g(i)\}$ is closed in X for all $i \in \alpha$. Suppose $f : \alpha \to X$ is a map such that $g(i) \in \overline{\{f(i)\}}$ for all $i \in \alpha$. Then $F = f \cup \{\infty_\alpha, g(\infty_\alpha)\}$ is continuous.*

Proof. By hypothesis, $F|\alpha$ is continuous. To check F is continuous at ∞_α, suppose U is open in X and $g(\infty_\alpha) \in U$. By continuity of g, obtain K so that $g(i) \in U$ if $K \leq i$. Thus, $f(i) \in U$ if $K \leq i$. □

Theorem 7.10. *If X is strongly pseudoradial, then the following are equivalent:*

(1) *X is decently strongly pseudoradially compact;*
(2) *X is decently compact.*

Proof. Suppose $f : \alpha \to X$ is decent as shown by the map g_f.

(1) ⇒ (2) By (1), obtain β closed and cofinal in α and obtain $y \in X$ so that $G = g_f|\beta \cup \{(\infty_\alpha, y)\}$ is continuous. Since G is continuous at ∞_α, $y \in \overline{g_f([i, \infty_\alpha))}$ for all $i \in \alpha$. By Remark 7.6, $y \in \overline{f([i, \infty_\alpha))}$ for all $i \in \alpha$. Thus, X is decently compact.

(2) ⇒ (1) By Lemma 7.8, obtain β closed and cofinal in α and obtain $y \in X$ so that $G = g_f|\beta \cup \{(\infty_\alpha, y)\}$ is continuous. By Lemma 7.9, $f|\beta \cup \{\infty_\alpha, y\}$ is continuous. □

Corollary 7.11. *Suppose X is strongly pseudoradial and weakly Hausdorff. If X is compact, then X is strongly pseudoradially compact.*

Proof. Since X is compact, X is decently compact. Since X is weakly Hausdorff, each map $f : \alpha \to X$ is decent. Now apply Theorem 7.10. □

Example 7.12. *The main example of A. J. Ostaszewski* [17] *shows that the converse of Corollary is false. If X is strongly pseudoradial, weakly Hausdorff, and strongly pseudoradially compact, then X need not be compact.*

Proof. The main space X in [17] is Hausdorff, sequential, and sequentially compact, but not compact. Since X is sequential, X is strongly pseudoradial. To check that X is strongly pseudoradially compact, suppose α is an unbounded well-ordered space and $f : \alpha \to X$ is a map. If α is countable, obtain a sequence β cofinal in α and apply sequential compactness to obtain a continuous extension of $f|\beta$. Suppose α is uncountable. If $f^{-1}(x)$ is unbounded for some x, let $\beta = f^{-1}(x)$ and obtain the extension $\infty_\alpha \to x$. To obtain a contradiction, suppose $f^{-1}(x)$ is bounded for all x. By Lemma 3.1, obtain β closed and cofinal in α so that $f|\beta$ is one to one. Since X is Hausdorff, $f|C$ is an embedding for all compact $C \subset \beta$. Thus, $f(\beta)$ is closed in X (since if $f(\beta)$ is not closed in the sequential space X, we obtain a cofinal sequence in the uncountable well-ordered set β). Thus, f embeds β onto a closed subspace $f(\beta) \subset X$. Obtain a closed subspace $A \subset f(\beta)$ so that A and $f(\beta)\backslash A$ are uncountable. Then $X\backslash A$ is open. However, both $X\backslash A$ and A are uncountable, contradicting the fact that in X, every open subspace is countable or cocountable [17, Lemma 1.3]. □

Unfortunately, Theorem 7.10 is not a generalization of Proposition 7.1 as shown by the following example, in which X is vacuously decently compact since no map $f : \alpha \to X$ is decent.

Example 7.13. *Consider the countable set $X = \{1, 2, 3, \dots\}$ with topology generated by the closed sets $[n, \infty)$. Then X is sequential and decently compact, but X is not sequentially compact.*

Lemma 7.16 effectively shows that the phenomenon in Example 7.13 is the only obstruction preventing a sequential decently compact space from being sequentially compact. Thus, Theorem 7.10 generalizes Proposition 7.1 provided we restrict ourselves to spaces with the following useful weak property, which we call the UW property in the paper at hand.

Definition 7.14. The space X has the *UW property* if, for each $x \in X$, there exists $y \in X$ so that $y \in \overline{\{x\}}$ and $\overline{\{y\}} = \{y\}$.

Lemma 7.15. *If X is compact or T_1, then X has the UW property.*

Proof. Suppose $x \in X$. Obtain a maximal transfinite sequence (indexed by a well-ordered set) $x = x_0, x_1, x_2, \dots$ so that $x_i \in \overline{\{x_j\}}\backslash\{x_j\}$ for all $j < i$. Let $L_i = \overline{\{x_i\}}$. If X is compact, $\emptyset \neq \cap L_i$. Let $y \in \cap L_i$. Note that $y \in \overline{\{x\}}$, and by maximality $\overline{\{y\}} = \{y\}$ (since if there exists $z_j \in \overline{\{y\}}\backslash\{y\}$, we obtain the contradiction $y \notin L_j$). If X is T_1, let $y = x$. □

Lemma 7.16. *Suppose the sequential space X has the UW property. Then X is sequentially compact if and only if X is decently compact.*

Proof. Suppose X is sequentially compact and $f : \alpha \to X$ is decent as shown by the map $g_f : \alpha \to X$. If there exists $x \in X$ so that $g_f^{-1}(x)$ is unbounded, let $\beta_1 = g_f^{-1}(x)$ and observe that $g_f|\beta_1$ is continuously extendable at ∞_α. If no such x exists, obtain by Lemma 3.1, β closed and cofinal in α so that $g_f|\beta$ is one to one. By Lemma 3.4, we may assume

β is a regular cardinal. If β is countable, then β has the discrete topology. Since X is sequentially compact, there exists β_1 closed and cofinal in β so that $g_f|\beta_1$ is continuously extendable at ∞_α. If β_1 is uncountable, we obtain a contradiction as follows. Since X is sequential, since $g_f|C$ is closed for all compact $C \subset \beta_1$, and since β_1 is uncountable, $g_f(\beta_2)$ is closed for all β_2 closed and cofinal in β_1. In particular, the uncountable well-ordered space β_1 is homeomorphic to the sequential space $g_f(\beta_1)$, and we have a contradiction. Thus, by Theorem 7.10, X is decently compact.

Conversely, suppose X is decently compact. Consider the minimal infinite well-ordered space $\alpha = \{1, 2, 3 \ldots\}$ and suppose $f : \alpha \to X$ is a map. Since X has the UW property, obtain for each $n \in \alpha$, $y_n \in \overline{\{f(n)\}}$ so that $\{y_n\} = \overline{\{y_n\}}$. Let $g_f(n) = y_n$. The map g_f shows f is decent. Thus, by Theorem 7.10, there exists β closed and cofinal in α so that $f|\beta$ is continuously extendable at ∞_α. Thus, $f|\beta$ is the desired convergent subsequence, and hence X is sequentially compact. □

We conclude with the following problem.

Problem 7.17. Suppose the compact T_1 space $X \in$ **SPsRad**. Must X be strongly pseudoradially compact?

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